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Heterogeneous Asynchronous time integrators for coupling methods

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Introduction

General formalism

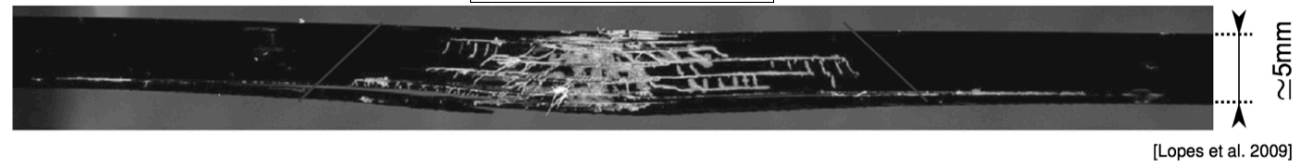
Stability

HATI

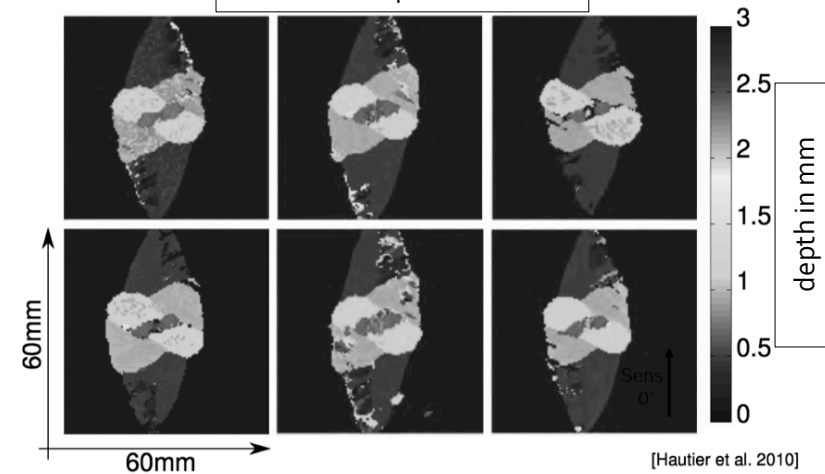
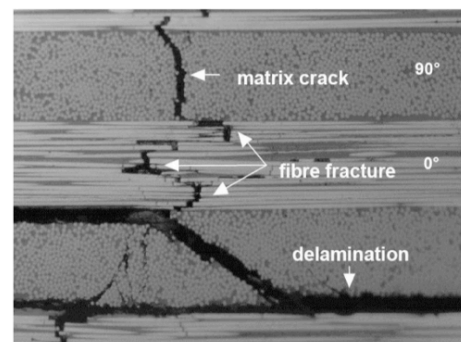
Conclusions

● Low energy impact on composite panels

Cone of damage



6 different experiments



→ spatial localization of damage and thickness of the ply $\ll$ size of the structure (helical cracks)

→ implicit-explicit co-simulation (Zebulon / Europlexus) [Chantrai 2014]

Introduction

General formalism

Stability

HATI

Conclusions



Introduction

General formalism

Stability

HATI

Conclusions



Introduction

General formalism

Stability

HATI

Conclusions

Introduction

● *Nonlinear transient dynamics*

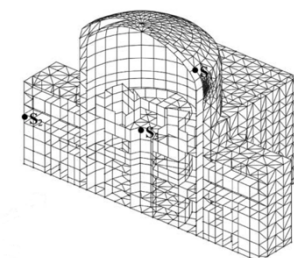
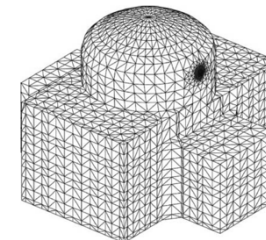
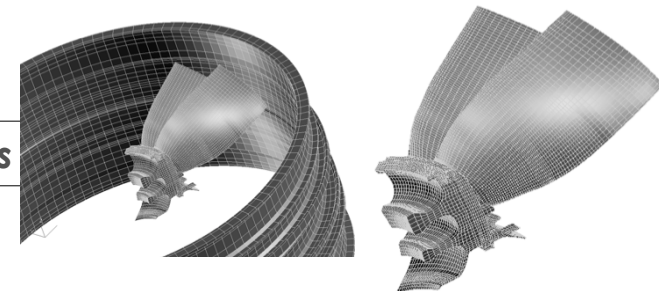
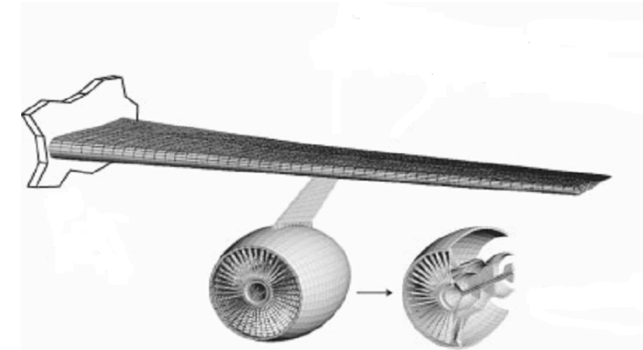
- *Earthquake*
- *Crash*
- *Impacts on critical components*
- *Fluid structure interaction*
- *Multi-physics*

● *Taking into account localized nonlinearities*

- *Nonlinear behavior (concrete, steel)*
- *Contact, friction*
- *Large deformations*

⇒ *Multi-scale problems
both in space and time*

⇒ *Main goal: overcoming limitation of direct time integrators (a unique integrator and a unique time step for the global mesh)*



Outline

Introduction

General formalism

Stability

HATI

Conclusions

- 1 ***A general formalism of direct time integrators combined with Lagrange multipliers***
- 2 ***Stability analysis***
- 3 ***A unified strategy to control « locally » the accuracy and the time step – Application to a bridge crane under earthquake***
- 4 ***Conclusions & prospects***

Introduction

General formalism

Stability

HATI

Conclusions

1

***A general formalism of direct time integrators combined
with Lagrange multipliers***

● *Direct time integrators dedicated to transient dynamics are developed for a long time:*

- ||→ *A first result in the linear case: Lax theorem [Lax, Richtmyer 1956]
Stability and consistency are necessary conditions for convergence*
- ||→ *Linear stability: spectral stability based on amplification matrix
Nonlinear stability: norms based on energy metrics*
- ||→ *Explicit integrators (central difference) and implicit ones (Houbolt, Newmark, HHT,...) were studied and utilized with a great success when combined with the finite element method for a wide range of applications in linear and nonlinear computational dynamics [Hilber, Hughes and Taylor 1977, Underwood & Belytschko 1983, Belytschko & Lieu 1989, Chung & Hulbert 1994]*
- ||→ *Still today Newmark time integrators are the most used in commercial softwares for nonlinear computational dynamics [Newmark 1959, Zienckiewicz, Wodd, Hine, Taylor 1984]*
- ||→ *A first ask: what are the choice criteria for time integrators dedicated to computational dynamics?*

➡ *In a first step, it consists in controlling the ACCURACY*

● *5 properties are generally identified:*

- *Unconditional stability for linear analysis*
- *Second order accuracy (Dahlquist barrier for linear multi steps methods (LMS)) [Dahlquist 1963]*
- *A unique implicit equation to solve at each time step*
- *« self-starting » integrators*
- *Control of the numerical dissipation in the "high frequency" range [Hilber & Hughes 1978 (overshoot), Hulbert 2004]*

➡ *Example 1 (limited to Dahlquist barrier): α time integrators (HHT- α , Wood-Bossak-Zienkiewicz (WBZ)- α , Chung-Hulbert (CH)- α) allow to ensure a second order accuracy and numerical dissipation of high frequency spurious oscillations [Hilber Hughes 1977, Wood Bossak Zienkiewicz 1980, Wiberg Zeng Li 1992, Saffan Oden 1993, Chung Hulbert 1993, Pegon 2001, Krenk 2005, Masuri Hoitink Zhou Tamma 2009]*

➡ *Example 2 (not limited by Dahlquist barrier): time discontinuous Galerkin (TDG), time extended finite elements (T-XFEM) allow strong time discontinuities with high order accuracy (greater than 2!) [Yin & al 2000, Kanapady Tamma 2003, Réthoré & al 2005]; space-time finite elements allow a space-time adaptation for controlling both the accuracy and the time step [Hughes Hulbert 1988 and 1990, Palaniappan & al 2004, Abedi & al 2005, Cavin & al 2005]*

● *Historically, the extension to non-linear or non-smooth problems has highlighted the following challenges:*

- *Loss of unconditional stability for implicit time integrators*
- *Loss of accuracy (from order 2 to order 1)*
- *Energy properties to check / enforce*
- *Overshoot, high frequency numerical behavior*
- *Numerical integration of internal forces*

⇒ *Many contributions for 50 years until today propose to overcome these difficulties:*

- *Integrators ensuring balance of linear momentum, angular momentum, energy, entropy [Simo et Tarnow 1992, Armero et Petocz 1998, Betsch et Steinmann 2001, Botasso et al 2001, Krenk 2006, Romero 2009]*
- *Symplectic integrators (Hamiltonian approaches, conservative problems) [Simo et al 1992, Marsden et West 2001, Chhay Hoarau Hamdouni Sagaut 2011]*
- *Variational and α -variational integrators [Cannarozzi & al 1995, Laursen & Chawla 1997, Kuhl & Crisfield 1999, Armero & Romero 2001, Masuri Hoitink Zhou Tamma 2009]*
- *Variational integrators for discrete mechanics [Marsden & West 2001, Hauret Letaltec 2006]*

⇒ *For example, symplectic variational integrators can exactly conserve a discrete Lagrangian with a symplectic structure and have better numerical properties over long periods, port-Hamiltonian approach, Q-structures [Hamdouni et al 2011]; applications also to non-smooth dynamics [Moreau 1999, Jean 1999, Pandolfi Kane Marsden Ortiz 2002, Acary 2008, West 2004, Betsch 2011, Renard 2013]*

⇒ *Secondly, it consists in controlling the TIME STEP*

● *3 conditions are generally identified:*

- *Frequency content of the structure and the prescribed loading*
- *Accuracy and stability properties of integrators (conditionally stable explicit schemes)*
- *Convergence of nonlinear solver for implicit integrators*

⇒ *Adaptive variable time steps strategies are often used to ensure the automatically the accuracy and efficiency in terms of CPU*
[Wood 1990, Zienkiewicz Taylor 1991, Wiberg Zeng Li 1992, Wiberg Li 1999, Schleupen Ramm 2000, Czekanski El-Abbasi Meguid 2001]

⇒ *A first conclusion: the main limitation of all these time discretizations is in using A UNIQUE TIME INTEGRATOR (homogeneous time integrator) and A UNIQUE TIME STEP for all the finite element mesh (synchronous time integrator).*

⇒ *One obvious improvement is to develop heterogeneous (each part of the mesh has its own integrator) asynchronous (each part of the mesh has its own time step) time integrators.*

Introduction

General formalism

Stability

HATI

Conclusions

➡ *Construction of a unified formalism for a large class of time integrators with Lagrange multipliers*

● *Transient non-linear dynamics equation semi-discretized by the finite element method:*

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{f}_{int}(\mathbf{u}(t)) = \mathbf{f}_{ext}(t) \quad \forall t \in [t_0, t_f]$$

$$\mathbf{u}(t_0) = \mathbf{u}_0, \dot{\mathbf{u}}(t_0) = \dot{\mathbf{u}}_0$$

● *Linear case:*

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}_{ext}(t) \quad \forall t \in [t_0, t_f]$$

$$\mathbf{u}(t_0) = \mathbf{u}_0, \dot{\mathbf{u}}(t_0) = \dot{\mathbf{u}}_0$$

➡ *These expressions have the following drawbacks:*

- *Second order hyperbolic in time*
- *Strong formulation in time with restrictive properties of differentiability*
- *Inadequate in presence of temporal discontinuities (non-smooth dynamics, dynamic crack propagation)*

➡ *Inadequate starting point for the construction of modern time integrators (difficult to impose conservation equations other than the equilibrium equation)*

➡ *unsuitable for building heterogeneous asynchronous time integrators*

Introduction

General formalism

Stability

HATI

Conclusions

► We consider as a starting point the simplest weak formulation in time for structural dynamics

● Variational formulation based on the following principle of stationary of the action integral:

$$\bullet \quad \mathcal{A}(\mathbf{u}, \dot{\mathbf{u}}) = \int_{t_0}^{t_f} \mathcal{L}(\mathbf{u}, \dot{\mathbf{u}}) dt \quad \mathcal{L}(\mathbf{u}, \dot{\mathbf{u}}) = \mathcal{T}(\dot{\mathbf{u}}) - \mathcal{V}(\mathbf{u})$$

$$\mathcal{T}(\dot{\mathbf{u}}) = \frac{1}{2} \dot{\mathbf{u}}^t \mathbf{M} \dot{\mathbf{u}}$$

$$\mathcal{V}(\mathbf{u}) = \mathcal{V}_{int} - \mathcal{V}_{ext} = \frac{1}{2} \mathbf{u}^t \mathbf{K} \mathbf{u} - \mathbf{F}_{ext}^t \mathbf{u}$$

$$\delta \mathcal{A} = 0 \quad \forall \delta \mathbf{u} \in \mathcal{U}_0(H^1(\Omega, [t_0, t_f])), \delta \mathbf{u}(\cdot, t_0) = \delta \mathbf{u}(\cdot, t_f) = 0$$

$$\bullet \quad \delta \mathcal{A} = 0 \Leftrightarrow \int_{t_0}^{t_f} \delta \mathbf{u}^t(t) (\mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{K} \mathbf{u}(t) - \mathbf{f}_{ext}(t)) dt = 0$$

► The previous semi-discretized formulation corresponds to the Euler-Lagrange equation of the principle of Hamilton above.

► The corresponding semi-discretized weak formulation also called acceleration form) can be considered as a starting point for the construction of temporal integrators. In this context, it "relaxes" the concept of equilibrium at a given time.

Introduction

General formalism

Stability

HATI

Conclusions

➡ *Passage from a second order time differential equation to a system of two first order time differential equations*

● *Two field Time action integral [Washizu 1969] (Legendre transform on the velocity):*

$$\begin{aligned}\mathcal{A}_{HR}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{v}) &= \int_{t_0}^{t_f} \mathcal{L}_{HR}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{v}) dt & \mathcal{L}_{HR}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{v}) &= \mathcal{T}_{HR}(\dot{\mathbf{u}}, \mathbf{v}) - \mathcal{V}(\mathbf{u}) \\ \delta \mathcal{A}_{HR} &= 0 & \mathcal{T}_{HR}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{v}) &= \mathbf{v}^t \mathbf{M} \dot{\mathbf{u}} - \frac{1}{2} \mathbf{v}^t \mathbf{M} \mathbf{v} \\ \Leftrightarrow \int_{t_0}^{t_f} & -\delta \mathbf{u}^t(t) (\mathbf{M} \dot{\mathbf{v}}(t) + \mathbf{K} \mathbf{u}(t) - \mathbf{f}_{ext}(t)) dt \\ & + \int_{t_0}^{t_f} \delta \mathbf{v}^t(t) \mathbf{M} (\dot{\mathbf{u}}(t) - \mathbf{v}(t)) dt = 0\end{aligned}$$

● *Corresponding Euler-Lagrange equation and symplectic structure: ($\mathbf{p} = \mathbf{M}\mathbf{v}$)*

$$\begin{aligned}\mathbf{M} \dot{\mathbf{v}}(t) + \mathbf{K} \mathbf{u}(t) &= \mathbf{f}_{ext}(t) \quad \forall t \in [t_0, t_f] \\ \dot{\mathbf{u}}(t) - \mathbf{v}(t) &= 0 \quad \forall t \in [t_0, t_f] \\ \mathbf{u}(t_0) &= \mathbf{u}_0, \mathbf{v}(t_0) = \mathbf{v}_0\end{aligned} \quad \Rightarrow \quad \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{u}} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{J}} \begin{bmatrix} \frac{\partial \mathcal{L}_{\mathbf{p}}}{\partial \mathbf{p}} \\ \frac{\partial \mathcal{L}_{\mathbf{p}}}{\partial \mathbf{u}} \end{bmatrix}$$

➡ *The corresponding weak formulation (also called velocity form) introduces the velocity $\mathbf{v}$ as a new variable.
The fundamental state vector becomes:*

$$\mathbf{X} \equiv (\mathbf{u}, \mathbf{v})$$

Introduction

General formalism

Stability

HATI

Conclusions

➡ *This weak formulation in time is the basis of many modern time integrators, for instance rigid solid dynamics [Cardona Geradin 1989, Simo & Wong 1991, Botasso et al 2001, Krenk & Hogsberg 2005, Krenk 2006].*

➡ *On the other hand, the stationary principle highlights the central role of the velocity in structural dynamics. In other-words, the good duality bracket is based on velocity and momentum/impulse, not acceleration and force.*

➡ *Thereafter, the kinematic constraints will be introduced in velocity. For example:*

$$\mathbf{L}\dot{\mathbf{u}}(t) = 0$$

➡ *This condition can be introduced easily into the time action integral with a Lagrange multiplier:*

$$\begin{aligned}\tilde{\mathcal{A}}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{v}, \boldsymbol{\lambda}) &= \int_{t_0}^{t_f} \tilde{\mathcal{L}}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{v}, \boldsymbol{\lambda}) \, dt \\ &= \int_{t_0}^{t_f} (\mathcal{L}_{HR}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{v}) + \boldsymbol{\lambda}^t(t) \mathbf{L}\dot{\mathbf{u}}(t)) \, dt\end{aligned}$$

Introduction

General formalism

Stability

HATI

Conclusions

- We obtain the following three fields stationary principle:

$$\begin{aligned} \delta \tilde{\mathcal{A}} &= 0 \\ \Leftrightarrow \int_{t_0}^{t_f} & -\delta \mathbf{u}^t(t) (\mathbf{M} \dot{\mathbf{v}}(t) + \mathbf{K} \mathbf{u}(t) - \mathbf{f}_{ext}(t) - \mathbf{L}^t \Lambda(t)) dt \\ & + \int_{t_0}^{t_f} \delta \mathbf{v}^t(t) \mathbf{M} (\dot{\mathbf{u}}(t) - \mathbf{v}(t)) dt \\ & + \int_{t_0}^{t_f} \delta \boldsymbol{\lambda}^t(t) \mathbf{L} \dot{\mathbf{u}}(t) dt = 0 \end{aligned}$$

- We can then deduce the corresponding Euler-Lagrange equations:

$$\mathbf{M} \dot{\mathbf{v}}(t) + \mathbf{K} \mathbf{u}(t) = \mathbf{f}_{ext}(t) + \mathbf{L}^t \Lambda(t) \quad \forall t \in [t_0, t_f]$$

$$\dot{\mathbf{u}}(t) - \mathbf{v}(t) = 0 \quad \forall t \in [t_0, t_f]$$

$$\mathbf{L} \dot{\mathbf{u}}(t) = 0 \quad \forall t \in [t_0, t_f]$$

$$\mathbf{u}(t_0) = \mathbf{u}_0, \dot{\mathbf{u}}(t_0) = \dot{\mathbf{u}}_0$$

➡ This gives the physical meaning of the Lagrange multiplier (interface loading $\mathbf{L}^t \Lambda$) ($\Lambda = -\dot{\boldsymbol{\lambda}}$) (generalized moment [Benes Matous 2010])

➡ Unlike Ordinary Differential Equations without multipliers (ODE), we now get a system of Differential Algebraic Equations (DAE).

Introduction

General formalism

Stability

HATI

Conclusions

⇒ We can now introduce the time discretization:

$$t_0 < t_1 < \dots < t_n < t_{n+1} < \dots < t_f \quad h = t_{n+1} - t_n$$

⇒ Now, from the previous three fields time action integral, we assume that the time discretized equilibrium equation is checked only at a given time within h . This time strongly depends on the chosen time integrator. This idea is not new and follows the very general concepts developed in the $G-\alpha$ methods [Kuhl Crisfield 1999, Erlicher Bonaventura Bursi 2002] or more generally by Tamma [Hoitink et al 2008, Masuri et al 2009] (particular cases: HHT- α , WBZ- α , CH- α)

$$\mathbf{M}\dot{\mathbf{v}}_{n+\xi_g} + \mathbf{K}\mathbf{u}_{n+\xi_f} = \mathbf{f}_{n+\xi_f} + \mathbf{L}^t \boldsymbol{\Lambda}_{n+\xi_f} \quad (\xi_g, \xi_f) \in [0, 1]^2$$

$$\int_{t_0}^{t_f} \mathbf{L}\dot{\mathbf{u}}(t) dt = 0$$

⇒ External, internal forces and multipliers are expressed at time $t_{n+\xi_f}$. As indicated by Tamma, a specific parameter is dedicated to the acceleration in order to find a second order accuracy for acceleration. In fact, many well-known 2nd order time integrators (for displacements and velocity) are only of order 1 for the acceleration [Hulbert Hughes 1988, Erlicher et al 2002, Masuri et al 2009]. Thus, the order 2 for all quantities can be obtained if an equilibrium in a weak sense in time is introduced (all LMS integrators have this specific time for acceleration).

