Variational approach to fracture:
Formulation, general properties and examples

Jean-Jacques Marigo

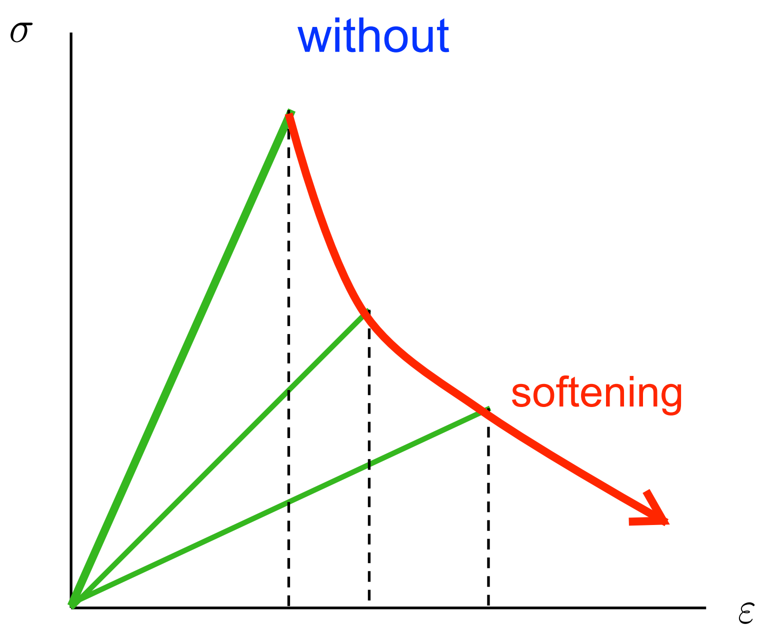
(Palaiseau, Ecole Polytechnique)

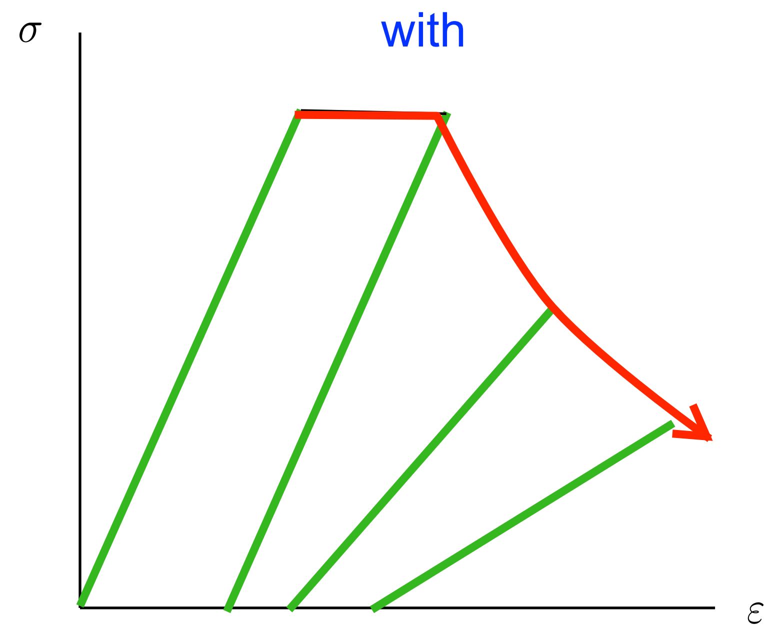
joint work with

Gilles Francfort (Villetaneuse, Paris 13)
Blaise Bourdin (Baton Rouge, LSU)
Corrado Maurini (Paris, UPMC)

Damage models without or with plasticity

- quasi-static, rate independent evolution law
- scalar damage variable
- variational approach



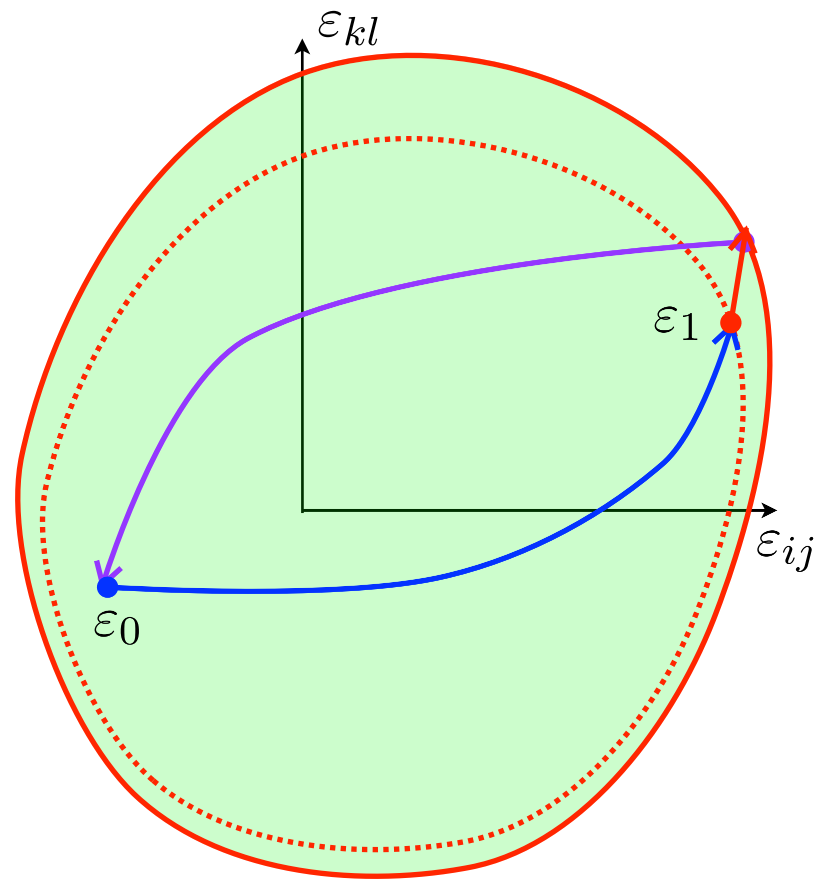


Justification of "standard" laws

✓ Drucker-Ilyushin Postulate

The strain work must be non negative in every strain cycle

$$\oint_{\mathcal{C}} \sigma \cdot d\varepsilon \geq 0, \qquad \forall \mathcal{C}$$



✓ In perfect plasticity

The D-I postulate is equivalent to the Hill principle of maximal plastic work which is equivalent to the convexity of the yield surface and the normality rule

$$\text{Drucker-Ilyushin} \iff \text{Hill}$$

✓ For brittle scalar damage laws

- stress-strain relation

$$\sigma = \frac{\partial \psi}{\partial \varepsilon}(\varepsilon, \alpha), \qquad \alpha \in [0, \alpha_M)$$

- yield criterion : damage grows only when the strains (or the stresses) reach some yield surface which is damage dependent
- Théorem (JJM, '89)

$$\text{Drucker-Ilyushin} \iff \text{Standard Law}$$

$$\text{yield criterion : } -\frac{\partial \psi}{\partial \alpha}(\varepsilon, \alpha) \leq w'(\alpha)$$

Endommagement sans plasticité

rupture fragile

General form of standard non regularized damage laws

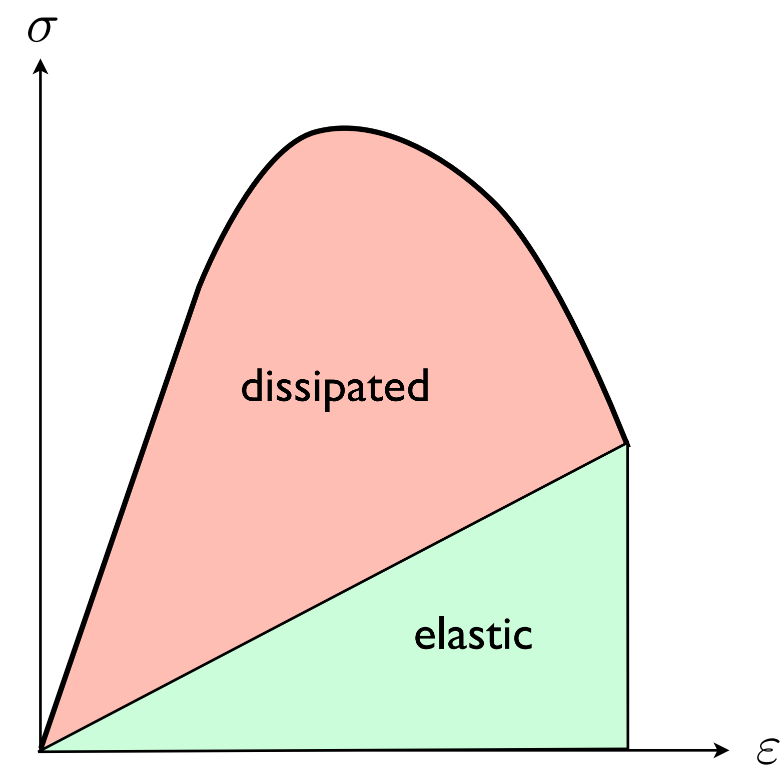
✓ constitutive relations

σ–ε relation : $\sigma = \dfrac{\partial \psi}{\partial \varepsilon}(\varepsilon, \alpha)$

irreversibility : $\dot{\alpha} \geq 0$

yield criterion : $-\dfrac{\partial \psi}{\partial \alpha}(\varepsilon, \alpha) \leq w'(\alpha)$

energy balance : $\left(\dfrac{\partial \psi}{\partial \alpha}(\varepsilon, \alpha) + w'(\alpha)\right) \dot{\alpha} = 0$



✓ *energetic interpretation*

the strain work is a state function equal to the sum of the elastic energy and the dissipated energy

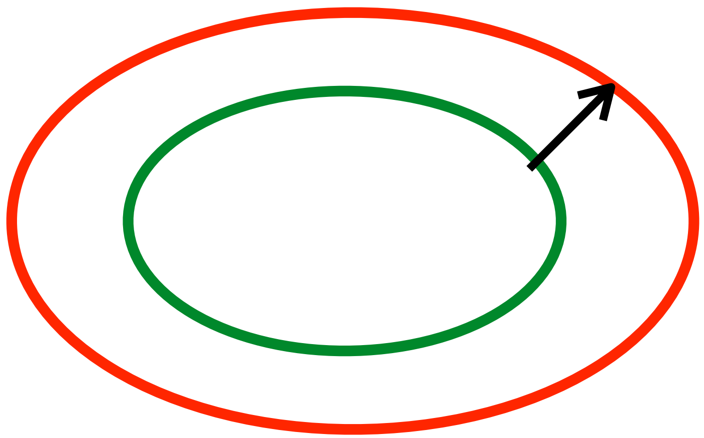
$$\int_{\overrightarrow{0\varepsilon}} \sigma \cdot d\varepsilon = W(\varepsilon, \alpha) = \psi(\varepsilon, \alpha) + w(\alpha)$$

$$\psi(\varepsilon, \alpha) = \tfrac{1}{2}\mathsf{E}(\alpha)_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$$