Introduction

General formalism

Stability

HATI

Conclusions

- The previous equation allows now to introduce a general formalism for modern time integrators in the presence of Lagrange multipliers (prerequisite in order to introduce HATI).

$$\begin{cases} \dot{\mathbf{v}}_{n+\xi_g} = (1 - \xi_g)\dot{\mathbf{v}}_n + \xi_g\dot{\mathbf{v}}_{n+1} \\ \dot{\mathbf{u}}_{n+\xi_f} = (1 - \xi_f)\dot{\mathbf{u}}_n + \xi_f\dot{\mathbf{u}}_{n+1} \\ \mathbf{u}_{n+\xi_f} = (1 - \xi_f)\mathbf{u}_n + \xi_f\mathbf{u}_{n+1} \\ \mathbf{f}_{n+\xi_f} = (1 - \xi_f)\mathbf{f}_n + \xi_f\mathbf{f}_{n+1} \\ \Lambda_{n+\xi_f} = (1 - \xi_f)\Lambda_n + \xi_f\Lambda_{n+1} \end{cases}$$

- Closed system of eq.: (9 unknowns 9 eq.):

$$\begin{cases} \mathbf{u}_{n+1} = \mathbf{u}_n + h\dot{\mathbf{u}}_n + \left(\frac{1}{2} - \beta\right)h^2\ddot{\mathbf{v}}_n + \beta h^2\ddot{\mathbf{v}}_{n+1} \\ \dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + (1 - \gamma)h\ddot{\mathbf{v}}_n + \gamma h\ddot{\mathbf{v}}_{n+1} \end{cases}$$

$$\mathbf{K}^* \Delta \dot{\mathbf{u}}_{n+1} = \mathbf{g}_{n+1} + \mathbf{L}^t \Lambda_{n+\xi_f}$$

$$\int_{t_0}^{t_f} \mathbf{L} \dot{\mathbf{u}}(t) dt = 0$$

$$\mathbf{K}^* = \xi_g \frac{1}{\gamma h} \mathbf{M} + \xi_f \frac{\beta h}{\gamma} \mathbf{K}$$

$$\mathbf{g}_{n+1} = \mathbf{f}_{n+\xi_f} - \mathbf{K}\mathbf{u}_n - \xi_f h \mathbf{K} \dot{\mathbf{u}}_n - \left(1 - \frac{\xi_g}{\gamma}\right) \mathbf{M} \ddot{\mathbf{v}}_n - \xi_f \left(\frac{\gamma - 2\beta}{2\gamma}\right) h^2 \mathbf{K} \ddot{\mathbf{v}}_n$$

It may be noted here the fundamental role played by the two parameters in the operator $\mathbf{K}^*$ by a linear combination of the mass and stiffness.

Introduction

General formalism

Stability

HATI

Conclusions

● *General compact form with Lagrange multipliers:*

$$\mathbf{K}^* \Delta \mathbf{U}_{n+1} = \mathbf{G}_{n+1} + \mathbf{L}^t \boldsymbol{\Lambda}_{n+\xi_f}$$

$$\mathbf{G}_{n+1} = \mathbf{F}_{n+\xi_f} - \mathbf{N} \mathbf{U}_n$$

$$\mathbf{K}^* = \begin{bmatrix} \mathbf{K}^* & \mathbf{0} & \mathbf{0} \\ -\frac{\beta h}{\gamma} \mathbf{I} & \mathbf{I} & \mathbf{0} \\ -\frac{1}{\gamma h} \mathbf{I} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{L}^t = \begin{bmatrix} \mathbf{L}^t \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{U}_n = \begin{bmatrix} \dot{\mathbf{u}}_n \\ \mathbf{u}_n \\ \dot{\mathbf{v}}_n \end{bmatrix}, \quad \Delta \mathbf{U}_{n+1} = \begin{bmatrix} \Delta \dot{\mathbf{u}}_{n+1} \\ \Delta \mathbf{u}_{n+1} \\ \Delta \dot{\mathbf{v}}_{n+1} \end{bmatrix}$$

$$\mathbf{F}_{n+\xi_f} = \begin{bmatrix} \mathbf{f}_{n+\xi_f} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \xi_f h \mathbf{K} & \mathbf{K} & \xi_f \left(\frac{\gamma-2\beta}{2\gamma} \right) h^2 \mathbf{K} + \left(1 - \frac{\xi_g}{\gamma} \right) \mathbf{M} \\ h \mathbf{I} & \mathbf{0} & \left(\frac{\gamma-2\beta}{2\gamma} \right) h^2 \mathbf{I} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{\gamma} \mathbf{I} \end{bmatrix}$$

➡ *This compact form contains all the equations required for the complete solving at a given time step (for all known α -integrators), to which we add the weak kinematic constraint in time:*

$$\int_{t_0}^{t_f} \mathbf{L} \dot{\mathbf{u}}(t) dt = 0$$

Introduction

General formalism

Stability

HATI

Conclusions

- *Special case of CH- α time integrators introduced by Chung and Hulbert 1993:*

$$\begin{aligned} (1 - \alpha_g)\mathbf{M}\dot{\mathbf{v}}_{n+1} + \alpha_g\mathbf{M}\dot{\mathbf{v}}_n + (1 - \alpha_f)\mathbf{K}\mathbf{u}_{n+1} + \alpha_f\mathbf{K}\mathbf{u}_n \\ = (1 - \alpha_f)\mathbf{f}_{n+1} + (1 - \alpha_f)\mathbf{f}_n + \mathbf{L}^t \boldsymbol{\Lambda}_{n+1-\alpha_f} \end{aligned} \quad \begin{cases} \xi_g = 1 - \alpha_g \\ \xi_f = 1 - \alpha_f \end{cases}$$

Scheme	α_g	α_f	γ	β
HHT- α	0	$-\alpha_{HHT} = \frac{1-\rho_\infty}{1+\rho_\infty}$	$\frac{1}{2} - \alpha_{HHT}$	$\frac{1}{4}(1 - \alpha_{HHT})^2$
WBZ- α	$\alpha_{WBZ} = \frac{\rho_\infty-1}{1+\rho_\infty}$	0	$\frac{1}{2} - \alpha_{WBZ}$	$\frac{1}{4}(1 - \alpha_{WBZ})^2$
CH- α	$\frac{2\rho_\infty-1}{1+\rho_\infty}$	$\frac{\rho_\infty}{1+\rho_\infty}$	$\frac{3}{2} - 2\alpha_f$	$(1 - \alpha_f)^2$

➡ *In practice, the four parameters are set to achieve unconditional stability, second order accuracy, control of spurious high frequency oscillations, while minimizing the numerical dissipation at low frequencies. Both α parameters can be defined as functions of the spectral radius to the high frequency limit, in order to control the numerical dissipation rate in this interval.*

- *Particular case of Newmark: $\xi_g = \xi = \gamma$ $\xi_f = \xi = \gamma$*

➡ *No time integrator of Newmark allows the control of the spurious high frequency oscillations while ensuring second-order accuracy.*

➡ *This formalism also contains Simo , Krenk, Verlet integrators as special cases in the presence of Lagrange multipliers.*

Introduction

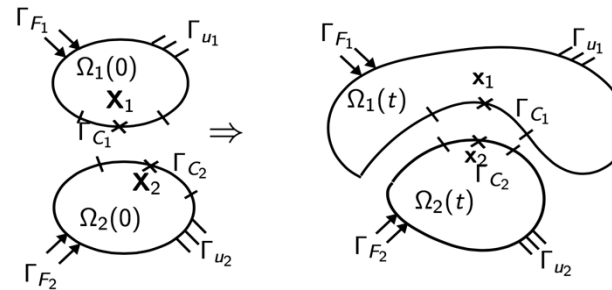
General formalism

Stability

HATI

Conclusions

● *Special case of non-smooth dynamics:*



● *Hertz-Signorini-Moreau conditions (HSM):*

Impenetrability: $g_N = [(\mathbf{X}_2 + \mathbf{u}_2) - (\mathbf{X}_1 + \mathbf{u}_1)] \cdot \mathbf{n}_1 \geq 0$

Contact pressure: $t_N = \boldsymbol{\sigma}_\alpha \cdot \mathbf{n}_\alpha \cdot \mathbf{n}_\alpha \leq 0 \quad , \quad \alpha = 1, 2$

Complementarity: $g_N \cdot t_N = 0$

● *Moreau lemma [Moreau 1978] (velocity-impulse formulation of HSM):*

$$\left\{ \begin{array}{l} g_N \geq 0 \\ t_N \leq 0 \\ g_N \cdot t_N = 0 \end{array} \right. \iff \left\{ \begin{array}{ll} \text{if } g_N > 0 & \text{then } i_N = 0 \\ \text{if } g_N = 0 & \text{then } \left\{ \begin{array}{l} \dot{g}_N \geq 0 \\ i_N \leq 0 \\ \dot{g}_N \cdot i_N = 0 \end{array} \right. \end{array} \right.$$

Introduction

General formalism

Stability

HATI

Conclusions

- *Four fields time action integral with impact with velocity HSM conditions at time t_c [Cirak and West 2005]*

$$\tilde{A}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{v}, t_c, \lambda) = \int_{t_0}^{t_c} L(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{v}) dt + \int_{t_c}^{t_f} L(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{v}) dt + \lambda^T(t_c) \mathbf{g}_N(t_c)$$

- *Stationarity of action integral:* $\delta \tilde{A} = 0 \Rightarrow$

*Smooth dynamics
Equation:*

$\delta \mathbf{u}$

:

$$\mathbf{M} \dot{\mathbf{v}} + \mathbf{F}_{int} = \mathbf{F}_{ext}, \quad \forall t \in [t_0, t_c^-] \cup [t_c^+, t_f]$$

$\delta \mathbf{v}$

:

$$\dot{\mathbf{u}} = \mathbf{v}$$

Impact equation:

δt_c

:

$$\left[\mathbf{M} \mathbf{v}(t) \right]_{t_c^-}^{t_c^+} = \nabla \mathbf{g}_N^T(t_c) \lambda(t_c)$$

Kinetic energy balance:

$\delta \mathbf{u}(t_c)$

:

$$\left[(\mathbf{M} \mathbf{v})^T \mathbf{M}^{-1} (\mathbf{M} \mathbf{v}) \right]_{t_c^-}^{t_c^+} = 0$$

- *Unified HSM non-smooth dynamics equation [Moreau 1999, Chen 2013, Acary 2014]*

$$d\mathbf{v} = d\mathbf{v}_s + d\mathbf{v}_{ns}$$

$$\mathbf{M} d\mathbf{v} + \mathbf{F}_{int} dt = \mathbf{F}_{ext} dt + d\mathbf{I}$$

$$d\mathbf{v}_s = \dot{\mathbf{v}} dt$$

$$d\mathbf{v}_{ns} = \mathbf{v}(t_c^+) - \mathbf{v}(t_c^-)$$

$$d\mathbf{I}(t) = \begin{cases} 0 & \forall t \in [t_0, t_c^-] \cup [t_c^+, t_f] \\ \mathbf{L}^T(t_c) \lambda(t_c) & \end{cases}$$

$$\mathbf{L} = \nabla \mathbf{g}_N$$

Introduction

General formalism

Stability

HATI

Conclusions

- **A new central difference method with Lagrange multipliers for non-smooth dynamics [Belytschko 2000, Casadei 2009, Heinstein 2010, Geradin and Rixen 2014, Fekak Brun Gravouil 2017] (compatible with the previous general formalism):**

$$\begin{aligned} \mathbf{M}_{lump} \mathbf{v}_{n+\frac{3}{2}} &= \mathbf{M}_{lump} \mathbf{v}_{n+\frac{1}{2}} + \Delta t (\mathbf{F}_{ext,n+1} - \mathbf{F}_{int,n+1} - \mathbf{C} \mathbf{v}_{n+\frac{1}{2}}) + \mathbf{I}_{n+1} \\ \mathbf{u}_{n+1} &= \mathbf{u}_n + \Delta t \mathbf{v}_{n+\frac{1}{2}} \\ \mathbf{l}_{n+1} &= \mathbf{L}_{n+1}^T \boldsymbol{\lambda}_{n+\frac{3}{2}} \\ \mathbf{v}_{n+\frac{3}{2}} &= \mathbf{L}_{n+1} \mathbf{v}_{n+\frac{3}{2}} \\ \left\{ \begin{array}{l} \text{if } g_N^l(t_{n+1}) > 0 \text{ then } \lambda_{n+\frac{3}{2}}^l = 0 \\ \text{if } g_N^l(t_{n+1}) \leq 0 \text{ then } \left\{ \begin{array}{l} v_{n+\frac{3}{2}}^l \geq 0 \\ \lambda_{n+\frac{3}{2}}^l \geq 0 \\ v_{n+\frac{3}{2}}^l \lambda_{n+\frac{3}{2}}^l = 0 \end{array} \right. \end{array} \right. \quad \forall l \in \{1, \dots, p\} \end{aligned}$$

- **Corresponding Delassus or Steklov-Poincaré operator at time $tn+3/2$:**

$$\left(\mathbf{L}_{n+1} \mathbf{M}_{lump}^{-1} \mathbf{L}_{n+1}^T \right) \boldsymbol{\lambda}_{n+\frac{3}{2}} = \mathbf{b}$$

$$\mathbf{b} = \mathbf{v}_{n+\frac{3}{2}} - \mathbf{L}_{n+1} \left(\mathbf{v}_{n+\frac{1}{2}} + \Delta t \mathbf{M}_{lump}^{-1} (\mathbf{F}_{ext,n+1} - \mathbf{F}_{int,n+1} - \mathbf{C} \mathbf{v}_{n+\frac{1}{2}}) \right)$$

➡ **Comments: acceleration not required, time of impact detection not required**

Introduction

General formalism

Stability

HATI

Conclusions

➡ *Two kinds of Time integrators for non-smooth dynamics*

● *Event-tracking time integrators*

- *Strong formulation in time semi-discretized in space*
- *Solving impact equation at time t_c*
- *Solving smooth dynamic equation between to events (contact, free movement)*

● *Time-stepping time integrators*

- *Space-time weak formulation for non-smooth dynamics*
- *Solving a unique non-smooth dynamic equation for one time step*

➡ *Only Time-stepping integrators converge in time with infinite impacts in a finite time (non-smooth dynamic equation and velocity HSM ensured in a weak sense on a time step).
General proof of convergence with the Hausdorff Measure [Acary 2012].*

➡ *The new CD-Lagrange is a symplectic explicit time-stepping integrator [Fekak 2017]*

Introduction

General formalism

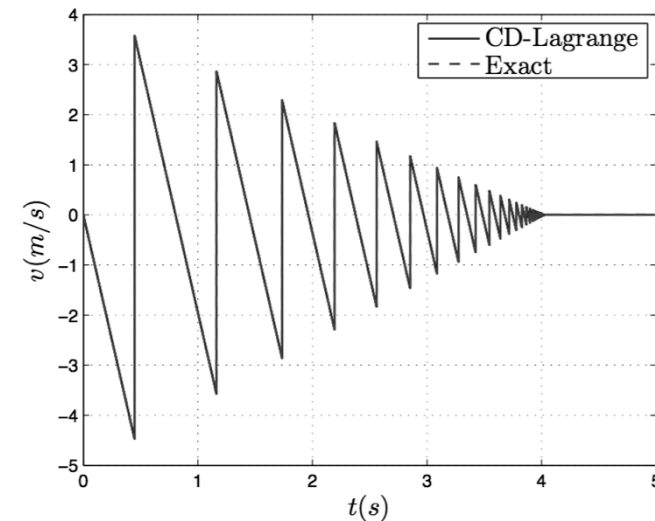
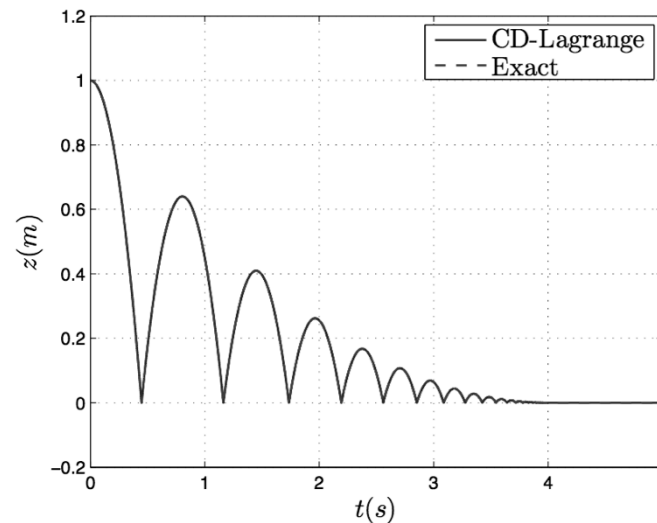
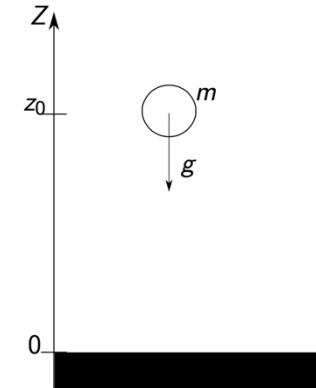
Stability

HATI

Conclusions

● **The bouncing ball example: rigid / rigid analytical solution**

- **First rebound:** $t_0 = \sqrt{\frac{2h}{g}}$
- **Nth rebound:** $t_n = t_0 \left(2 \frac{1 - e^n}{1 - e} - 1 \right)$
- **Downtime of the mass:** $T = t_0 \frac{1 + e}{1 - e}$



The mass does an infinity of rebounds before it stops in a finite time (Zeno Phenomenon) (restitution coefficient $e=0.8$)

Introduction

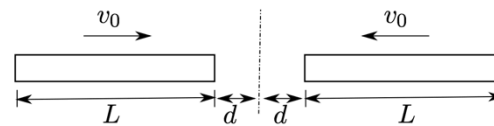
General formalism

Stability

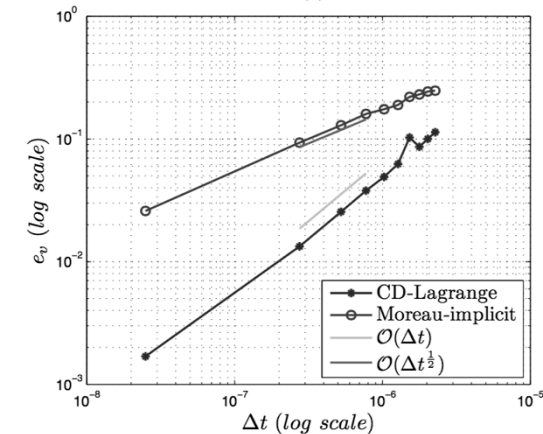
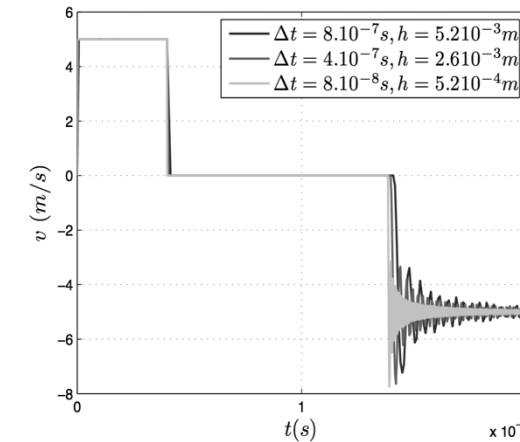
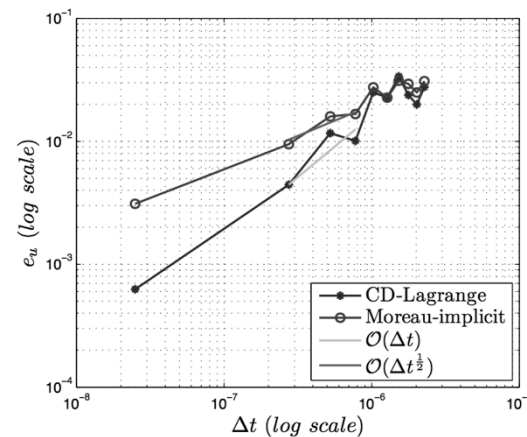
HATI

Conclusions

● **Contact / impact of two identical elastic bars : deformable / deformable analytical solution**



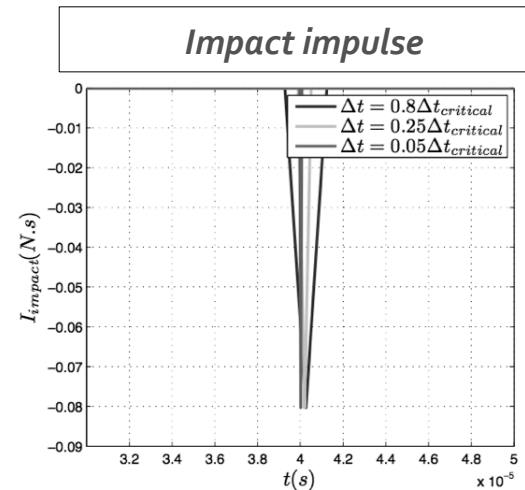
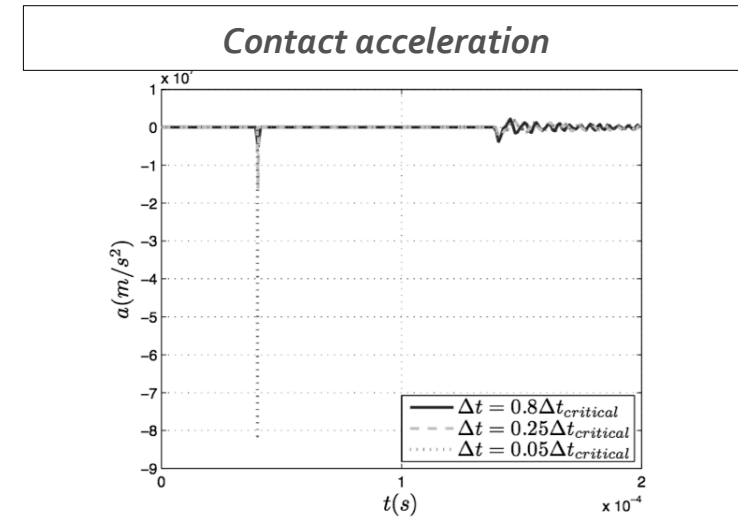
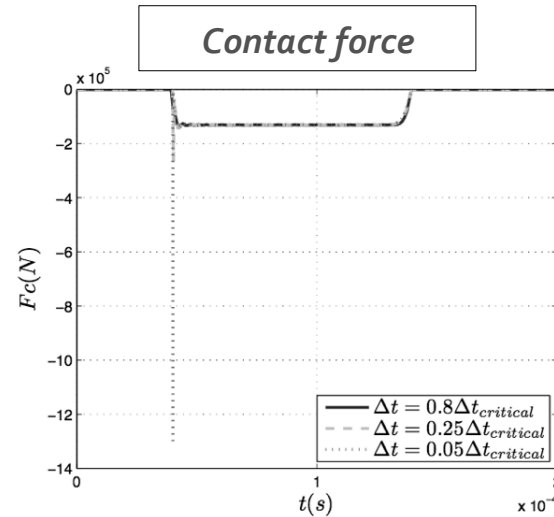
$$\begin{aligned} E &= 2.1 \cdot 10^{11} \text{ Pa} \\ \rho &= 7847 \text{ kg/m}^3 \\ L &= 0.254 \text{ m} \\ d &= 0.2 \cdot 10^{-3} \text{ m} \\ v_0 &= 5 \text{ m/s} \\ A &= 0.645 \cdot 10^{-3} \text{ m}^2 \end{aligned}$$



⇒ **Space-time global error indicator (Hausdorff measure) [Acary 2012]**

⇒ **Moreau implicit of order $\frac{1}{2}$, CD-Lagrange of order 1 [Fekak 2017]**

● *Contact / impact of two identical elastic bars : deformable / deformable analytical solution*



At impact time, acceleration and contact forces do not converge, however impulse converges.

Introduction

General formalism

Stability

HATI

Conclusions

2

Stability analysis

Introduction

General formalism

Stability

HATI

Conclusions

● *Numerical analysis of time integrators is essential to understand their advantages and disadvantages for structural dynamics*

⇒ *Stability, accuracy, numerical dissipation, overshoot are crucial.*

⇒ *The stability properties may be assessed by appropriate norms on the state vector $\mathbf{X} \equiv (\mathbf{u}, \mathbf{v})$ (displacements, velocities and accelerations possibly). They are generally based on the orthogonality property of the eigenvectors (decoupled differential systems, SDOF) [Hulbert Hughes 1987, Hulbert 2004]*

⇒ *Using of the amplification matrix: $\mathbf{X}(t_{n+1}) - \mathbf{A}\mathbf{X}(t_n) - \mathbf{L}_n = \Delta t \tau(t_n)$*

- *A-stability $\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n \quad |\mathbf{A}| < 1 \quad (\text{symplectic}) \quad \mathbf{A}^t \mathbf{J} \mathbf{A} = \mathbf{J}$*
- *consistency $|\tau(t)| \leq c \Delta t^k, \forall t \in [0, T], k > 0$*

⇒ *Convergence: if $\Delta t \rightarrow 0$ then $e(t_n) \rightarrow 0$ $e(t_{n+1}) = \mathbf{X}(t_{n+1}) - \mathbf{X}_{n+1}$ (S+C)*

⇒ *Dahlquist barrier for LMS integrators (unconditional stability smaller or equal to 2). Counterexample: TDG, T-XFEM*

⇒ *The majority of time integrators are A-stable (only in the linear case). What about transient nonlinear dynamics?*

Introduction

General formalism

Stability

HATI

Conclusions

● *A significant example: the HHT- α time integrator can show some inefficiency of the numerical dissipation in the nonlinear case with possible overshoot of the velocity and only first order accuracy for the acceleration [Bauchau Damilano Theron 1995, Hulbert Hughes 1987].*

➡ *A better property is L-stability (no overshoot) [Piché 1995].*

➡ *L-stability combined with an optimization of the numerical dissipation of high frequency modes illustrates the very good properties of the CH- α integrators [Chung Hulbert 1993].*

➡ *L-stability is connected to the high frequency numerical dissipation and it has been shown that this is a critical property for non-linear constraints dynamic problems. Also interest of Liapunov exponents [Nawrotzki Eller 2000].*

➡ *In the non-linear regime, energy-stability is also a norm of great interest [Belytschko Schoeberle 1975, Hughes 1976].*

$$\left[\frac{1}{2} \dot{\mathbf{v}}^t \mathbf{A} \dot{\mathbf{v}} + \frac{1}{2} \dot{\mathbf{u}}^t \mathbf{K} \dot{\mathbf{u}} \right]_n^{n+1} = \frac{1}{h} \Delta \dot{\mathbf{u}}^t \{(\mathbf{f}_{n+1} - \mathbf{f}_n)\} - \left(\gamma - \frac{1}{2} \right) \{ \Delta \dot{\mathbf{v}}^t \mathbf{A} \Delta \dot{\mathbf{v}} \}$$

$$\Delta E_{kin} + \Delta E_{int} = \Delta E_{ext} + \Delta E_{diss}$$

$$\mathbf{A} = \mathbf{M} + \left(\beta - \frac{\gamma}{2} \right) h^2 \mathbf{K} \quad \Rightarrow$$

Stability properties from the pseudo-energy method (equivalent to A-stability in the linear case) [Hughes 1987].

Introduction

General formalism

Stability

HATI

Conclusions

● *Some properties of the stability analysis by the pseudo-energy:*

- *Equivalent to the A-stability in the linear case*
- *Does not require any state vector (useful for HATI)*
- *In the presence of Lagrange multipliers, the following pseudo-energy interface is used in the right hand side (ODE -> DAE):*

$$\Delta E_{interface} = \frac{1}{h} \Delta \dot{\mathbf{u}}^t \{ \mathbf{L}^t (\boldsymbol{\Lambda}_{n+1} - \boldsymbol{\Lambda}_n) \}$$

➡ *This method has historically served to validate or invalidate the first attempts of HATI building.*

- *Distinguish the pseudo-energy norm (dedicated to stability) and the discrete energy balance of a time integrator (very useful in the non-linear case). Example of discretized energy balance of CH- α time integrators introduced by Chung and Hulbert 1993:*

$$\begin{aligned} & \left[\frac{1}{2} \dot{\mathbf{u}}^t \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2} \mathbf{u}^t \mathbf{K} \mathbf{u} + \left(\beta - \frac{\gamma}{2} \right) \frac{1}{2} h^2 \dot{\mathbf{v}}^t \mathbf{M} \dot{\mathbf{v}} \right]_n^{n+1} = \dots \\ & \Delta \mathbf{u}^t \left\{ \frac{1}{2} (\mathbf{f}_{n+1} + \mathbf{f}_n) + \left(\gamma - \frac{1}{2} \right) (\mathbf{f}_{n+1} - \mathbf{f}_n) \right\} \\ & - (1 - \gamma - \alpha_g) \Delta \mathbf{u}^t \mathbf{M} \Delta \dot{\mathbf{v}} - \left(\frac{1}{2} - \alpha_f \right) \Delta \mathbf{u}^t \mathbf{K} \Delta \mathbf{u} \\ & - \left(\gamma - \frac{1}{2} \right) \left(\beta - \frac{\gamma}{2} \right) h^2 \Delta \dot{\mathbf{v}}^t \mathbf{M} \Delta \dot{\mathbf{v}} \\ & \Delta W_{kin} + \Delta W_{int} + \Delta W_{comp} = \Delta W_{ext} + \Delta W_{diss} \end{aligned}$$

Introduction

General formalism

Stability

HATI

Conclusions

3

***A unified strategy to control « locally » the accuracy and
the time step***

Introduction

General formalism

Stability

HATI

Conclusions

➡ A state of the art on Heterogeneous Asynchronous Time Integrators (HATI)

● *In order to go beyond the conventional time integrators in nonlinear computational dynamics, many attempts in the past consisted in developing HATI within the finite element method. We can highlight the pioneering work of Belytschko and co-authors in the 70s.*

- *Mixed methods (exp / imp) or Multi-time steps (exp / exp or imp / imp) element or node partitions [Mullen Kennedy Belytschko 1976, Belytschko Mullen 1978, Hughes Liu 1978, Hughes Pister Taylor 1979, Smolinski Belytschko 1988]*
- *Mixed methods and multi-time-step (exp / imp) [Liu Belytschko 1982]*
- *A first proof of stability by energy method (exp / exp multi-time) [Smolinski 1992, Smolinski Sleith Belytschko 1996] and in the case (exp/imp same time) [Hughes 1987]*
- *An original strategy for element time integration (with a unique time integrator); explicit time integrator [Casadei Halleux 2009] variational time integrator [Ryckman Lew 2012]*

➡ *Two difficulties: no evidence of stability in the general case (in particular multi-time exp / imp) and difficulties in the explicit case (multi-time exp / exp) with possibilities of instability lower than the CFL condition [Belytschko Lu 1993, Daniel 1997, Daniel 2003]; statistic stability condition [Sotelino 1994]*

➡ *A first remark: it seems that no evidence of stability will be available for these methods because counterexamples exist (unstable multi-time exp / imp)
A second note: all these methods are equivalent to primal domain decomposition techniques (displacement continuity on the interface)*

Introduction

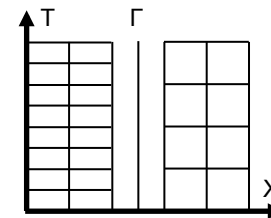
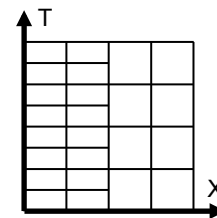
General formalism

Stability

HATI

Conclusions

From a theoretical point of view, space-time weak formulations allow to choose displacement, velocity or acceleration interface continuity. Subsequently, we show that a velocity dual approach (with Lagrange multipliers) is the (only?) one able to provide a general framework for any time integrator coupling with their own time scale (HATI)



$$\Omega_A \cap \Omega_B = \emptyset$$

$$h_A = [t_0; t_m]$$

$$h_A = m h_B$$

Let the three-fields space-time weak formulation previously introduced with a velocity continuity between two subdomains A and B (subsequently be omitted weak relations between $\dot{\mathbf{u}}$ and $\mathbf{v}$ for clarity):

$$\delta \tilde{\mathcal{A}} = 0$$

$$\begin{aligned} \Leftrightarrow & - \int_{t_0}^{t_m} \delta \mathbf{u}^{A^t}(t) (\mathbf{M}_A \dot{\mathbf{v}}^A(t) + \mathbf{K}_A \mathbf{u}^A(t) - \mathbf{f}_A(t) - \mathbf{L}_A^t \Lambda(t)) dt \\ & - \int_{t_0}^{t_m} \delta \mathbf{u}^{B^t}(t) (\mathbf{M}_B \dot{\mathbf{v}}^B(t) + \mathbf{K}_B \mathbf{u}^B(t) - \mathbf{f}_B(t) - \mathbf{L}_B^t \Lambda(t)) dt \\ & + \int_{t_0}^{t_m} \delta \lambda^t(t) (\mathbf{L}_A \dot{\mathbf{u}}^A(t) + \mathbf{L}_B \dot{\mathbf{u}}^B(t)) dt = 0 \end{aligned}$$

Introduction

General formalism

Stability

HATI

Conclusions

- In order to consider a wide class of time integrators, we follow the approach of the first part inspired by Crisfield, Tamma and co-authors. We derive a system of 7 equations with 8 parameters $\gamma_A, \beta_A, \xi_{A,f}, \xi_{A,g}$ and $\gamma_B, \beta_B, \xi_{B,f}, \xi_{B,g}$:

$$\begin{cases} \mathbf{K}_A^* \Delta \dot{\mathbf{u}}_m^A = \mathbf{g}_m^A + \mathbf{L}_A^t \boldsymbol{\Lambda}_{n+\xi_{A,f}} \\ \Delta \mathbf{u}_m^A = \frac{\beta_A h_A}{\gamma_A} \Delta \dot{\mathbf{u}}_m^A + h_A \dot{\mathbf{u}}_0^A + \frac{\gamma_A - 2\beta_A}{2\gamma_A} h_A^2 \ddot{\mathbf{v}}_0^A \\ \Delta \dot{\mathbf{v}}_m^A = \frac{1}{\gamma_A h_A} \Delta \dot{\mathbf{u}}_m^A - \frac{1}{\gamma_A} \ddot{\mathbf{v}}_0^A \end{cases}$$

$$\begin{cases} \mathbf{K}_B^* \Delta \dot{\mathbf{u}}_j^B = \mathbf{g}_j^B + \mathbf{L}_B^t \boldsymbol{\Lambda}_{j-1+\xi_{B,f}} \\ \Delta \mathbf{u}_j^B = \frac{\beta_B h_B}{\gamma_B} \Delta \dot{\mathbf{u}}_j^B + h_B \dot{\mathbf{u}}_{j-1}^B + \frac{\gamma_B - 2\beta_B}{2\gamma_B} h_B^2 \ddot{\mathbf{v}}_{j-1}^B \\ \Delta \dot{\mathbf{v}}_j^B = \frac{1}{\gamma_B h_B} \Delta \dot{\mathbf{u}}_j^B - \frac{1}{\gamma_B} \ddot{\mathbf{v}}_{j-1}^B \\ \forall j \in \{1, m\} \end{cases}$$

$$\int_{t_0}^{t_m} \mathbf{L}_A \dot{\mathbf{u}}^A(t) + \mathbf{L}_B \dot{\mathbf{u}}^B(t) dt = 0$$

⇒ Each subdomain A and B has its own time scale and time integrator

Introduction

General formalism

Stability

HATI

Conclusions

● *With a more compact form:*

$$\begin{cases} \mathbb{K}_A^* \Delta \mathbf{U}_m^A = \mathbf{F}_{\gamma_A}^A - \mathbf{N}_A \mathbf{U}_0^A + \mathbf{L}_A^t \boldsymbol{\Lambda}_{\xi_A, f} \\ \mathbb{K}_B^* \Delta \mathbf{U}_j^B = \mathbf{F}_{j-1+\gamma_B}^B - \mathbf{N}_B \mathbf{U}_{j-1}^B + \mathbf{L}_B^t \boldsymbol{\Lambda}_{j-1+\xi_B, f} \quad \forall j \in \{1, m\} \\ \int_{t_0}^{t_m} \mathbf{L}_A \dot{\mathbf{u}}^A(t) + \mathbf{L}_B \dot{\mathbf{u}}^B(t) dt = 0 \end{cases}$$

⇒ *The last step consists in discretizing the weak time continuity of the velocity at the interface between A and B.*

⇒ *For that purpose, we consider the following pseudo-interface energy between 2 subdomains:*

$$\Delta E_{kin,m}^A + \Delta E_{int,m}^A + \sum_{j=1}^m \{ \Delta E_{kin,j}^B + \Delta E_{int,j}^B \} = \dots$$

$$\left(\Delta E_{ext,m}^A + \sum_{j=1}^m \Delta E_{ext,j}^B \right) + \left(\Delta E_{diss,m}^A + \sum_{j=1}^m \Delta E_{diss,j}^B \right) + \Delta E_{interface}$$

$$\Delta E_i = \frac{1}{h_A} \Delta \dot{\mathbf{u}}_m^{A^t} \{ \mathbf{L}_A^t (\boldsymbol{\Lambda}_m - \boldsymbol{\Lambda}_0) \} + \sum_{j=1}^m \left\{ \frac{1}{h_B} \Delta \dot{\mathbf{u}}_j^{B^t} \{ \mathbf{L}_B^t (\boldsymbol{\Lambda}_j - \boldsymbol{\Lambda}_{j-1}) \} \right\}$$

Introduction

General formalism

Stability

HATI

Conclusions

- *Historically, the first HATI based on a dual velocity formulation consisted in the introduction of an additional hypothesis on the Lagrange multipliers within the Newmark time integrators: linear time interpolation from the macro to the micro scale [Gravouil 2000, Combescure Gravouil 2001]:*

$$\Lambda_j = \left(1 - \frac{j}{m}\right) \Lambda_0 + \frac{j}{m} \Lambda_m$$

- *Nevertheless, in the context of α methods with Lagrange multipliers (HTTP- α , α -WBZ, CH- α), this expression seems a natural extension of the general expression introduced at the beginning of the presentation:*

$$\Lambda_{n+\xi_f} = (1 - \xi_f) \Lambda_n + \xi_f \Lambda_{n+1}$$

⇒ *Finally we can also write:*

$$\Lambda_j - \Lambda_{j-1} = \frac{1}{m} (\Lambda_m - \Lambda_0)$$

⇒ *The pseudo-interface energy becomes:*

$$\Delta E_{interface} = \left[\frac{1}{h_A} \Delta \dot{\mathbf{u}}_m^{A^t} \mathbf{L}_A^t + \sum_{j=1}^m \left\{ \frac{1}{mh_B} \Delta \dot{\mathbf{u}}_j^{B^t} \mathbf{L}_B^t \right\} \right] (\Lambda_m - \Lambda_0)$$

Introduction

General formalism

Stability

HATI

Conclusions

⇒ In this context, two options for canceling the pseudo-interface energy are available (to ensure stability of the overall problem) [Brun Gravouil Combescure 2014]:

$$\mathbf{L}_A \Delta \dot{\mathbf{u}}_m^A + \sum_{j=1}^m \mathbf{L}_B \Delta \dot{\mathbf{u}}_j^B = 0 \quad \frac{1}{m} \mathbf{L}_A \Delta \dot{\mathbf{u}}_m^A + \mathbf{L}_B \Delta \dot{\mathbf{u}}_j^B = 0$$

⇒ The first relation defines the family of BGC-macro (dual HATI) , because velocity continuity is ensured at the macro time scale between A and B.

⇒ The second relation defines the family of BGC-micro (dual HATI) , because velocity continuity is ensured at the micro time scale between A and B.

● Some properties for these two classes of HATI:

- They guarantee by definition the overall stability of the coupling of α -type time integrators with their own time scale (proof of stability by Energy method)
- The BGC-micro family includes as a special case the GC method (HATI micro dedicated to Newmark time integrators) [Gravouil Combescure 2001].
- The BGC-macro family includes as a special case the PH method (HATI macro dedicated to Newmark time integrators) [Prakash Hjelmstad 2004].