"linear" case

Hardening and softening conditions

✓ Strain hardening



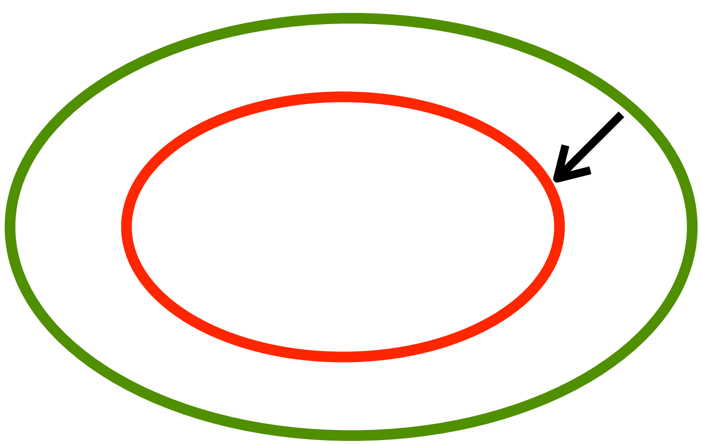
strain space

$$-\frac{1}{2}\,\mathsf{E}'(\alpha)\varepsilon\cdot\varepsilon \leq \mathsf{w}'(\alpha)$$

$$\alpha \mapsto \mathsf{E}'(\alpha)/\mathsf{w}'(\alpha) \ \text{increasing}$$

$$\varepsilon = \mathsf{S}(\alpha)\sigma$$

✓Stress softening



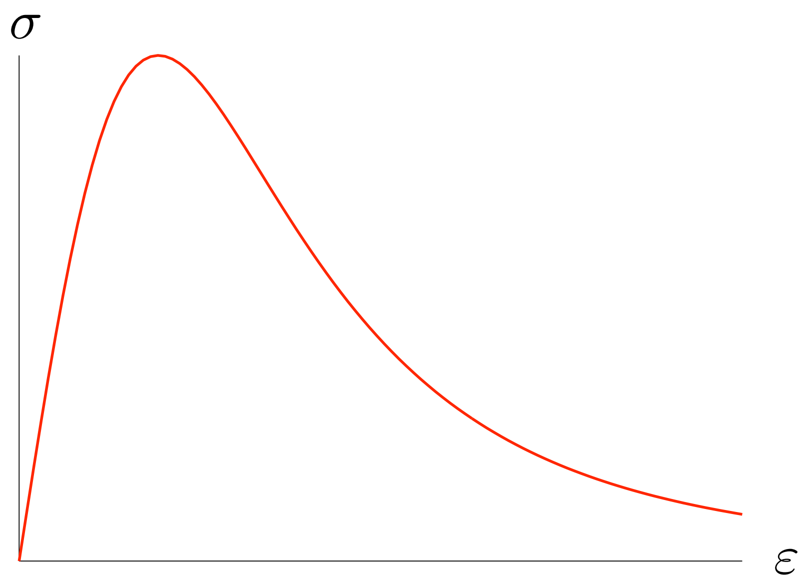
stress space

$$\frac{1}{2}\,\mathsf{S}'(\alpha)\sigma\cdot\sigma \leq \mathsf{w}'(\alpha)$$

$$\alpha \mapsto \mathsf{S}'(\alpha)/\mathsf{w}'(\alpha) \ \text{increasing}$$

✓Examples

- Ambrosio-Tortorelli model

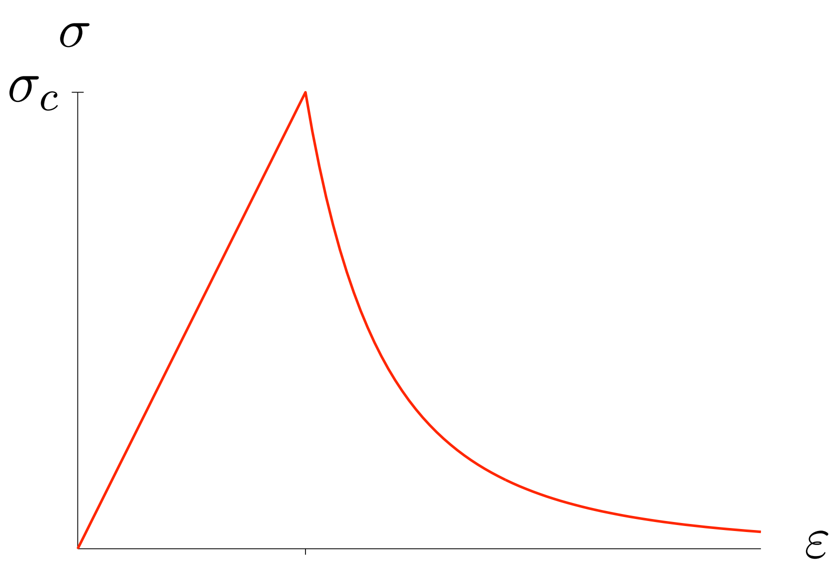


$$\mathsf{E}(\alpha) = (1-\alpha)^2 \mathsf{E}_0$$

$$\mathsf{w}(\alpha) = \mathsf{w}_1\, \alpha^2$$

no elastic range

- A model with finite critical stress and stress softening



$$\mathsf{E}(\alpha) = (1-\alpha)^2 \mathsf{E}_0$$

$$\mathsf{w}(\alpha) = \frac{\sigma_c^2}{\mathsf{E}_0}\alpha$$

Construction of the gradient damage models

✓ Definition of the strain work density function

$$W(\varepsilon, \alpha, \nabla\alpha) = \frac{1}{2}\,\mathsf{E}(\alpha)(\varepsilon - \varepsilon^{th}) \cdot (\varepsilon - \varepsilon^{th}) + \mathsf{w}(\alpha) + \boxed{\frac{1}{2}\,\mathsf{w}_1 \ell(\alpha)^2 \nabla\alpha \cdot \nabla\alpha}$$

$$\ell(\alpha) = \text{material characteristic length}$$

✓ Choice of the damage parameter

$$\boxed{W(\varepsilon, \alpha, \nabla\alpha) = \mathsf{w}(\alpha) + \tfrac{1}{2}\mathsf{w}_1 \ell^2 \nabla\alpha \cdot \nabla\alpha + \tfrac{1}{2}\mathsf{E}(\alpha)(\varepsilon - \varepsilon^{th}) \cdot (\varepsilon - \varepsilon^{th})} \qquad \boxed{\alpha \in [0, 1]}$$