● In practice, we can prove that the BGC-micro family – although stable – can numerically dissipate energy at the interface, which is not the case for the BGC-macro family:

$$\Delta E_{i \text{ BGC-macro}} = 0$$

$$\Delta E_{i \text{ BGC-micro}} \leq 0$$

● BGC-macro HATI numerical implementation:

$$\begin{bmatrix} \mathbb{K}_B^* & & & & \\ \mathbb{N}_B & \mathbb{K}_B^* & & & \\ \mathbb{N}_B & \mathbb{N}_B & \mathbb{K}_B^* & & \\ & \ddots & \ddots & \ddots & \\ \mathbb{N}_B & \mathbb{N}_B & \mathbb{N}_B & \mathbb{N}_B & \mathbb{K}_B^* \\ \hline & & & & \mathbb{K}_A^* & -\mathbb{C}_{A,m}^t \\ -\mathbb{L}_B & -\mathbb{L}_B & -\mathbb{L}_B & \cdots & -\mathbb{L}_B & -\mathbb{L}_A & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbb{C}_{B,1}^t \\ -\mathbb{C}_{B,2}^t \\ -\mathbb{C}_{B,3}^t \\ \vdots \\ -\mathbb{C}_{B,m}^t \\ \hline \mathbb{K}_A^* & -\mathbb{C}_{A,m}^t \\ \hline \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{U}_1^B \\ \Delta \mathbf{U}_2^B \\ \Delta \mathbf{U}_3^B \\ \vdots \\ \Delta \mathbf{U}_m^B \\ \hline \Delta \mathbf{U}_m^A \\ \hline \boldsymbol{\Lambda}_m \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\xi B,f}^B - \mathbb{N}_B \mathbf{U}_0^B + \mathbb{E}_{B,1}^t \boldsymbol{\Lambda}_0 \\ \mathbf{F}_{1+\xi B,f}^B - \mathbb{N}_B \mathbf{U}_0^B + \mathbb{E}_{B,2}^t \boldsymbol{\Lambda}_0 \\ \mathbf{F}_{2+\xi B,f}^B - \mathbb{N}_B \mathbf{U}_0^B + \mathbb{E}_{B,3}^t \boldsymbol{\Lambda}_0 \\ \vdots \\ \mathbf{F}_{m-1+\xi B,f}^B - \mathbb{N}_B \mathbf{U}_0^B + \mathbb{E}_{B,m}^t \boldsymbol{\Lambda}_0 \\ \hline \mathbf{F}_{\xi A,f}^A - \mathbb{N}_A \mathbf{U}_0^A + \mathbb{E}_{A,m}^t \boldsymbol{\Lambda}_0 \\ \hline \mathbf{0} \end{bmatrix}$$

$$\begin{bmatrix} K & -L^t \\ B & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{U} \\ \boldsymbol{\Lambda} \end{bmatrix} = \begin{bmatrix} F \\ \mathbf{0} \end{bmatrix}$$

[Prakash Hjelmstad 2004, Mahjoubi Gravouil Combescure 2009, Brun Gravouil Combescure 2014]

$$\Rightarrow \begin{cases} K \Delta \mathbf{U}_{\text{free}} = F \\ H \boldsymbol{\Lambda} = -B \Delta \mathbf{U}_{\text{free}} \\ K \Delta \mathbf{U}_{\text{link}} = L^t \boldsymbol{\Lambda} \end{cases} \quad \text{with } H = [BK^{-1}L^t]$$

The BGC-macro family requires the calculation of a single Lagrange multiplier on each macro time step

● *BGC-micro HATl numerical implementation:*

$$\begin{cases} \mathbf{K}_A^* \Delta \mathbf{U}_m^A = \mathbf{F}_{\gamma_A}^A - \mathbf{N}_A \mathbf{U}_0^A + \mathbf{L}_A^t \boldsymbol{\Lambda}_{\xi_A, f} \\ \mathbf{K}_B^* \Delta \mathbf{U}_j^B = \mathbf{F}_{j-1+\gamma_B}^B - \mathbf{N}_B \mathbf{U}_{j-1}^B + \mathbf{L}_B^t \boldsymbol{\Lambda}_{j-1+\xi_B, f} \quad \forall j \in \{1, m\} \\ \frac{1}{m} \mathbf{L}_A \Delta \dot{\mathbf{u}}_m^A + \mathbf{L}_B \Delta \dot{\mathbf{u}}_j^B = 0 \end{cases}$$

- *The BGC-micro family requires the calculation of m Lagrange multipliers on each micro time step*

$$\mathbf{H} \Delta \boldsymbol{\Lambda}_j = -\mathbf{b}_j$$

With:

$$\begin{cases} \mathbf{H} = [\xi_{A, f} \mathbf{L}_A (\mathbf{K}_A^*)^{-1} \mathbf{L}_A^t + \xi_{B, f} \mathbf{L}_B (\mathbf{K}_B^*)^{-1} \mathbf{L}_B^t] \\ \mathbf{b}_j = \frac{1}{m} \mathbf{L}_A \Delta \dot{\mathbf{u}}_{free, m}^A + \mathbf{L}_B \Delta \dot{\mathbf{u}}_{free, j}^B - \frac{1}{m} \mathbf{L}_A (\mathbf{K}_A^*)^{-1} \mathbf{L}_A^t \boldsymbol{\Lambda}_0 - \mathbf{L}_B (\mathbf{K}_B^*)^{-1} \mathbf{L}_B^t \boldsymbol{\Lambda}_{j-1} \end{cases}$$

- *In practice, we can prove that the BGC-micro family – although stable – can numerically dissipate energy at the interface, which is not the case for the BGC-macro family:*

$$\begin{aligned} \Delta E_{i BGC-macro} &= 0 \\ \Delta E_{i BGC-micro} &\leq 0 \end{aligned}$$

- Recently, the A-stability analysis (amplification matrix) and the consistency analysis have been conducted for the BGC family with the following state vector [Confalonieri et al 2013, Brun Gravouil Combescure 2014]:
- Stability analysis:

$$\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n$$

$$\begin{bmatrix} \mathbf{X}_{n+1}^A \\ \mathbf{X}_{n+1}^B \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{AA} & \mathbf{A}^{AB} \\ \mathbf{A}^{BA} & \mathbf{A}^{BB} \end{bmatrix} \begin{bmatrix} \mathbf{X}_n^A \\ \mathbf{X}_n^B \end{bmatrix}$$

$$\mathbf{X}_n = [\mathbf{X}_n^A \quad \mathbf{X}_n^B]^T = [\mathbf{u}_n^A \quad \dot{\mathbf{u}}_n^A \quad h_A \dot{\mathbf{v}}_n^A \quad h_A \lambda_n \quad \mathbf{u}_n^B \quad \dot{\mathbf{u}}_n^B \quad h_A \dot{\mathbf{v}}_n^B]^T$$

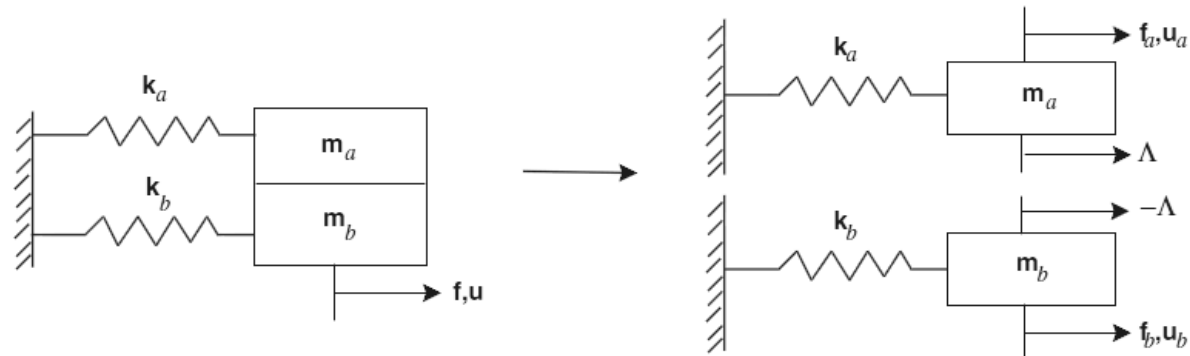
- Truncation error analysis (consistency)

$$\tau_n = \mathbf{A}\mathbf{X}(t_n) - \mathbf{X}(t_{n+1}) \quad \tau_n = \alpha h_A^k + O(h_A^{k+1})$$

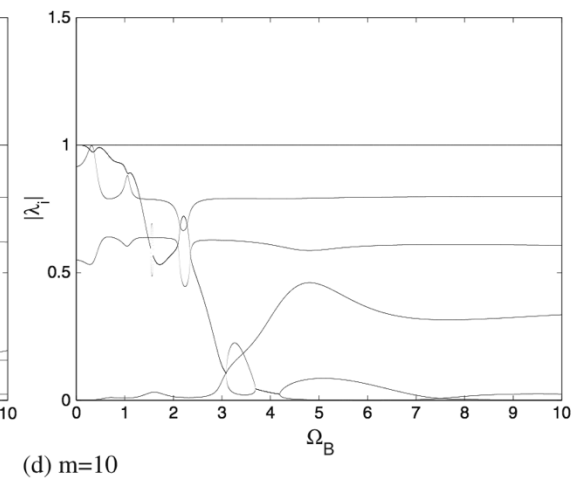
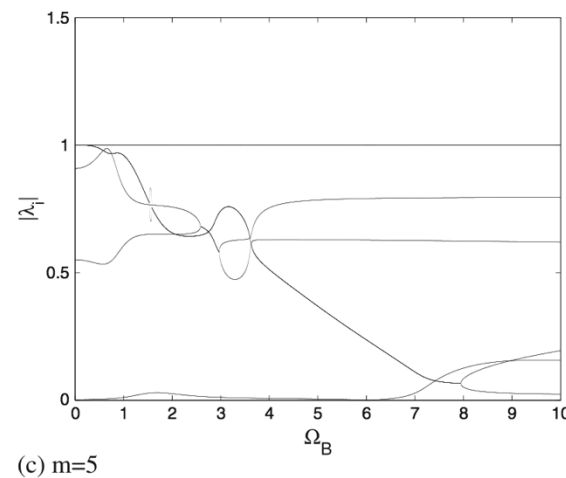
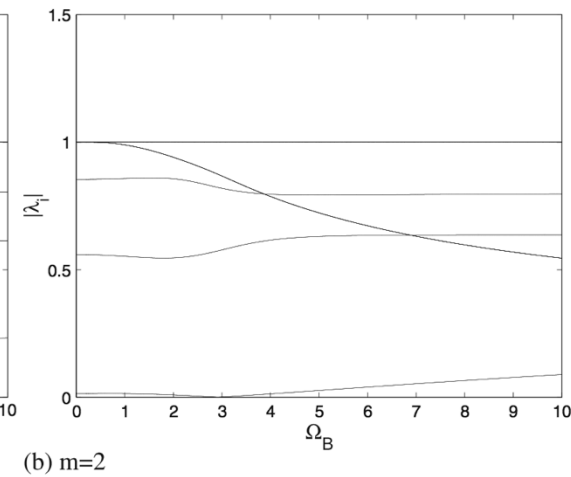
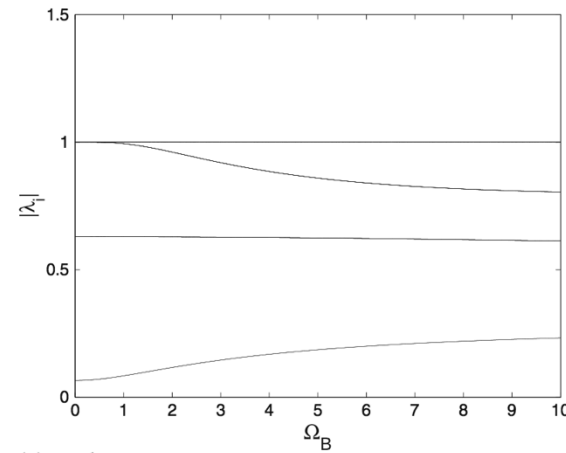
$$\tau_u^A = O(h_A^4), \tau_v^A = O(h_A^3), \tau_a^A = O(h_A^3), \tau_\lambda^A = O(h_A^3)$$

$$\tau_u^B = O(h_A^4), \tau_v^B = O(h_A^3), \tau_a^B = O(h_A^3)$$

- *Analysis of BGC-macro and BGC-micro methods for a double oscillator*
[Brun Gravouil Combescure 2014]

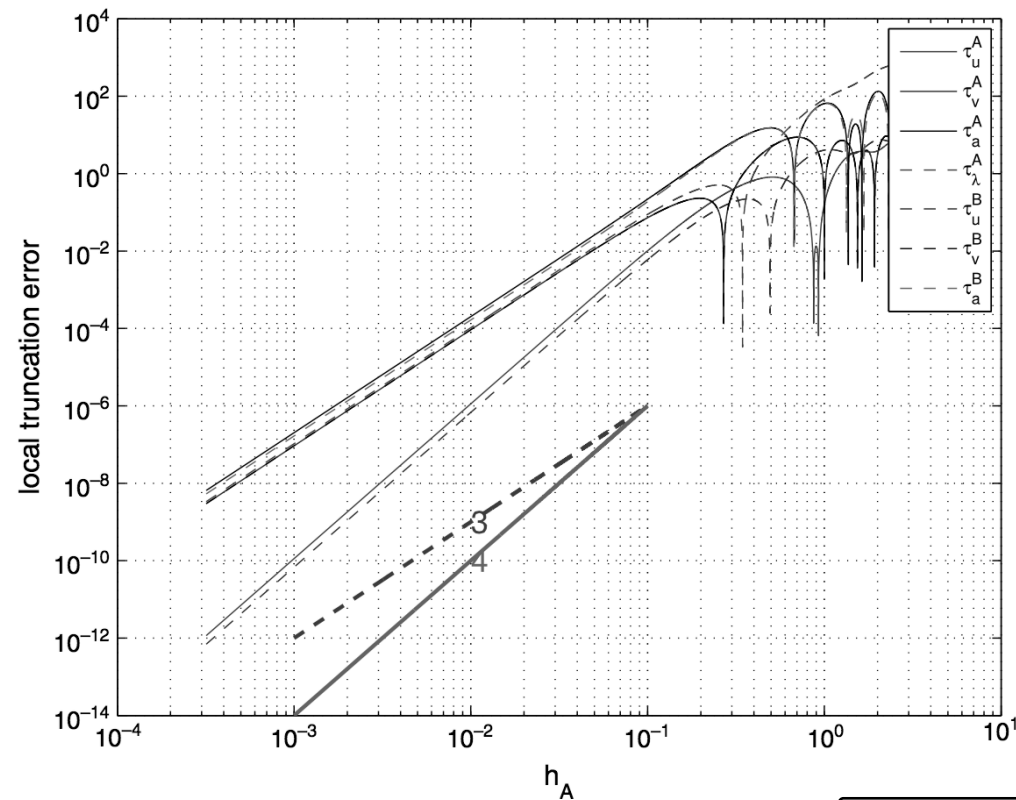


- **Absolute value of eigenvalues of a double oscillator with the BGC-macro ($CH-\alpha$ $\rho_\infty = 0.8$ / $CH-\alpha$ $\rho_\infty = 0.5$) for different ratio of time steps**



- **Truncation error τ of a double oscillator with the BGC-macro method ($CH-\alpha$ $\rho_\infty = 0.8$ / $CH-\alpha$ $\rho_\infty = 0.5$) with a ratio of time steps of 2**

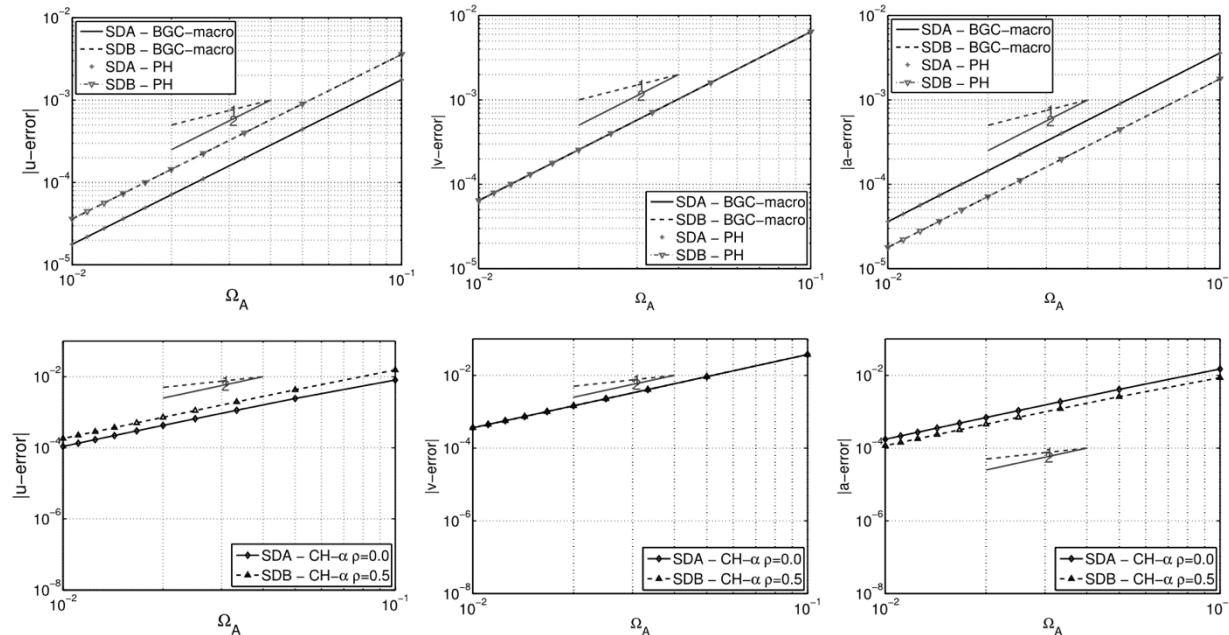
$$k = \frac{\ln(\tau_n(h_{2,A})) - \ln(\tau_n(h_{1,A}))}{\ln(h_{2,A}) - \ln(h_{1,A})}$$



$$\tau_u^A = O(h_A^4), \tau_v^A = O(h_A^3), \tau_a^A = O(h_A^3), \tau_\lambda^A = O(h_A^3)$$

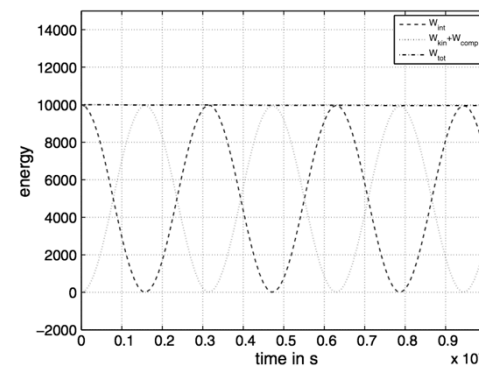
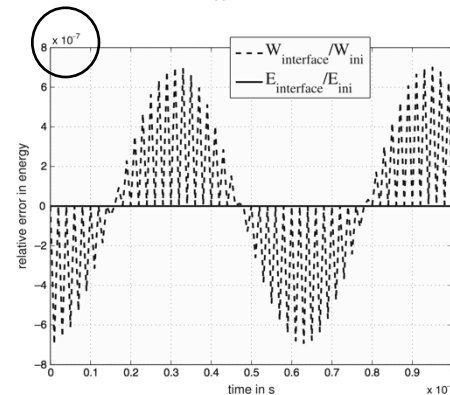
$$\tau_u^B = O(h_A^4), \tau_v^B = O(h_A^3), \tau_a^B = O(h_A^3)$$

● Convergence rate of the BGC-macro with $m=20$



AA /CD

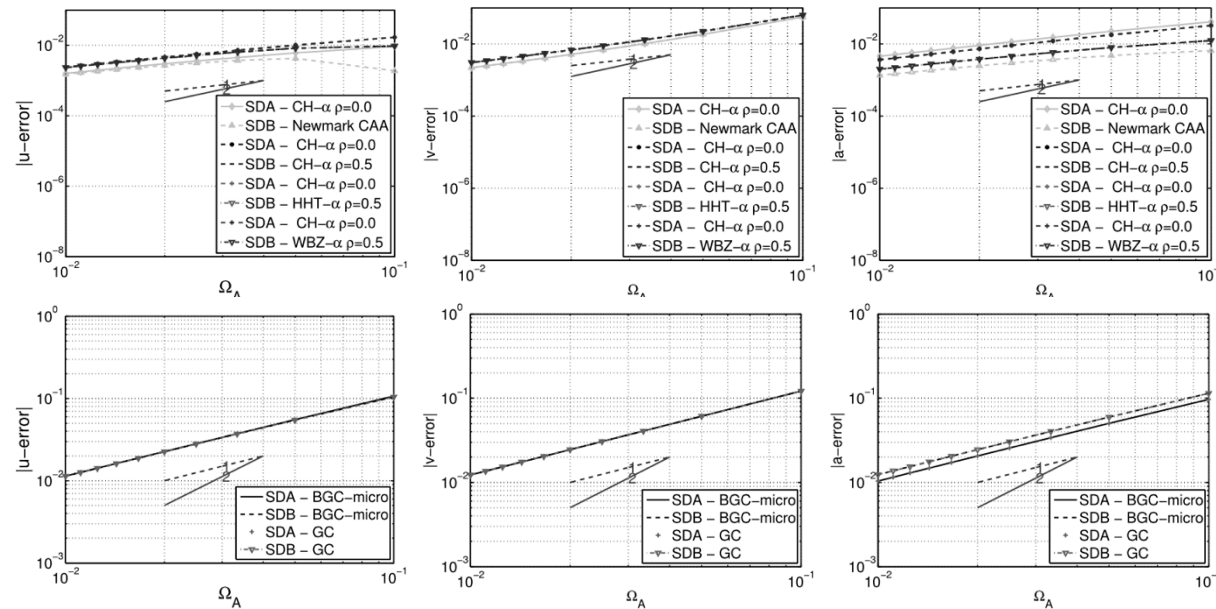
CH- α $\rho_\infty = 0.8$ /
CH- α $\rho_\infty = 0.5$



Discretized energy
balance

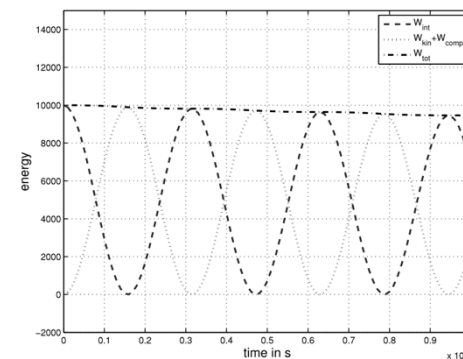
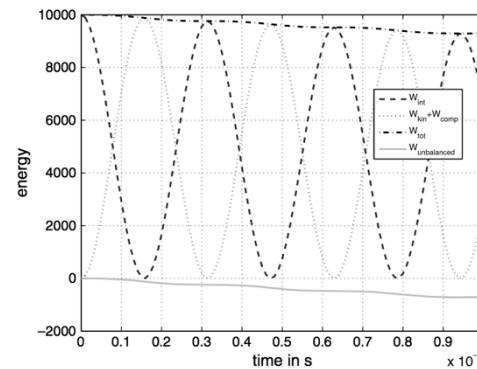
$$\Delta E_{iBGC-macro} = 0$$