✓ Constitutive inequalities

$$\mathsf{E}(0) = \mathsf{E}_0 > 0, \qquad \mathsf{E}(1) = 0 \qquad \mathsf{E}(\alpha) > 0, \qquad \mathsf{E}'(\alpha) < 0$$

$$\mathsf{w}(0) = 0 \qquad \mathsf{w}'(\alpha) > 0 \qquad \mathsf{w}_1 = \mathsf{w}(1) < +\infty$$

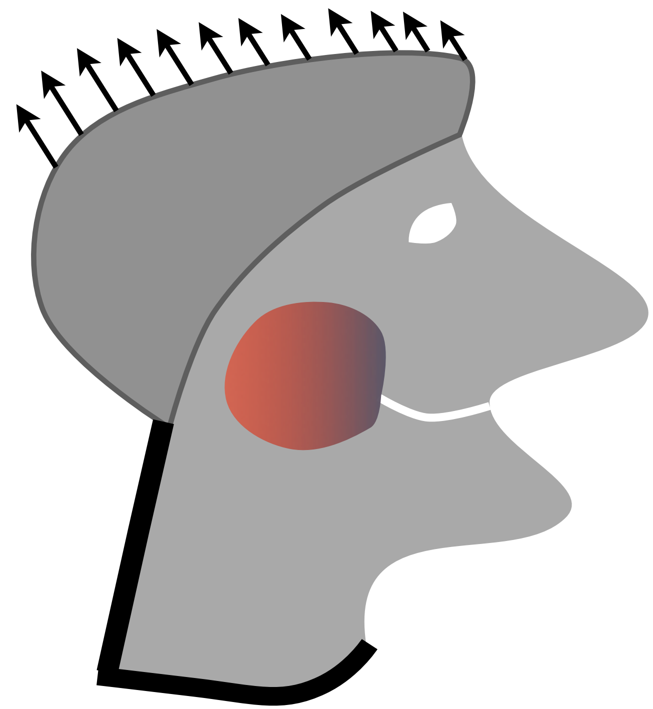
$$\text{stress softening} = \boxed{\alpha \mapsto \mathsf{S}'(\alpha)/\mathsf{w}'(\alpha)\ \text{increasing}}$$

$$\mathsf{S}(\alpha) = \mathsf{E}(\alpha)^{-1} = \text{compliance tensor}$$

✓the global evolution problem

▸the global total energy

$$\mathcal{E}_t(u,\alpha) = \int_\Omega W_t(\varepsilon(u),\alpha,\nabla\alpha)dV - f_t(u)$$



▸the evolution problem in its variational form

1. Irreversibility

$$\dot{\alpha}_t \geq 0$$

2. First order stability condition

$$\mathcal{E}'_t(u_t,\alpha_t)(v-u_t,\beta-\alpha_t) \geq 0, \quad \forall v \in \mathcal{C}_t, \quad \forall \beta \ : \ \alpha_t \leq \beta \leq 1$$

2'. Complete stability condition

$$\forall (v,\beta) \text{ admissible and } h \text{ small enough}, \quad \mathcal{E}_t(u_t,\alpha_t) \leq \mathcal{E}(u_t + hv, \alpha_t + h\beta)$$

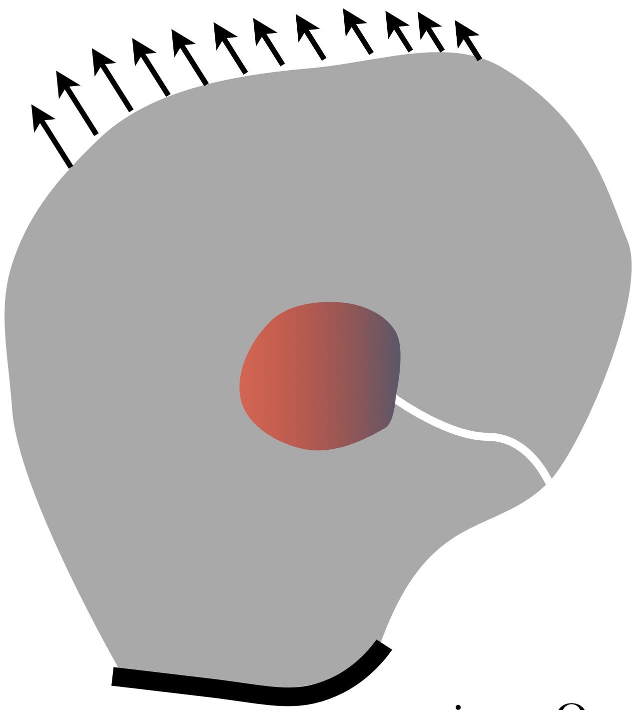
$$\beta \geq 0$$

3. Global energy balance

$$\frac{d}{dt}\mathcal{E}_t(u_t,\alpha_t) = \frac{\partial \mathcal{E}_t}{\partial t}(u_t,\alpha_t)$$

▸the evolution problem in its local form

$$\begin{cases} \operatorname{div} \sigma_t + f_t = 0 & \text{in} \quad \Omega \\ \sigma_t n = F_t & \text{on} \quad \partial_F \Omega \\ u_t = U_t & \text{on} \quad \partial_D \Omega \end{cases}$$