● Convergence rate of the BGC-micro with $m=20$



AA /CD

CH- α $\rho_\infty = 0.8$ /
CH- α $\rho_\infty = 0.5$

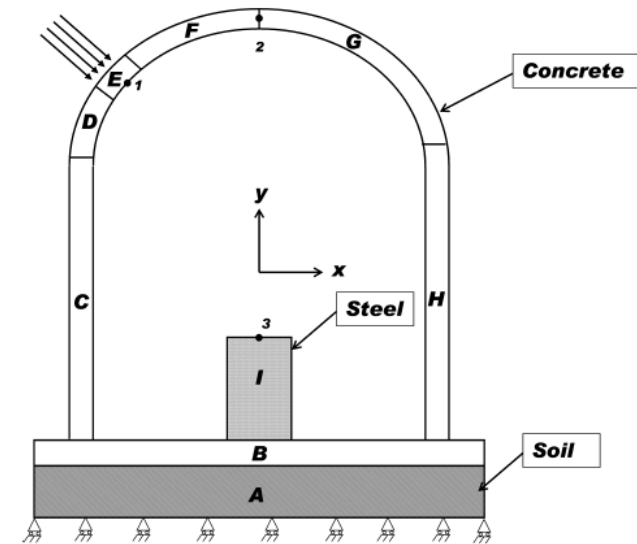


Discretized energy
balance

$$\Delta E_{iBGC-micro} \leq 0$$

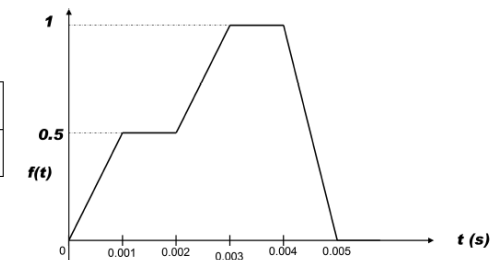
A second example (BGC macro) [Mahjoubi et al 2010]

- Subdomain A : Newmark $\gamma = 0.5, \beta = 0.25$
- Subdomain B : Newmark $\gamma = 0.5, \beta = 0.66$
- Subdomain C : Krenk $\alpha = 0.25$
- Subdomain D : HHT $\alpha = -0.2, \gamma = 0.7, \beta = 0.36$
- Subdomain E : Newmark $\gamma = 0.5, \beta = 0.0$
- Subdomain F : HHT $\alpha = -0.1, \gamma = 0.6, \beta = 0.3025$
- Subdomain G : Newmark $\gamma = 0.7, \beta = 0.36$
- Subdomain H : Newmark $\gamma = 0.55, \beta = 0.2756$
- Subdomain I : Krenk $\alpha = 0.1111$

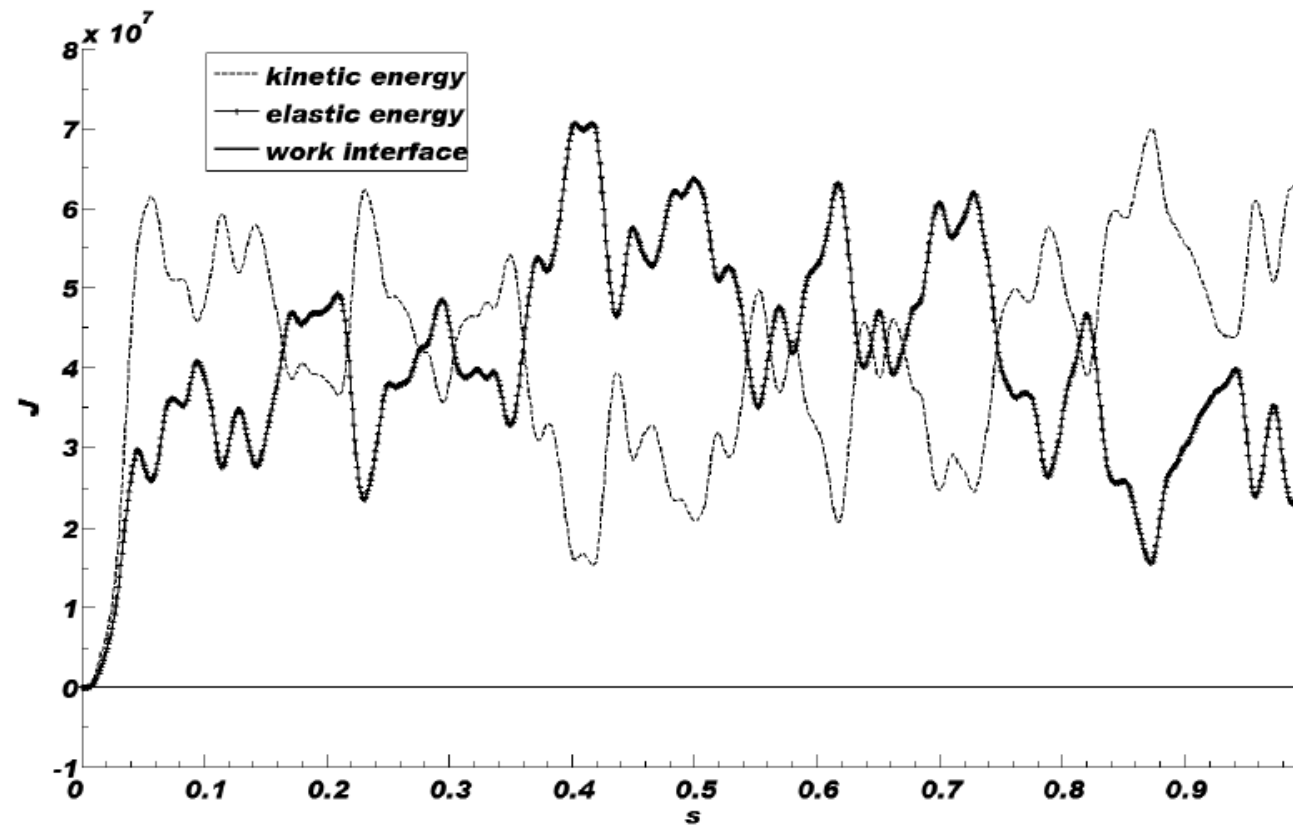


Sous-domaine	A	B	C	D	E	F	G	H	I
m_j	1	4	128	512	1024	256	128	32	16

Ratio between the different time scales



Discretized energy balance



Introduction

General formalism

Stability

HATI

Conclusions

● A first conclusion on BGC-type dual HATI

Applications in many fields of nonlinear dynamics: simulation of crash tests, impacts on safety structures, earthquakes, incompatible interfaces, experimental / numerical hybrid testing in real time, microsystems, electro-mechanical coupled problems, microelectronics fracture , fluid / structure interaction, co-simulation [Gravouil 2000, Combescure et al 2001, Herry et al 2002, Faucher et al 2003, Noels et al 2004, Pinto et al 2004, Bourel et al 2006, Bonnet et al 2008, Bonelli et al 2008, Mahjoubi et al 2009, Abaqus 6.13 users guide 2011, Batti et al 2011, Brun et al 2012, Ghanem et al 2012, Confalonieri et al 2012, Corigliano et al 2013, Bettinotti Allix Malherbe 2013, Li Combescure Leboeuf 2013, Karimi et al 2014, Chantrait Rannou Gravouil 2014, Prakash 2014]

➡ *Co-simulation: cast3m/cast3m; rad2rad; abaqus standard/abaqus explicit; cast3m/europlexus; zébulon/europlexus*

● *Here we can distinguish the well known 'partitioned' approaches used in fluid / structure interaction and the 'monolithic' approaches. Formally, the approaches developed here are equivalent to monolithic strategies and offer general stability results for any α -time integrator and any time scale.*

Introduction

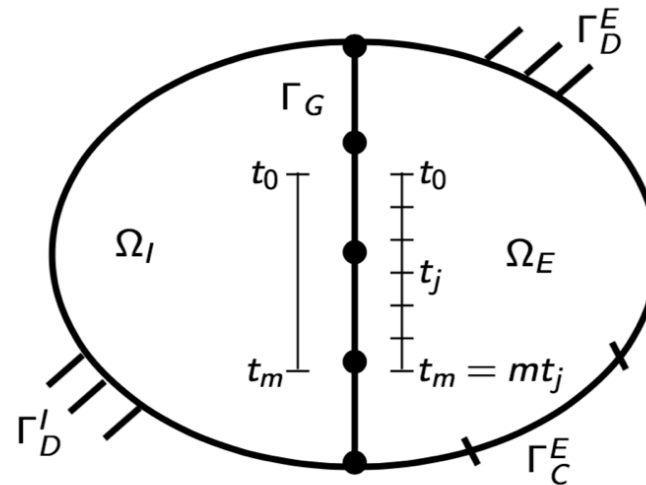
General formalism

Stability

HATI

Conclusions

● *Application to Heterogeneous Asynchronous Implicit-Explicit non-smooth dynamics*



Heterogeneous Asynchronous Time Integrator : BGC-micro
[Gravouil and Combescure, 2001]

- Space-time Velocity continuity

$$\mathbf{L}_G^E \mathbf{v}^E(t) + \mathbf{L}_G^I \mathbf{v}^I(t) = 0 \quad \forall t \in [t_0, t_m]$$

Where $\mathbf{L}_G^k$ ($k = I, E$) are restriction operators from Ω_k to Γ_G

Introduction

General formalism

Stability

HATI

Conclusions

● *Space-time weak form for implicit-explicit non-smooth dynamics*

- Action integral (HATI [Gravouil et al., 2015]) :

$$\begin{aligned}\tilde{\tilde{A}}(\mathbf{u}^I, \mathbf{v}^I, \mathbf{u}^E, \mathbf{v}^E, t_c) &= A^I(\mathbf{u}^I, \mathbf{v}^I) + \tilde{A}^E(\mathbf{u}^E, \mathbf{v}^E, t_c) \\ &+ \int_{t_0}^{t_m} \left(\mathbf{L}_G^E \mathbf{v}^E(t) + \mathbf{L}_G^I \mathbf{v}^I(t) \right)^T \lambda_G dt\end{aligned}$$

- Stationarity of the Action integral : $\delta \tilde{\tilde{A}} = 0 \Rightarrow$

- In Ω_I :

$$\mathbf{M}^I \dot{\mathbf{v}}^I(t) + \mathbf{F}_{int}^I(t) = \mathbf{F}_{ext}^I(t) + \mathbf{F}_{link}^I(t)$$

- In Ω_E :

$$\mathbf{M}^E d\mathbf{v}^E + \mathbf{F}_{int}^E dt = \mathbf{F}_{ext}^E dt + \mathbf{F}_{link}^E dt + d\mathbf{l}$$

+ HSM velocity conditions

- On interface Γ_G :

$$\mathbf{L}_G^E \mathbf{v}^E(t) + \mathbf{L}_G^I \mathbf{v}^I(t) = 0$$

Introduction

General formalism

Stability

HATI

Conclusions

● *Time discretization of the implicit-explicit non-smooth dynamics equations*

- Equilibrium equation in Ω_I at macro time step t_{n+m} :

$$\mathbf{M}^I \dot{\mathbf{v}}_{n+m}^I + \mathbf{C}^I \mathbf{v}_{n+m}^I + \mathbf{F}_{int,n+m}^I = \mathbf{F}_{ext,n+m}^I + \mathbf{F}_{D,n+m}^I + \mathbf{F}_{link,n+m}^I$$

- Non-smooth dynamic equation in Ω_E at micro time step t_{n+j} :

$$\begin{aligned} \mathbf{M}_{lump}^E \mathbf{v}_{n+j+\frac{1}{2}}^E &= \mathbf{M}_{lump}^E \mathbf{v}_{n+j-\frac{1}{2}}^E - \Delta t (\mathbf{F}_{int,n+j}^E + \mathbf{C}^E \mathbf{v}_{n+j-\frac{1}{2}}^E) \\ &\quad + \Delta t (\mathbf{F}_{ext,n+j}^E + \mathbf{F}_{D,n+j}^E + \mathbf{F}_{link,n+j}^E) + \mathbf{I}_{n+j} \end{aligned}$$

- contact equation at micro time step t_{n+j} :

$$\mathbf{H}_c \boldsymbol{\Lambda}_{c,n+j+\frac{1}{2}} = \mathbf{b}_c$$

- coupling equation at micro time step t_{n+j} (GC) :

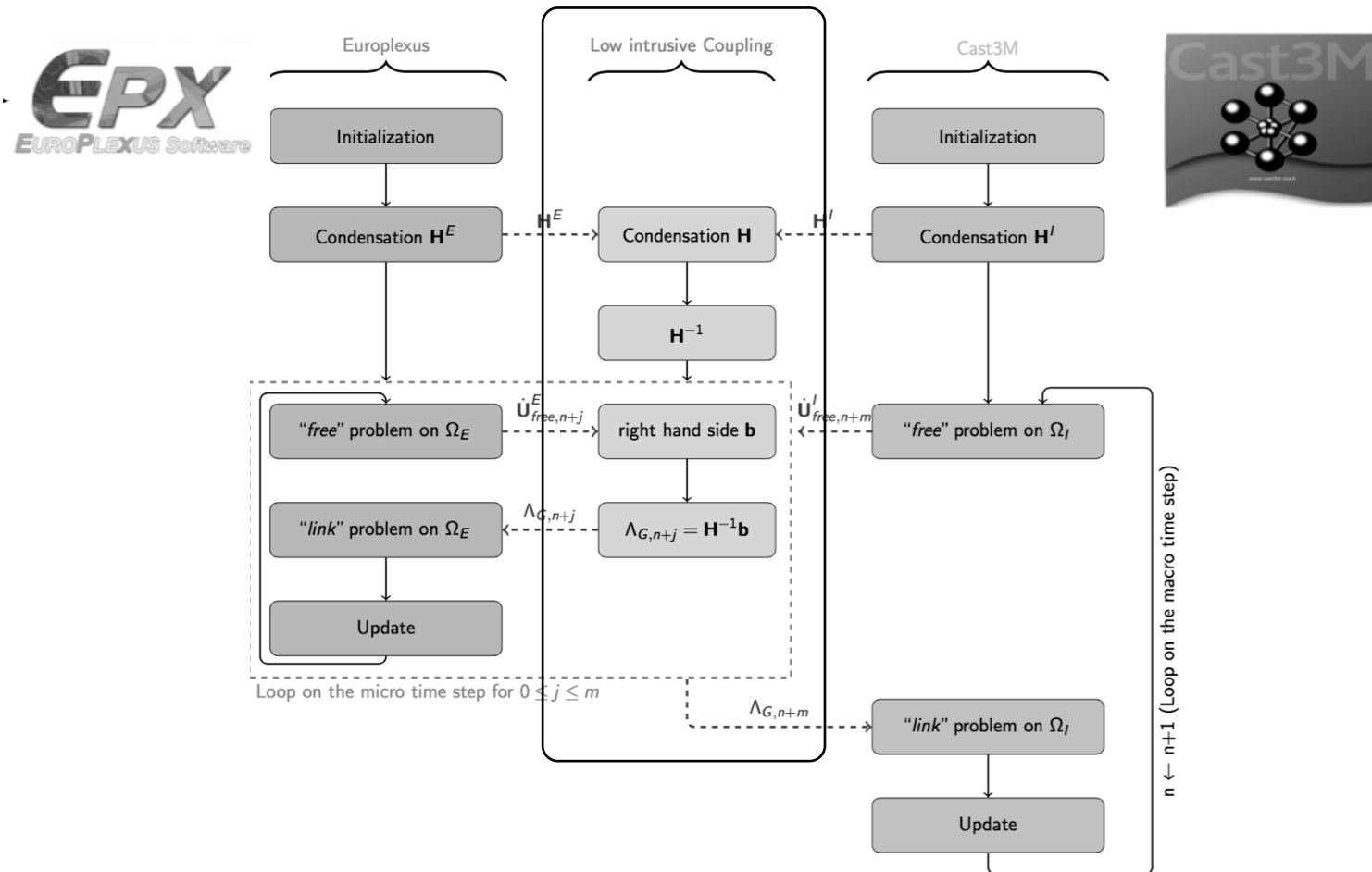
$$\mathbf{L}_G^E \mathbf{v}_{n+j}^E + \mathbf{L}_G^I \mathbf{v}_{n+j}^I = 0 \Rightarrow (\mathbf{H}^E + \mathbf{H}^I) \boldsymbol{\Lambda}_{G,n+j} = \mathbf{b}$$

where $\mathbf{H}^E = \frac{\Delta t}{2} \mathbf{L}_G^E (\mathbf{M}_{lump}^E)^{-1} (\mathbf{L}_G^E)^T$

and $\mathbf{H}^I = \Delta T \gamma_I \mathbf{L}_G^I (\mathbf{M}^I + \Delta T \gamma_I \mathbf{C}^I + \Delta T^2 \beta_I \mathbf{K}^I)^{-1} (\mathbf{L}_G^I)^T$

➡ *Very efficient for implicit linear / explicit non-linear co-simulation (H_c and H_{csts})*

● Co-simulation for implicit-explicit non-smooth dynamics [Fekak et al 2017]



➡ Low intrusive coupling and constant H and H_c operators (linear implicit and non-linear non-smooth explicit)

Introduction

General formalism

Stability

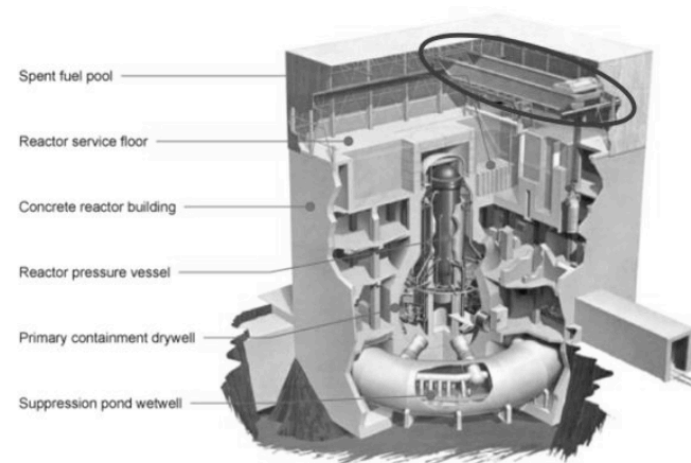
HATI

Conclusions

● Industrial context

➡ *During an earthquake, a bridge crane is exposed to multiple impacts: possible derailment of the bridge crane*

Inside a reactor in Fukushima



Izmit, 1999, derailment of wheels



➡ *Application of direct time integration methods to accurately model bridge cranes subjected to multiple impacts during an earthquake*

Introduction

General formalism

Stability

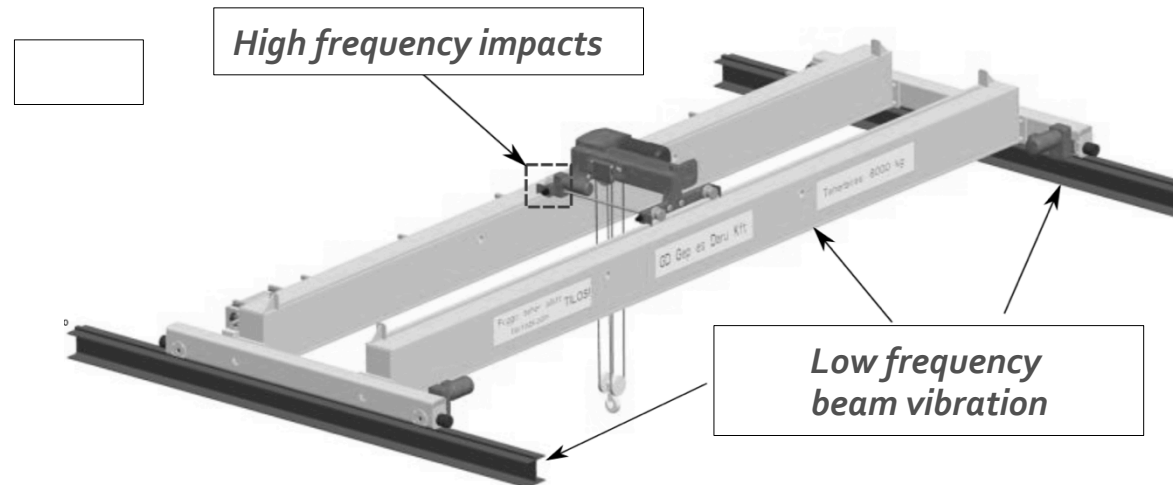
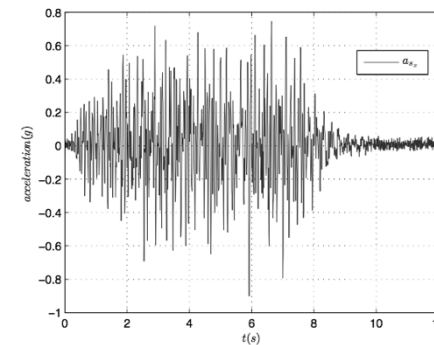
HATI

Conclusions

● Industrial context

➡ **Multi-contact, multi-scale
in time and space**

Given Earthquake (10s)



➡ **Application of implicit-explicit co-simulation to accurately model bridge cranes
subjected to multiple impacts during an earthquake**