Stress-strain relation : $\sigma_t = \mathsf{E}(\alpha_t)(\varepsilon_t - \varepsilon_t^{th})$ in Ω

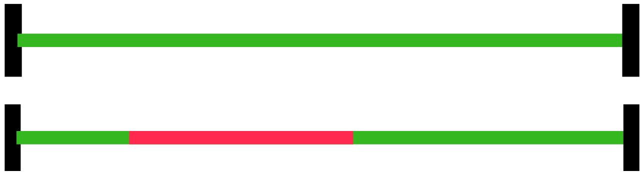
Irreversibility : $\dot{\alpha}_t \geq 0$ in Ω

Damage condition : $\frac{1}{2}\mathsf{S}'(\alpha_t)\sigma_t \cdot \sigma_t - w'(\alpha_t) + \boxed{w_1 \ell^2 \Delta \alpha_t} \leq 0$ in Ω

Consistency condition : $\left(\frac{1}{2}\mathsf{S}'(\alpha_t)\sigma_t \cdot \sigma_t - w'(\alpha_t) + \boxed{w_1 \ell^2 \Delta \alpha_t}\right) \dot{\alpha}_t = 0$ in Ω

Boundary condition : $\boxed{\frac{\partial \alpha_t}{\partial n} \geq 0}$, $\boxed{\frac{\partial \alpha_t}{\partial n} \dot{\alpha}_t = 0}$ on $\partial \Omega$

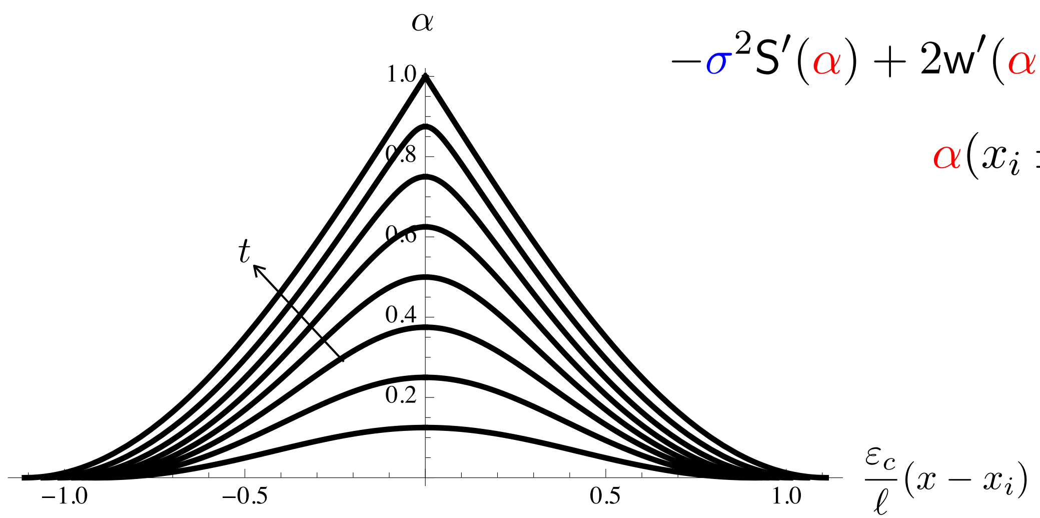
Construction of a solution with damage localization in 1D



At $t = 0$, $\sigma = \sigma_c$, $\alpha(x) = 0$

At $t > 0$
1. σ decreases from σ_c to 0,
2. damage localization in $(x_i - D, x_i + D)$

- damage localization



$$-\sigma^2 \mathsf{S}'(\alpha) + 2\mathsf{w}'(\alpha) - 2\mathsf{w}_1 \ell^2 \alpha'' = 0 \quad \text{in} \quad (x_i - D, x_i + D)$$

$$\alpha(x_i \pm D) = \alpha'(x_i \pm D) = 0$$

first integral

$$\mathsf{w}_1 \ell^2 \alpha'^2 = 2\mathsf{w}(\alpha) - \sigma^2(\mathsf{S}(\alpha) - \mathsf{S}(0))$$