Introduction

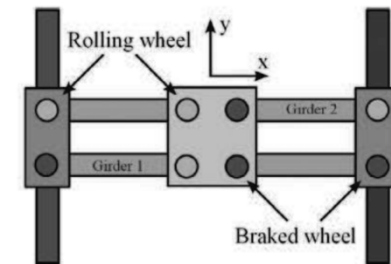
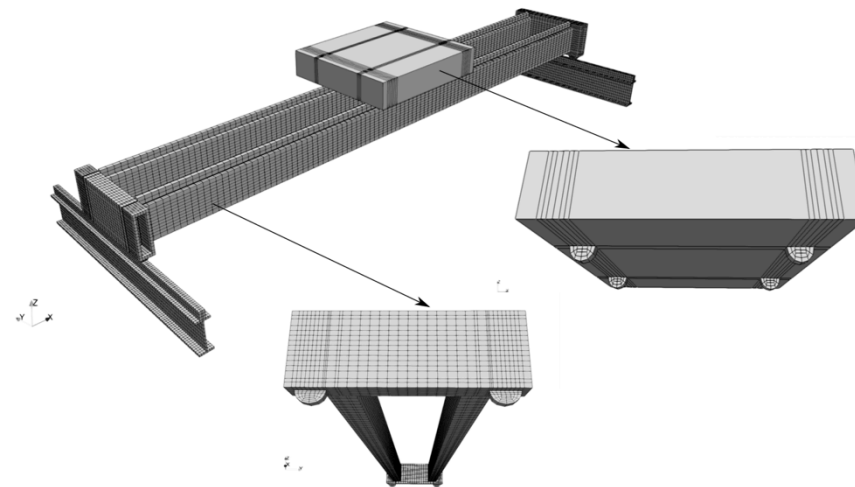
General formalism

Stability

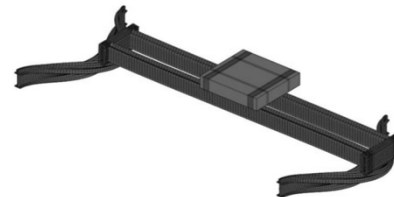
HATI

Conclusions

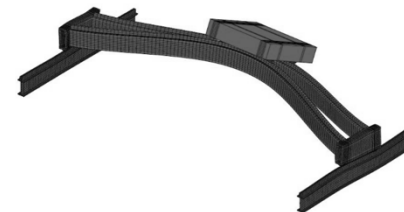
● *Tridimensional finite element model and wheels configuration [Fekak et al 2017]*



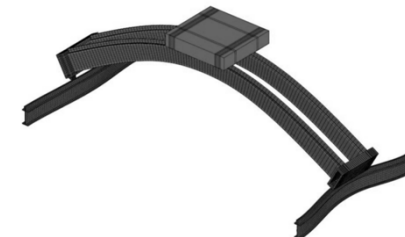
Mode x : exp 9.5 Hz, num 9.3 Hz



Mode y : exp 9.5 Hz, num 9.7 Hz



Mode z : exp 13 Hz, num 14 Hz



Validation of the boundary conditions modeling versus experiments

Introduction

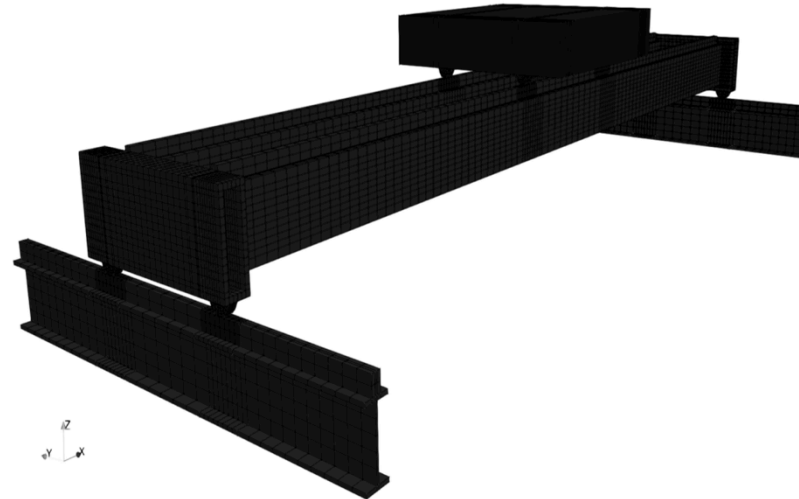
General formalism

Stability

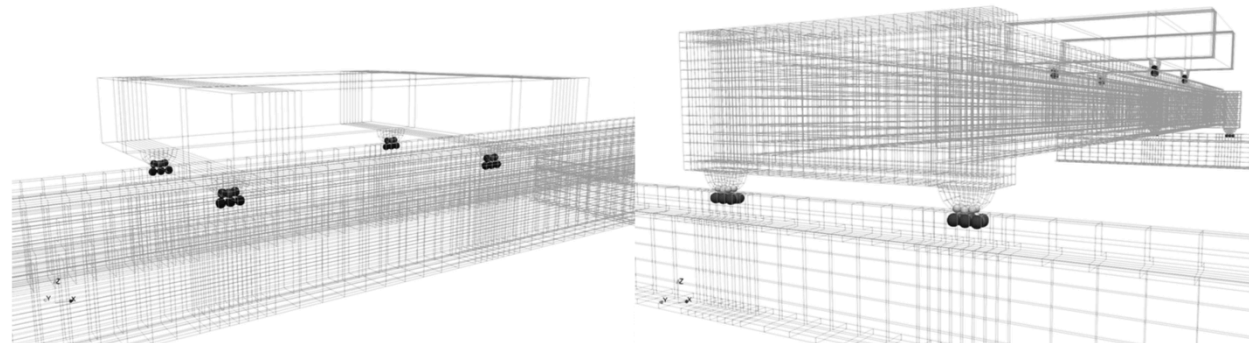
HATI

Conclusions

● *Explicit-implicit co-simulation (Europlexus / Cast3M) [Fekak et al 2017]*



→ *Subdomain decomposition (explicit: blue, implicit: red) according to high frequency area close to contact zones and pinballs [Belytschko 1993, Casadei 2002]*



● *Explicit-implicit co-simulation (Europlexus / Cast3M) [Fekak et al 2017]*

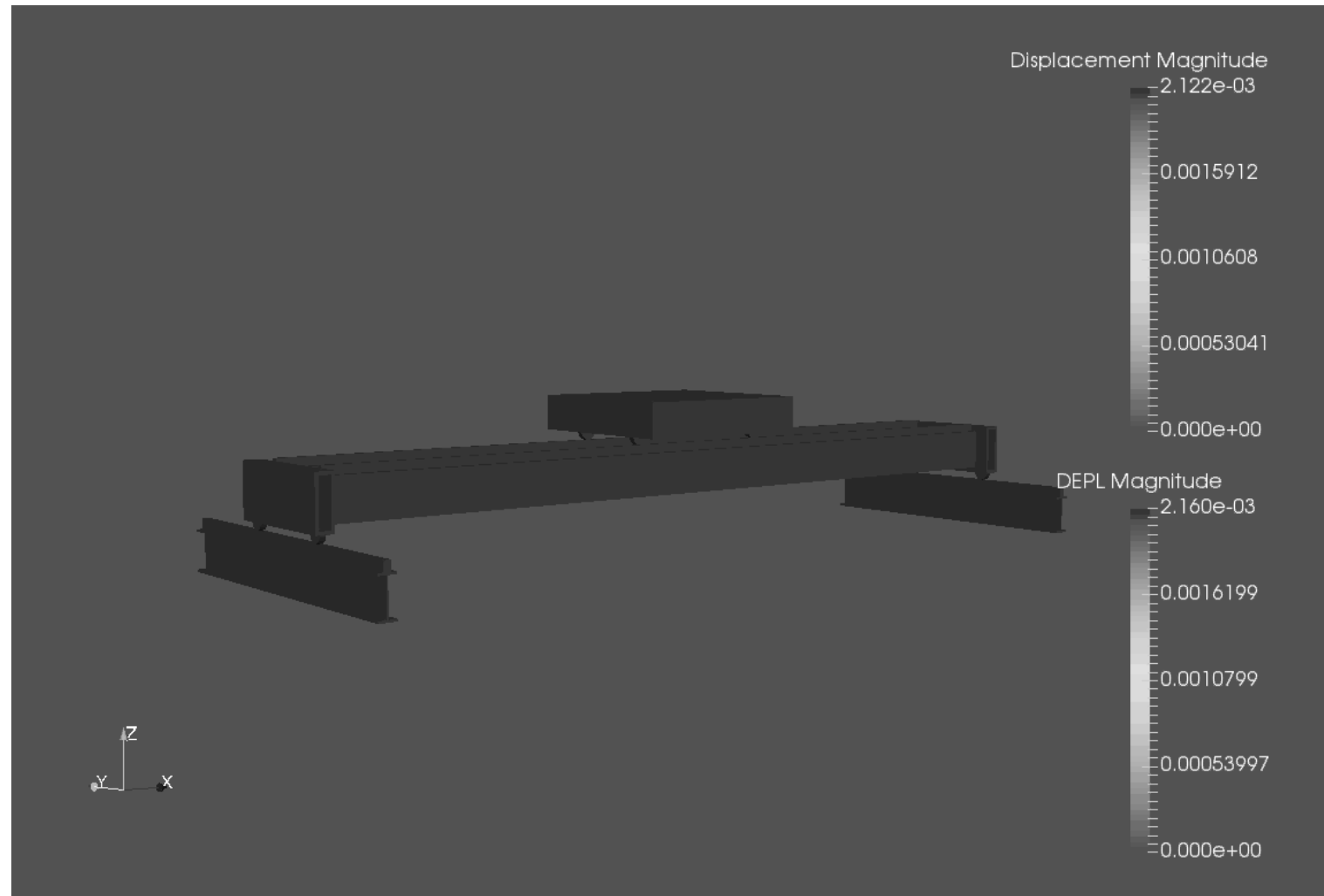
Introduction

General formalism

Stability

HATI

Conclusions



Introduction

General formalism

Stability

HATI

Conclusions

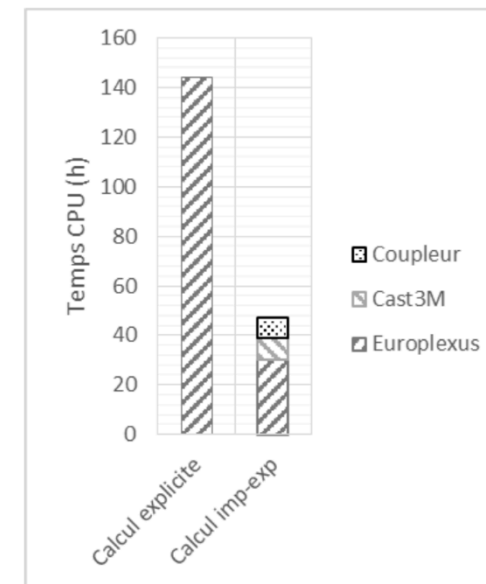
● *Numerical efficiency of co-simulation [Fekak et al 2017]*

	explicit SD	implicit SD	Interface
dofs	5088	70344	1296
time step	10^{-6} s	$100 \cdot 10^{-6}$ s	—

⇒ *Seismic loading of 10 s*

⇒ *Explicit: 10 millions time steps*
Implicit: 100 thousand time steps
 $m = 100$

⇒ *Optimization of the coupling
implementation between
Europlexus and Cast3m*



Introduction

General formalism

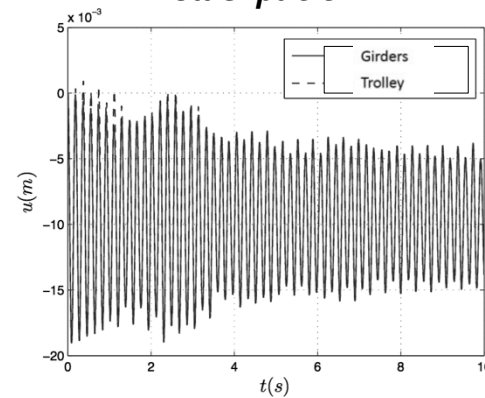
Stability

HATI

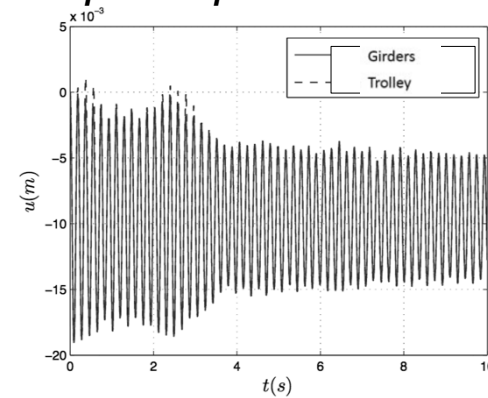
Conclusions

● *Comparison between full explicit and implicit-explicit co-simulation [Fekak et al 2017]*

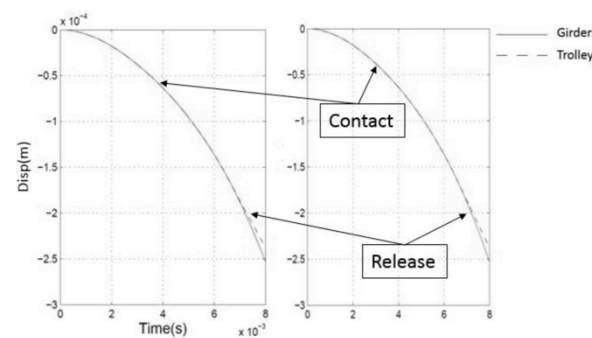
Full explicit



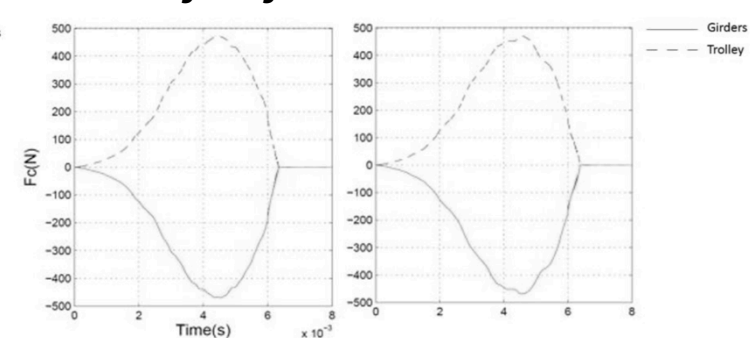
Explicit-implicit co-simulation



First rebound



Zoom of the first rebound



➡ *Rebound between trolley and girders are simulated accurately (6.35ms versus 6.39ms for full explicit and co-simulation respectively)*

Introduction

General formalism

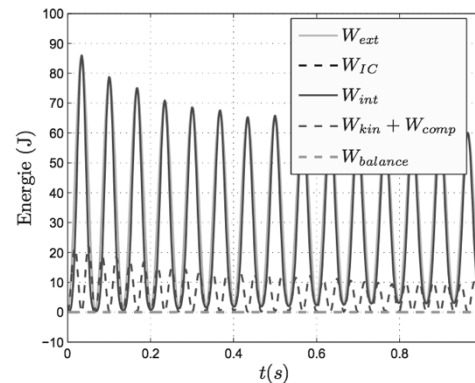
Stability

HATI

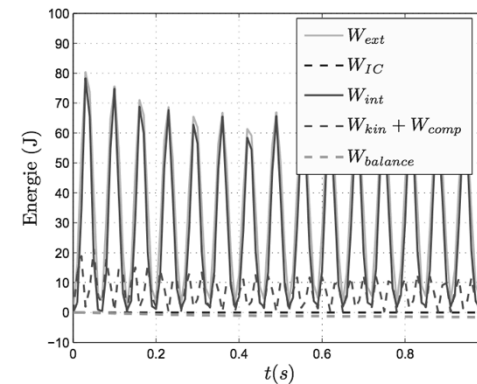
Conclusions

● **Comparison between full explicit and implicit-explicit co-simulation [Fekak et al 2017]**

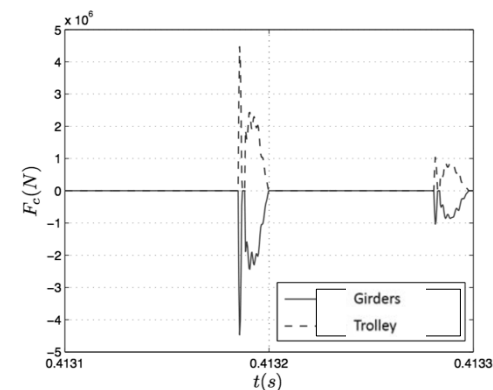
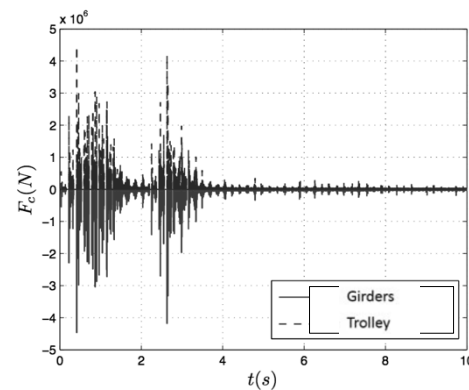
Full explicit



Explicit-implicit co-simulation



⇒ **Small numerical dissipation at the interface for the co-simulation (in agreement with the stability analysis) (here less than 2% with $m=100$)**



⇒ **Successive impacts and contacts between the trolley and girders**

⇒ **Observed Dirac correspond to impacts followed by smooth contact**

Introduction

General formalism

Stability

HATI

Conclusions

4

Conclusions & prospects

Introduction

General formalism

Stability

HATI

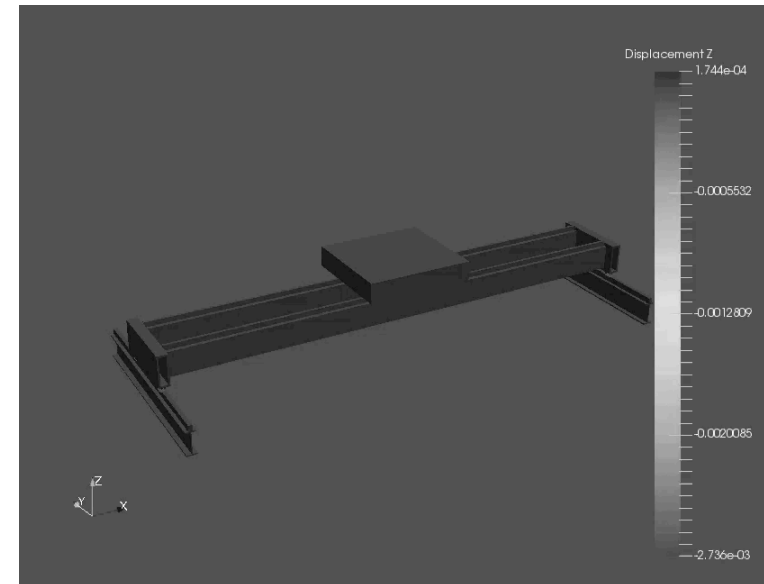
Conclusions

● Conclusions and prospects

- ➡ *Inspired from the pioneer works of Belytschko & co-authors, we have proposed a new class of dual Heterogeneous Asynchronous Time Integrators (HATI) based on interface velocity continuity: BGC-macro and BGC-micro*
- ➡ *A general framework of HATI for G - α integrators (particular cases: HHT- α , WBZ- α , CH- α , Krenk- α), and Newmark (BGC-macro: PH; BGC-micro: GC)*
- ➡ *Wide range of applications: non-linear computational dynamics, multi-physics, FSI, co-simulation*
- ➡ *Extension of BGC to non-smooth contact dynamics. Example: bridge crane under earthquake [Fekak Brun Gravouil 2017]*
- ➡ *Improvement of the bridge crane modeling mandatory (elastic-plastic behavior close to contact area, friction, natural accelerogram as input)*
- ➡ *Comparison between numerical and experimental results in progress*

Acknowledgements: CETIM and the French crane manufacturers (REEL, NFM TECHNOLOGIES, TRACTEL SOLUTIONS, MANITOWOC-POTAIN, ADC-FAYAT)

Thank you for your attention



● *CD-Lagrange with unilateral contact*

Require: $\mathbf{X}_0, \mathbf{U}_0, \dot{\mathbf{U}}_0, \Delta t, e$

1: Initialiser $\ddot{\mathbf{U}}_0, \dot{\mathbf{U}}_{\frac{1}{2}}$

2: $t \leftarrow 0$

3: **while** ($t \leq t_f$) **do**

4: $t \leftarrow t + \Delta t$

5: Calculer $\mathbf{U}_{n+1}, \mathbf{g}_{N,n+1}$

6: $\lambda_{n+\frac{3}{2}} \leftarrow 0$

7: **for all** ($l \leq p$) **do**

8: **if** ($g_N^l \leq 0$) **then**

9: $v_{n+\frac{3}{2}}^l \leftarrow -ev_{n+\frac{1}{2}}^l$

10: Calculer $\lambda_{n+\frac{3}{2}}^l$

11: **if** ($\lambda_{n+\frac{3}{2}}^l < 0$) **then**

12: $\lambda_{n+\frac{3}{2}}^l \leftarrow 0$

13: **end if**

14: **end if**

15: **end for**

16: Calculer $\mathbf{I}_{n+1}, \mathbf{W}_{n+1}$

17: Calculer $\dot{\mathbf{U}}_{n+\frac{3}{2}}, \mathbf{v}_{n+\frac{3}{2}}$

Ensure: si impact : $\mathbf{U}_{n+1}, \mathbf{I}_{n+1}, \dot{\mathbf{U}}_{n+1}$

Ensure: si contact : $\mathbf{U}_{n+1}, \mathbf{I}_{n+1}, \dot{\mathbf{U}}_{n+1}, \mathbf{F}_{c,n+1}, \ddot{\mathbf{U}}_{n+1}$