- until rupture

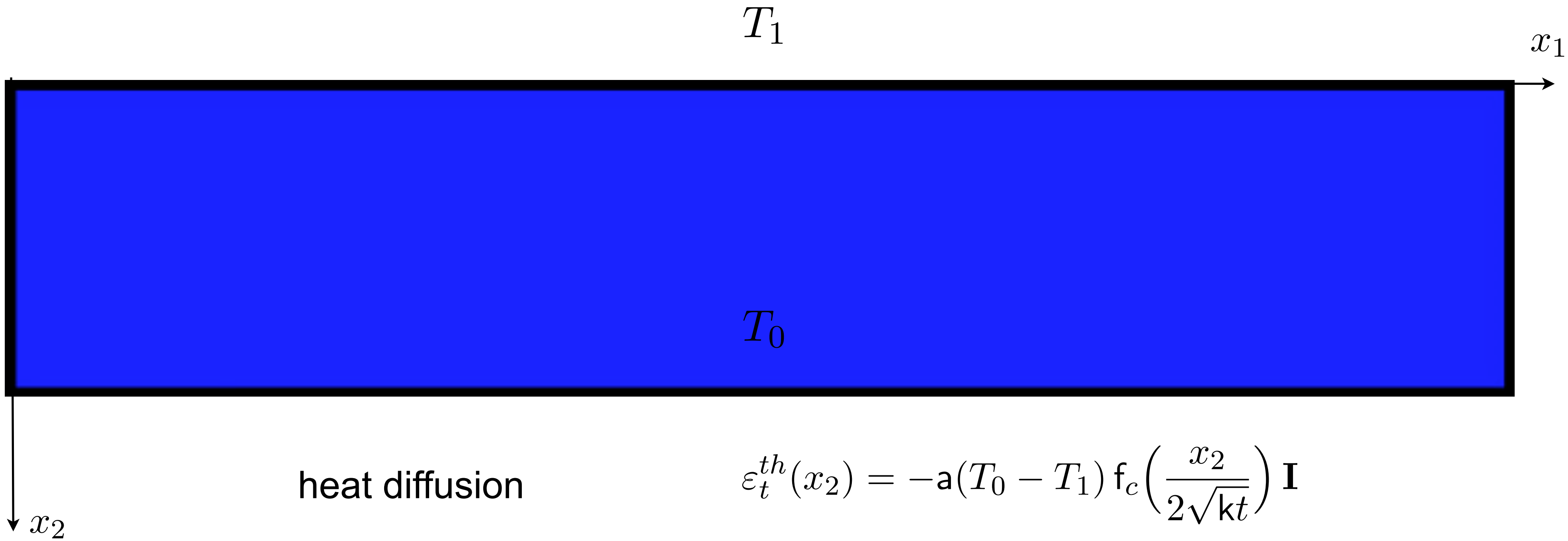
$$\alpha(x_i) = 1$$

damage profile
$$|x - x_i| = \ell \int_\alpha^1 \sqrt{\frac{\mathsf{w}_1}{2\mathsf{w}(\beta)}} d\beta, \qquad D = \ell \int_0^1 \sqrt{\frac{\mathsf{w}_1}{2\mathsf{w}(\alpha)}} d\alpha$$

dissipated energy
$$G_c = 2\ell \int_0^1 \sqrt{2\mathsf{w}_1 \mathsf{w}(\alpha)} d\alpha$$

Le problème modèle

✓ Problem of a thermal shock



heat diffusion $\quad \varepsilon_t^{th}(x_2) = -\mathsf{a}(T_0 - T_1)\, \mathsf{f}_c\left(\dfrac{x_2}{2\sqrt{\mathsf{k}t}}\right) \mathbf{I}$

✓ model

$\mathsf{E}(\alpha) = (1-\alpha)^2 \mathsf{E}_0 \qquad \mathsf{w}(\alpha) = \mathsf{w}_1 \alpha \qquad \ell$

$$\sigma_c = \sqrt{\mathsf{w}_1 \mathsf{E}_0}$$

✓ numerical method

- time discretization
- alternate minimization algorithm:

$$u_i^n = \mathrm{argmin}_u \mathcal{E}_i(u, \alpha_i^n)$$

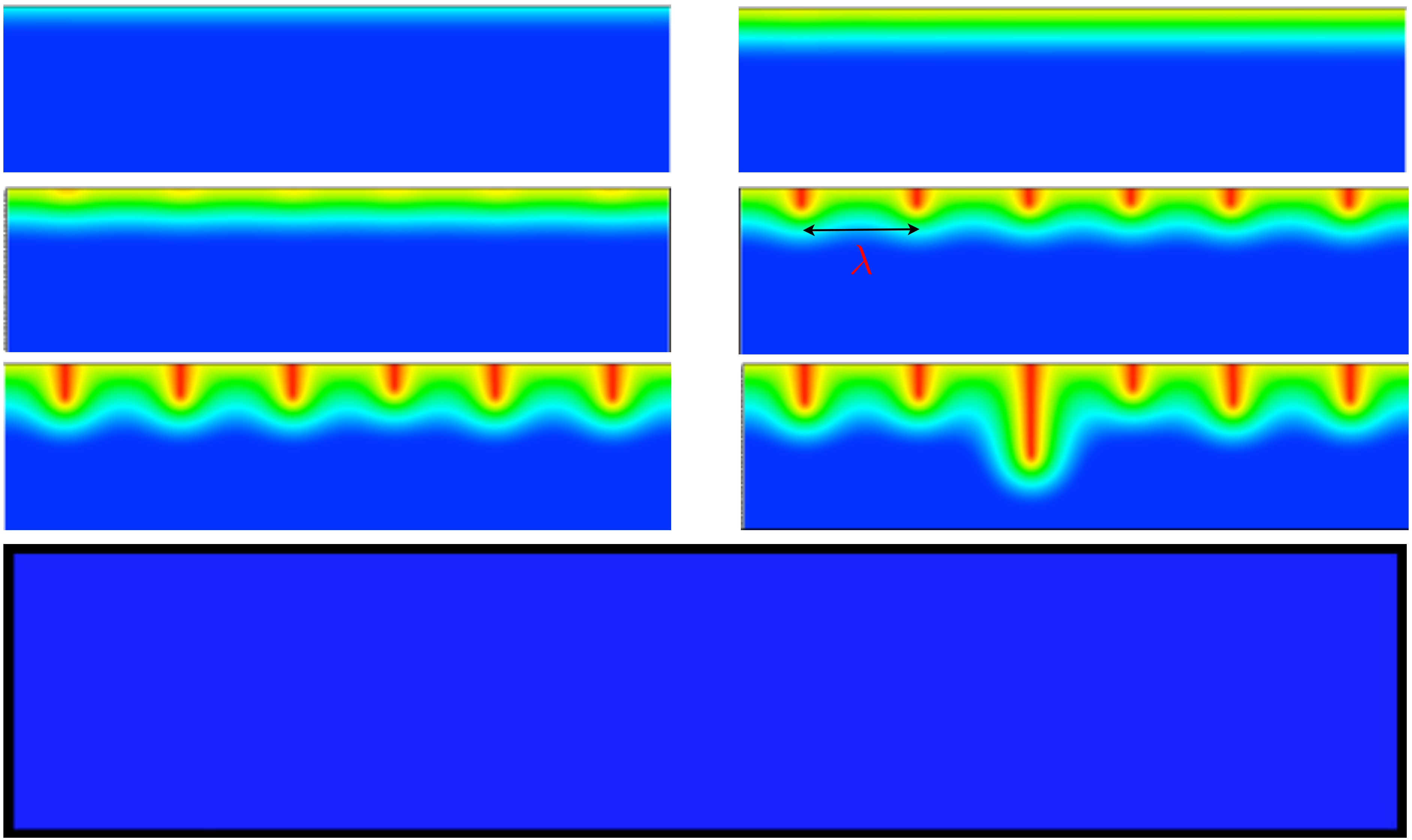
$$\alpha_i^{n+1} = \mathrm{argmin}_{\alpha \geq \alpha_{i-1}} \mathcal{E}_i(u_i^n, \alpha)$$

Case $\boxed{T_0 - T_1 \leq \frac{\sigma_c}{\mathsf{a}\mathsf{E}_0}}$: no damage, no crack

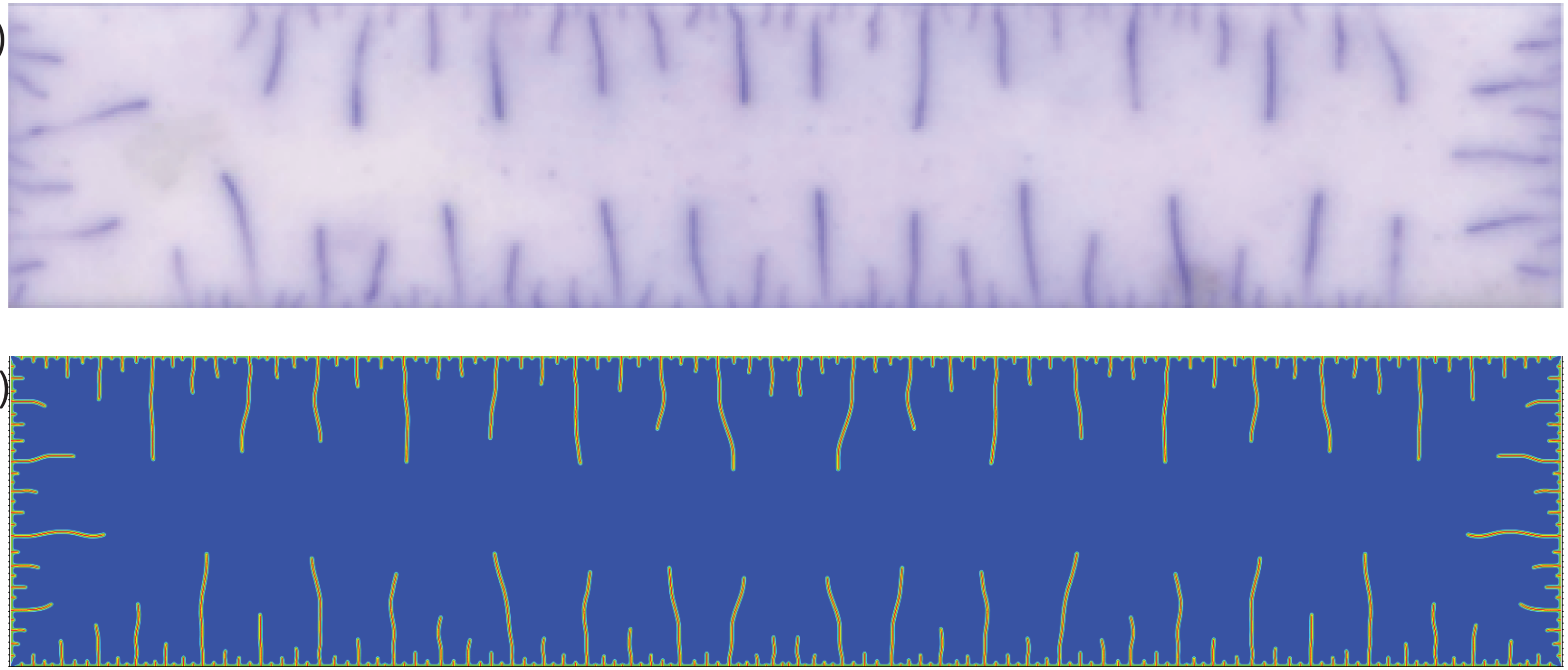
$$\boxed{\sigma_c = \sqrt{\mathsf{w}_1 \mathsf{E}_0}}$$

Case $\boxed{T_0 - T_1 > \frac{\sigma_c}{\mathsf{a}\mathsf{E}_0}}$

$$\boxed{\lambda \sim \frac{\sigma_c}{\mathsf{E}_0 \mathsf{a} (T_0 - T_1)} \ell}$$



a)
b)