18: **end while**

▷ Boucle sur tous les points de contact

● CD-Lagrange with friction

Require: $\mathbf{x}_0, \mathbf{u}_0, \dot{\mathbf{u}}_0, \Delta t, e, \mu$

1: Initialiser $\ddot{\mathbf{u}}_0, \dot{\mathbf{u}}_{\frac{1}{2}}$

2: $t \leftarrow 0$

3: while ($t \leq t_f$) do

4: $t \leftarrow t + \Delta t$

5: Calculer $\mathbf{u}_{n+1}, \mathbf{g}_{N,n+1}$

6: $\lambda_{N,n+\frac{3}{2}} \leftarrow 0, \lambda_{T,n+\frac{3}{2}} \leftarrow 0$

7: for all ($l \leq p$) do

8: if ($g_N^l \leq 0$) then

9: $\mathbf{v}_{N,n+\frac{3}{2}}^l \leftarrow -e \mathbf{v}_{N,n+\frac{1}{2}}^l, \mathbf{v}_{T,n+\frac{3}{2}}^l \leftarrow 0$

10: Calculer $\lambda_{N,n+\frac{3}{2}}^l, \lambda_{T,n+\frac{3}{2}}^l$

11: if ($\lambda_{N,n+\frac{3}{2}}^l < 0$) then

12: $\lambda_{N,n+\frac{3}{2}}^l \leftarrow 0, \lambda_{T,n+\frac{3}{2}}^l \leftarrow 0$

13: else

14: if $\|\lambda_{T,n+\frac{3}{2}}^l\| > \mu \lambda_{N,n+\frac{3}{2}}^l$ then

15: $\lambda_{T,n+\frac{3}{2}}^l \leftarrow -\frac{\mathbf{b}_T^l}{\|\mathbf{b}_T^l\|} \mu \lambda_{N,n+\frac{3}{2}}^l$

16: end if

17: end if

18: end if

19: end for

20: Calculer $\mathbf{l}_{n+1}, \mathbf{w}_{n+1}$

21: Calculer $\dot{\mathbf{u}}_{n+\frac{3}{2}}, \mathbf{v}_{n+\frac{3}{2}}$

Ensure: si impact : $\mathbf{u}_{n+1}, \mathbf{l}_{n+1}, \dot{\mathbf{u}}_{n+1}$

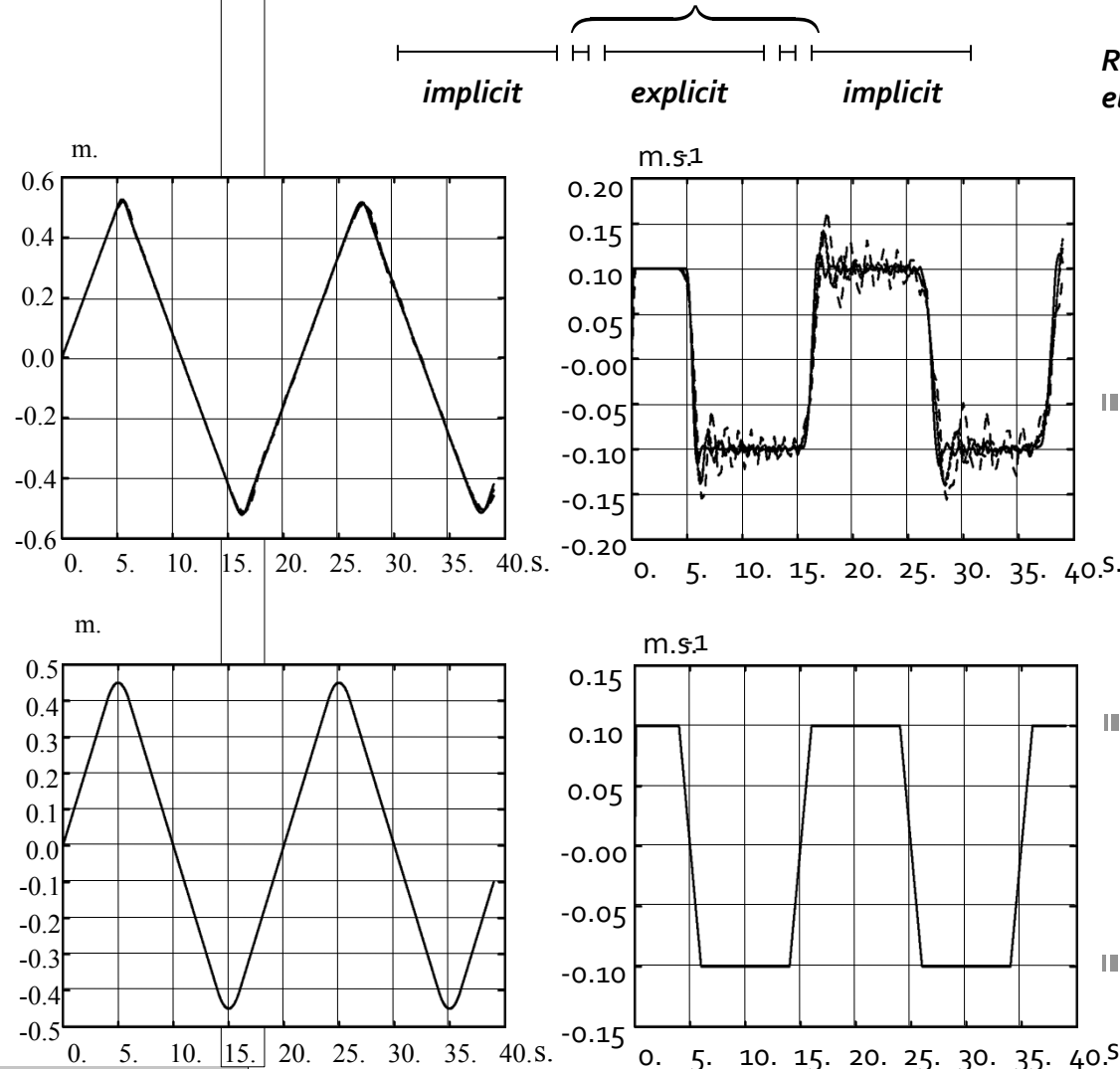
Ensure: si contact : $\mathbf{u}_{n+1}, \mathbf{l}_{n+1}, \dot{\mathbf{u}}_{n+1}, \mathbf{F}_{c,n+1}, \ddot{\mathbf{u}}_{n+1}$

22: end while

▷ Boucle sur tous les points de contact

▷ Pas de contact

● A dual mixed method in time [Gravouil Combescure 2000]



R : ratio between the size of transition elements and explicit elements

($m=3$)

--- Reference method
 Implicit / implicit
 - - - - - Implicit / explicit
 ——— Explicit / explicit

Results similar to the mixed method in time of [Liu, Belytschko 1982] but with any R .

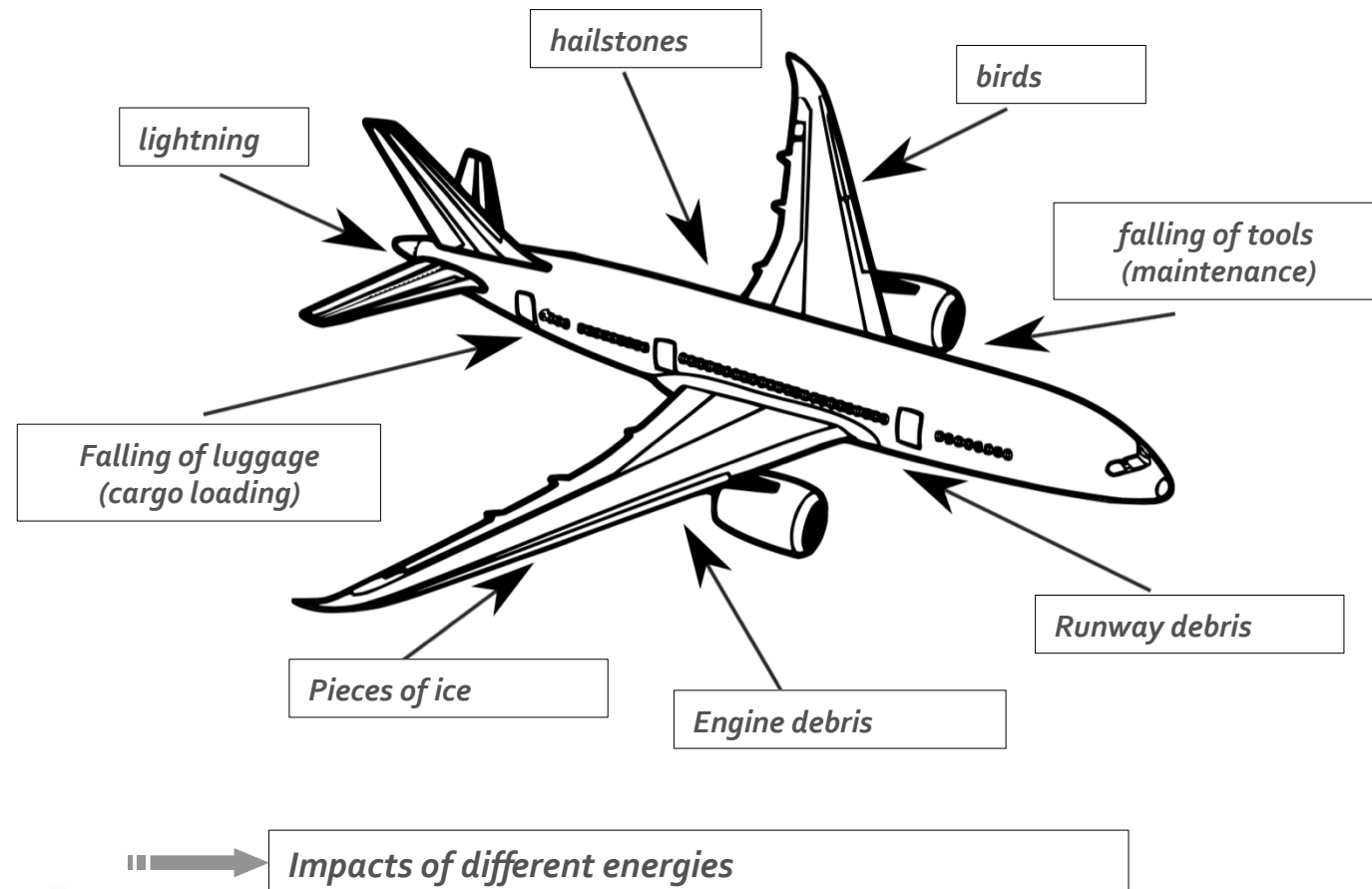
($m=10$)

——— explicit / explicit

« exact » solution with $R=1$ (explicit / explicit) and no numerical damping
 Still stable with $R=1$ for any coupling (implicit / implicit and explicit / implicit)

Primal method unstable in all case with $R=1$

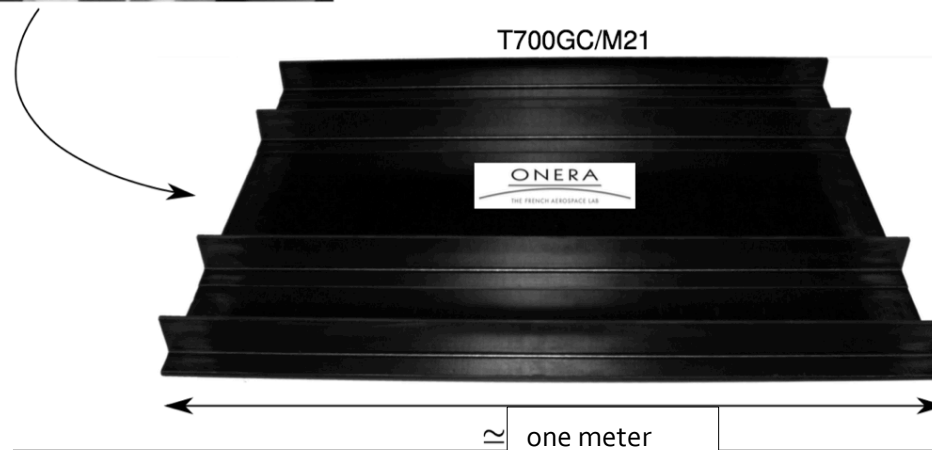
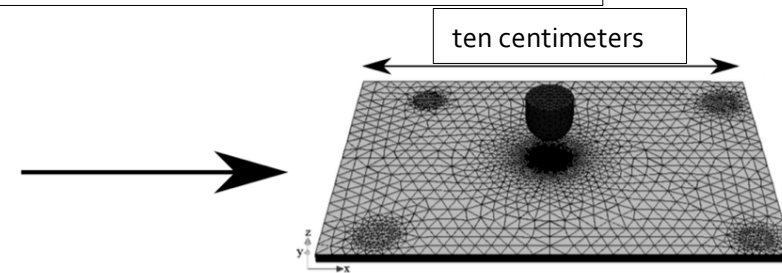
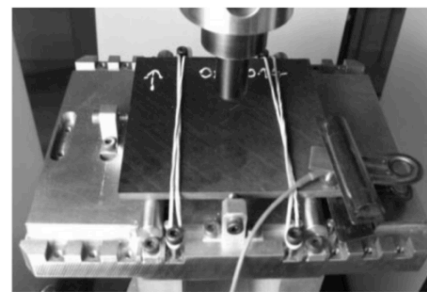
- **Industrial application of the BGC-micro (explicit / implicit co-simulation)**
[Chantrait Rannou Gravouil 2014]



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federating research project on static / dynamic transition

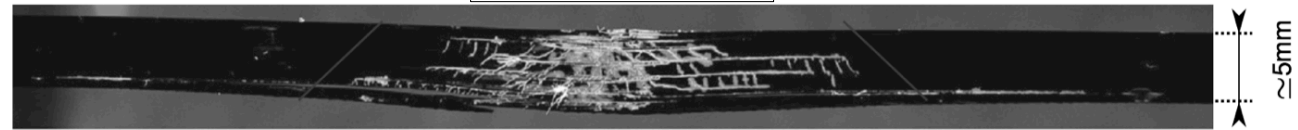
- *Emilie Trouset: modeling the impact on small structures (ten centimeters)*



||| → *modeling structures of larger sizes subjected to an impact*

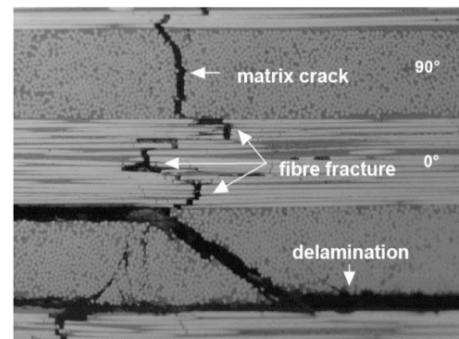
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Cone of damage

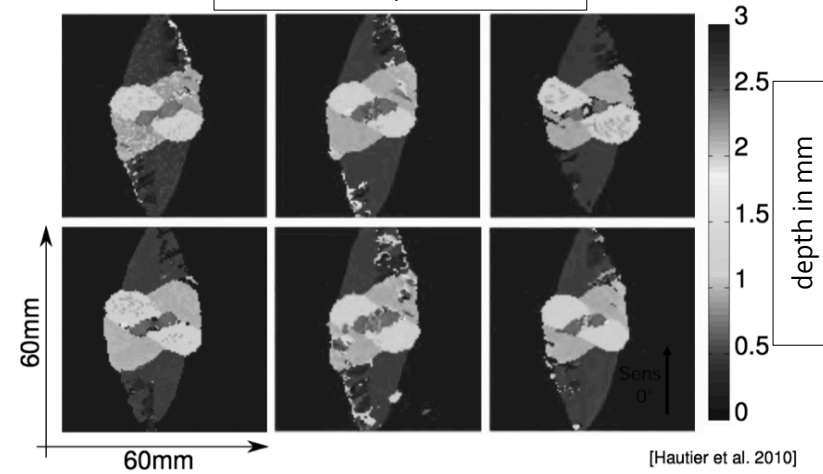


[Lopes et al. 2009]

6 different experiments



[Davies et al. 2004]



[Hautier et al. 2010]

→ **spatial localization of damage and thickness of the ply << size of the structure**
(helical cracks)

- **Industrial application of the BGC-micro (explicit / implicit co-simulation)**
[Chantrait Rannou Gravouil 2014]

- **Damage thermodynamically admissible**

$$\sigma = \underline{\tilde{S}}^{-1} \varepsilon \quad (1)$$

$$\underline{\tilde{S}}^{-1} = (\underline{S}_0 + d_1 \underline{H}_1 + d_2 \underline{H}_2 + d_3 \underline{H}_3)^{-1} \quad (2)$$

$\underline{\tilde{S}}$: Material flexibility

$\underline{H}_i$: Effect tensor

d_i : Damage variable

- **Deformation criteria of Hashin type:**

Fiber tensile
direction

$$f_1^t = \left(\frac{\varepsilon_{11}}{x_t} \right)^2$$

Fiber compression
direction

$$f_1^c = \left(\frac{\varepsilon_{11}}{x_c} \right)^2 + \frac{1}{2} \left(\frac{\varepsilon_{12}}{s_{12}} \right)^2 + \frac{1}{2} \left(\frac{\varepsilon_{13}}{s_{13}} \right)^2$$

[Carrère N., Maire J-F. 2007,
[Laurin 2005]

- **Damage kinetics with delayed effect**

$$\forall i \in \{1, 2, 3\}, d_i = \max \left(\max_t (d_i(t)), \tilde{d}_i \right) \text{ with} \quad (3)$$

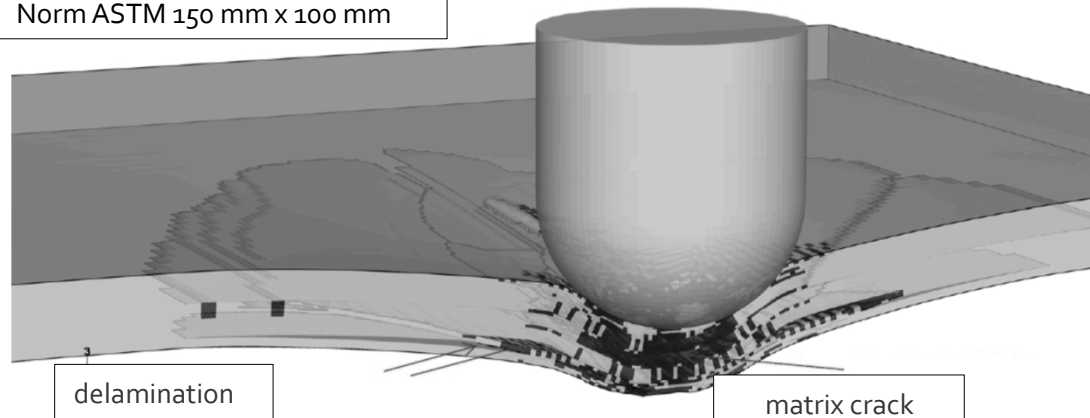
$$\begin{cases} \frac{\partial \tilde{d}_i}{\partial t} = \frac{1}{\tau_i} \left(\alpha_i \left(\sqrt{f_i} - 1 \right)^{p_i} - \tilde{d}_i \right) & \text{si } f_i > 1 \\ \frac{\partial \tilde{d}_i}{\partial t} = 0 & \text{si } f_i \leq 1 \end{cases}$$

➡ **simplified version of the model OPFM**

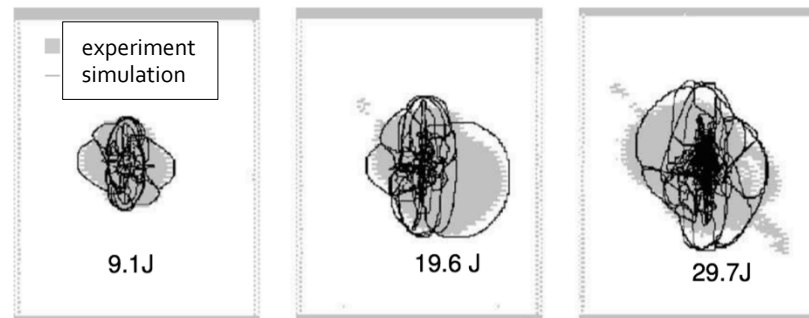


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[Chantrait Rannou Gravouil 2014]

Norm ASTM 150 mm x 100 mm



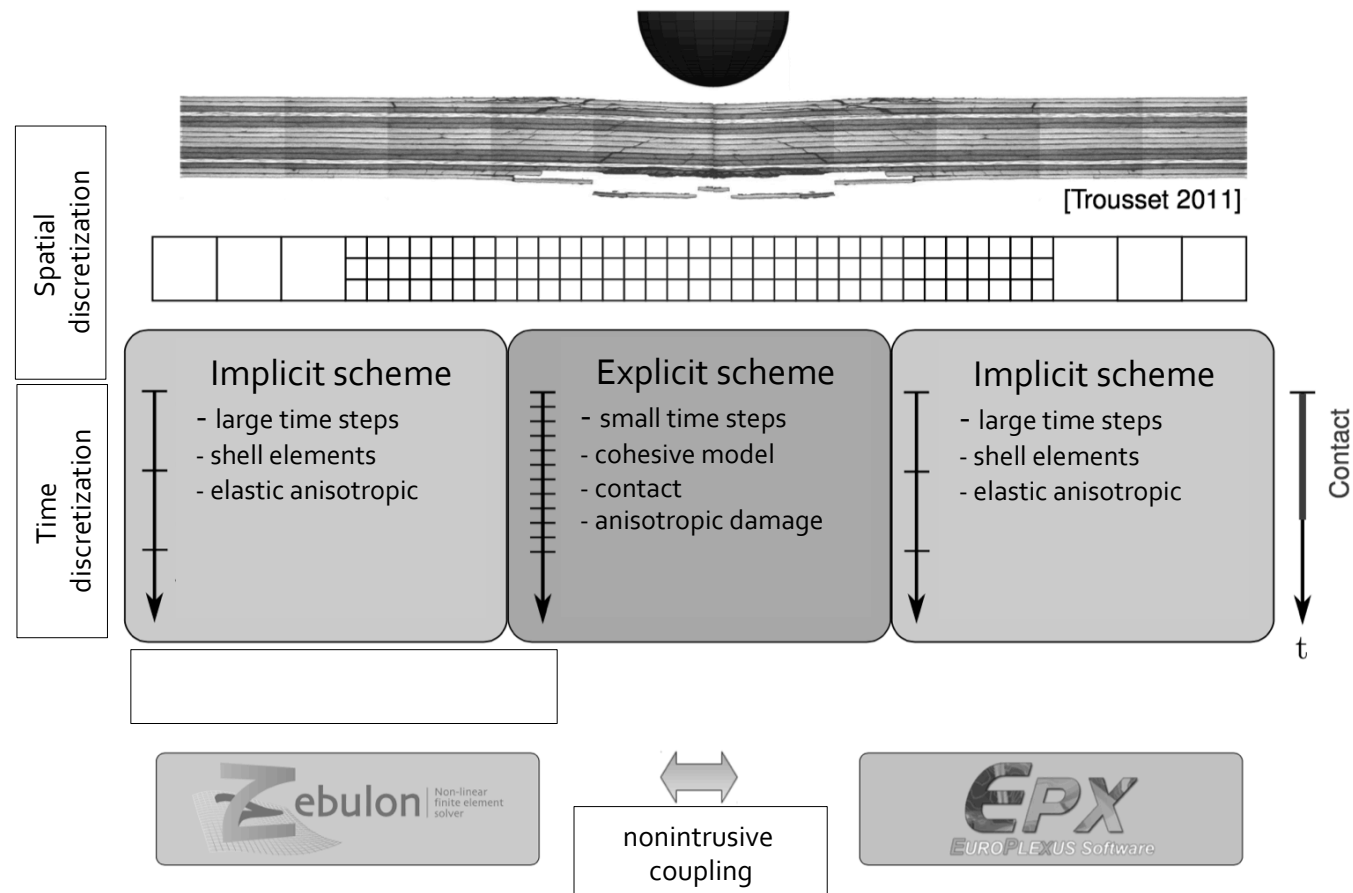
[Lopes et al. 2009]



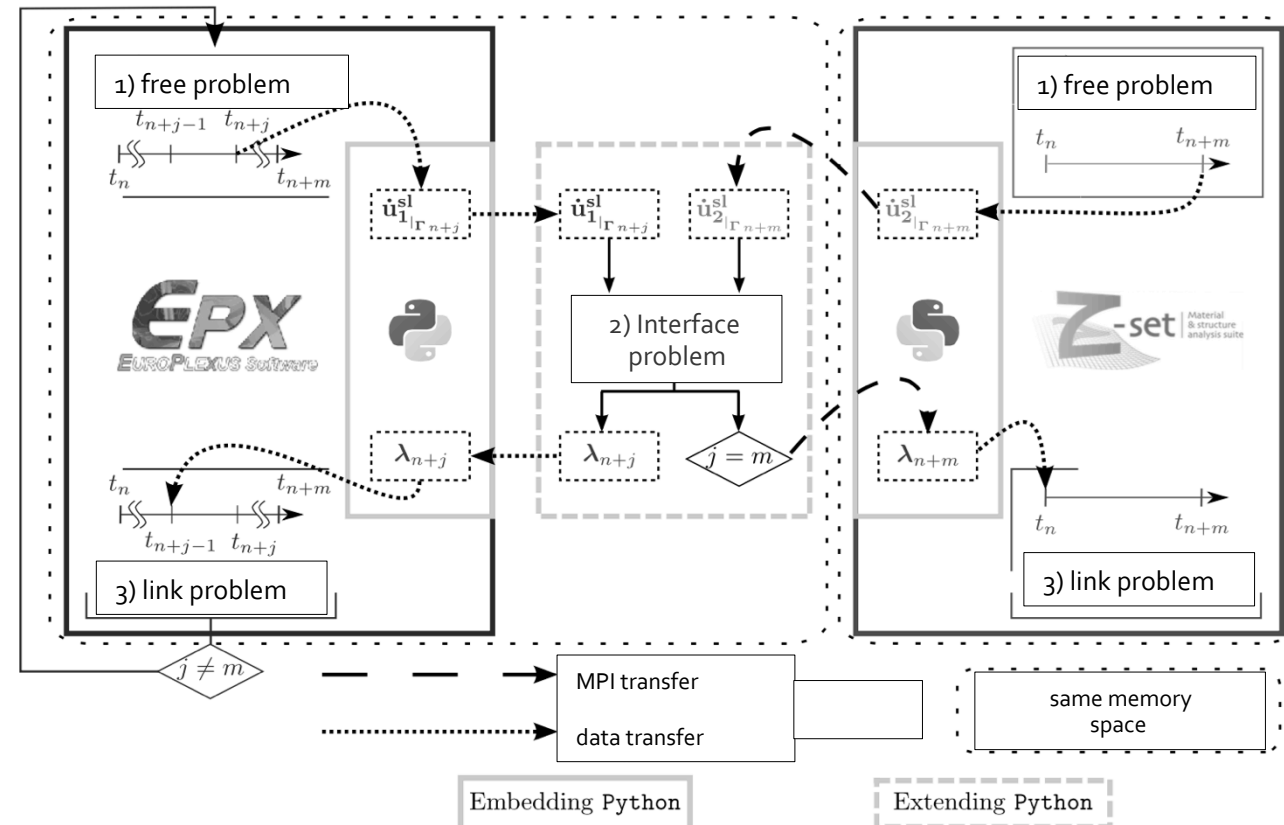
[Lopes et al. 2009]

➡ **satisfactory model but the calculation time is too high on small plates**

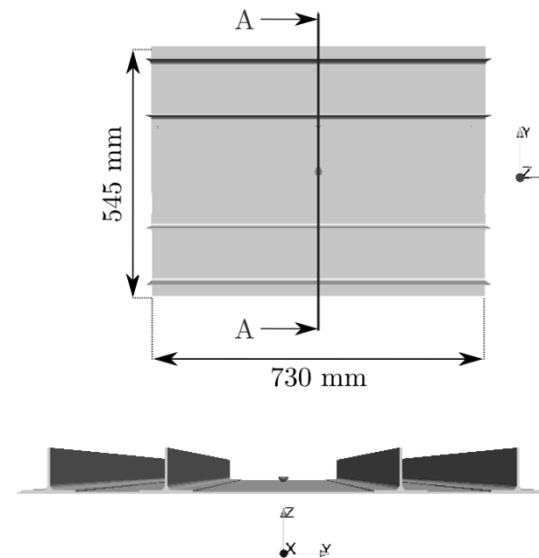
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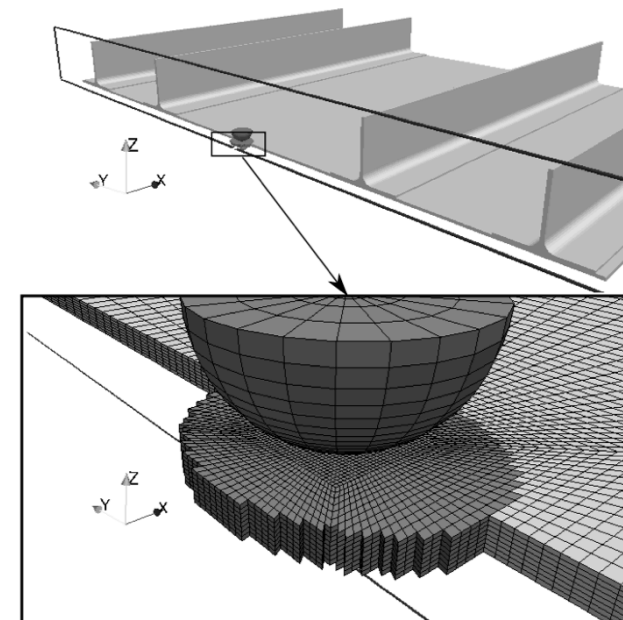


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Numerical specifications

- DOFs: 100000
- 90000 implicit DOFs
- 4500 interface DOFs

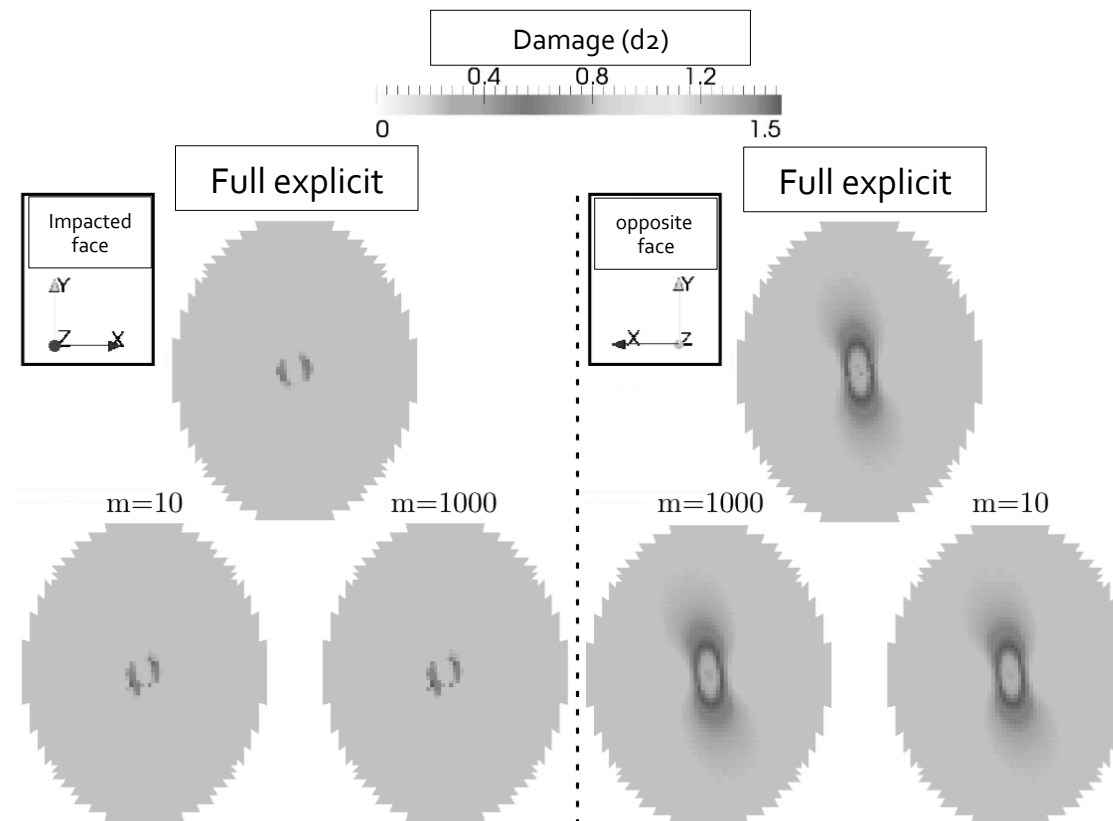


Ply sequence

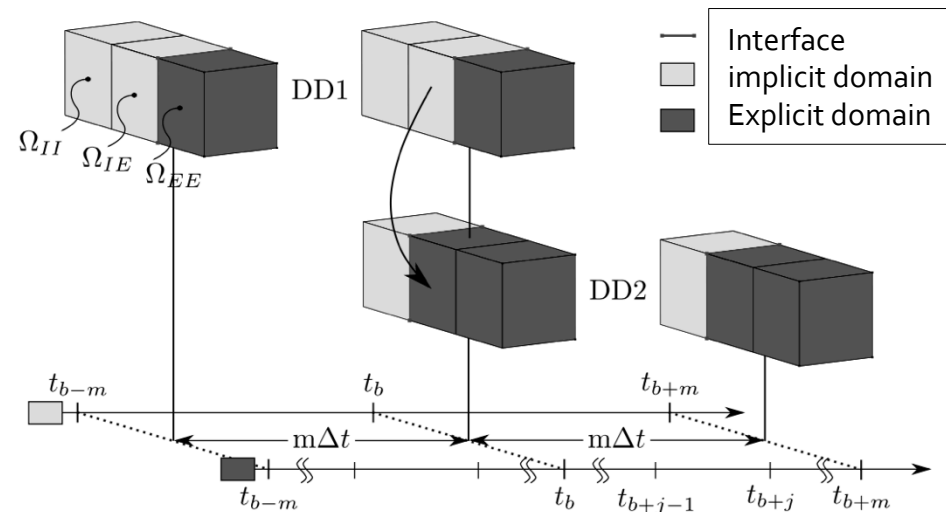
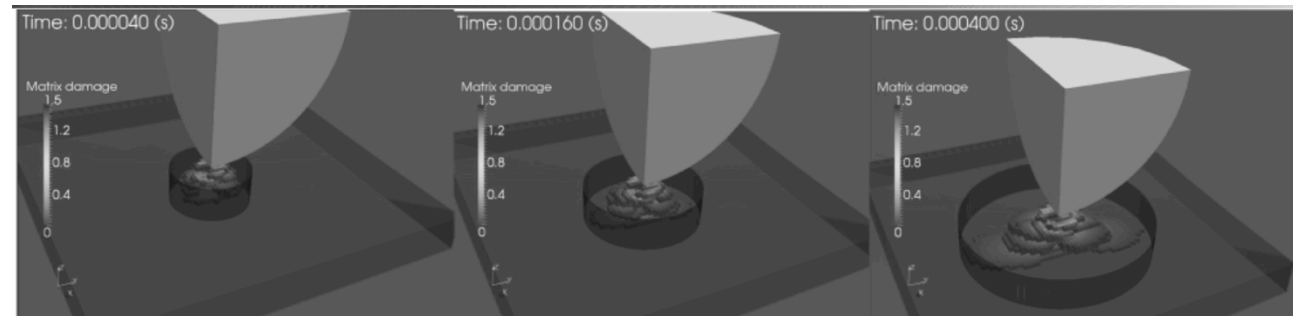
- Thickness $(90, 45, 0, -45)_s$
- Stiffener $(90, 45, 0, -45)_s$



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[Chantrait Rannou Gravouil 2014]

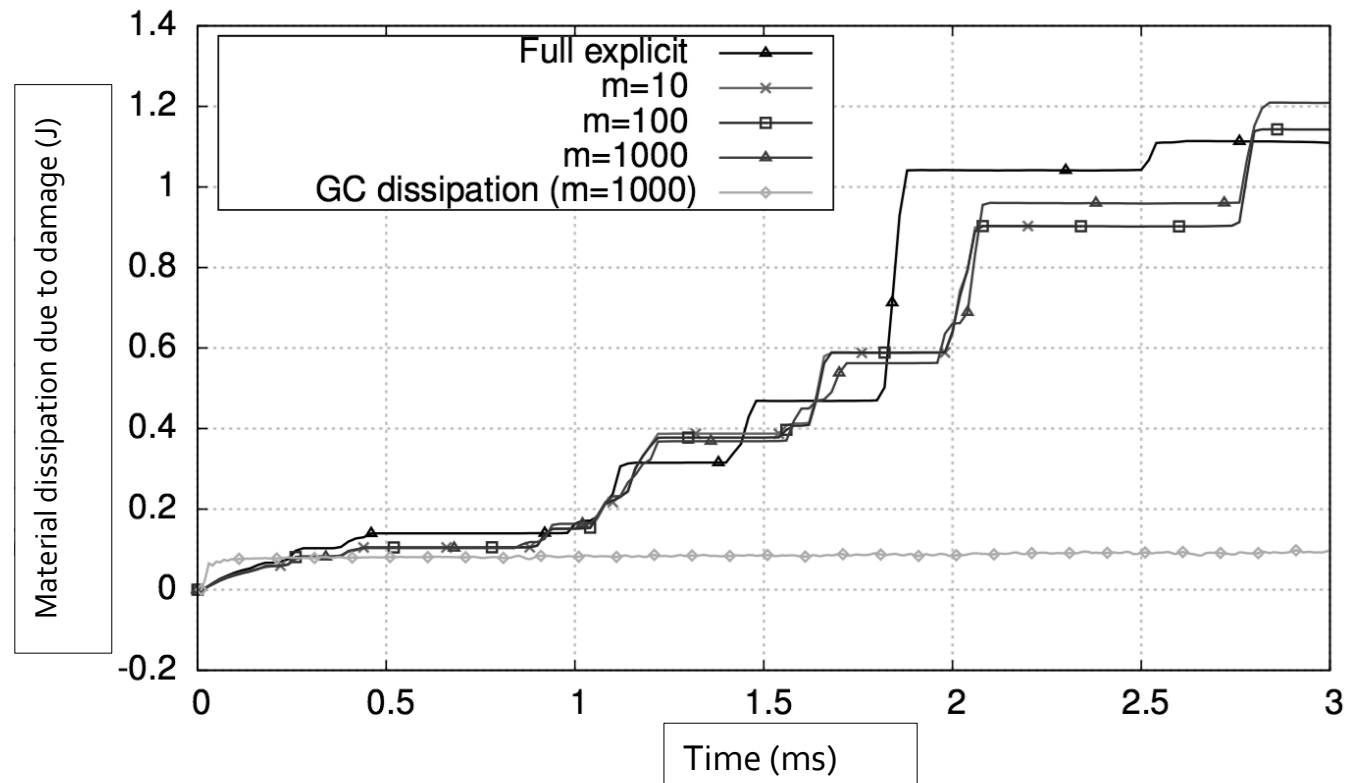


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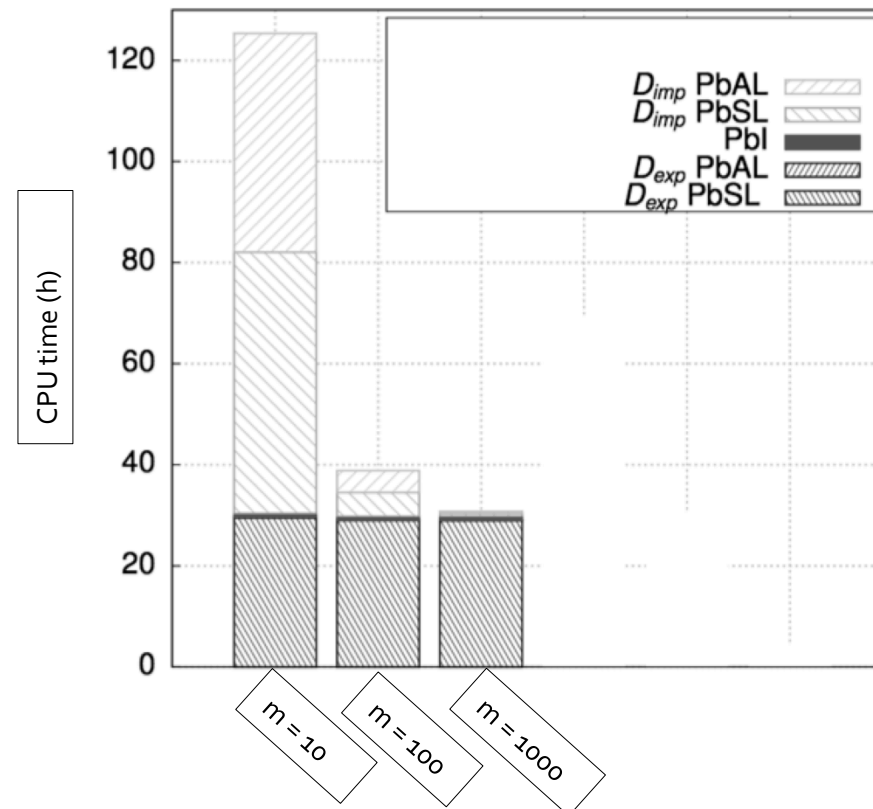


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interface numerical dissipation very low compare to mechanical dissipation

● **Industrial application of the BGC-micro (explicit / implicit co-simulation)**
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Schur complement
(20 mn)

Significant time saving with m high ($dt_{exp} = 1 \times 10^{-8} s$)
Main numerical effort in the explicit domain