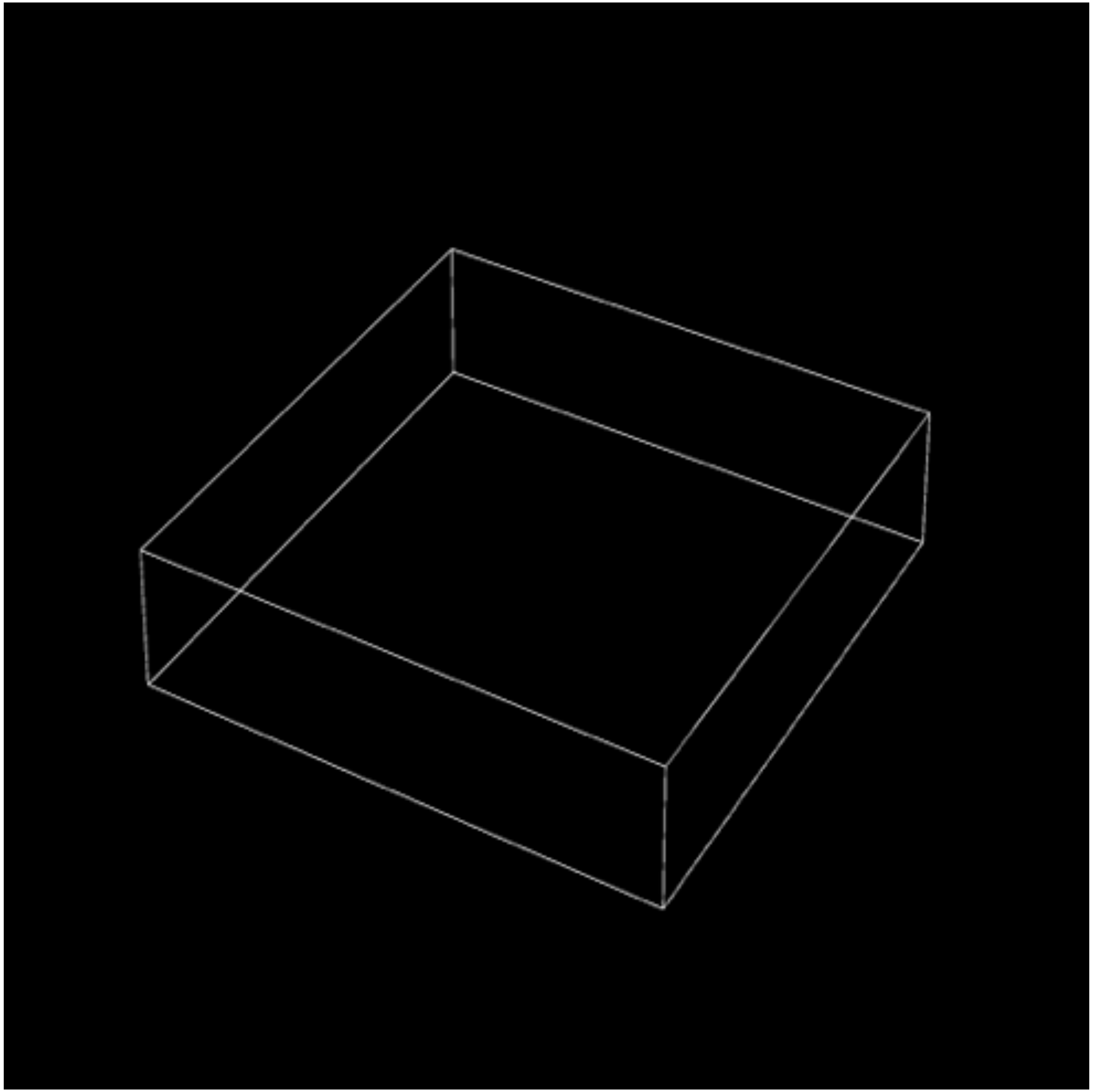


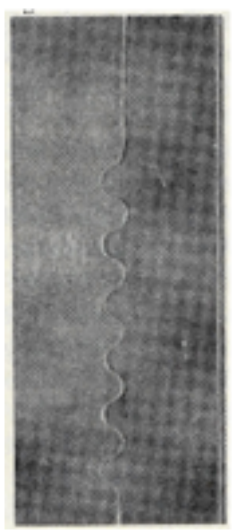
Ceramic parameters: E_0=340 GPa, G_c =42 J.m^{-2}, σ_c =340 MPa, ν = .22
(from G_c and σ_c one deduces ℓ = .05 mm)
Temperature gradient **T_0-T_1** = 380°.

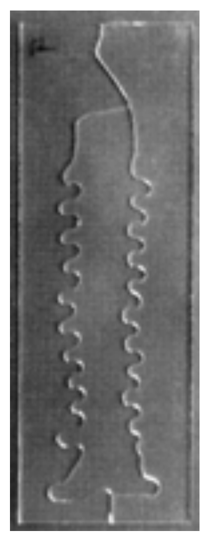
(a) Experimental crack pattern in a slab (10 mm × 50 mm × 1mm) after a thermal shock (from Jiang et al. [2012]).

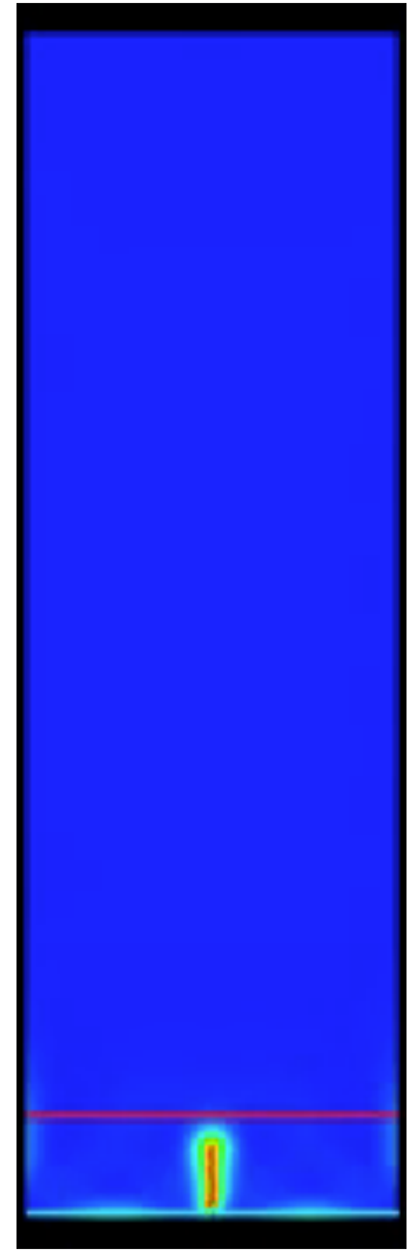
(b) Value of the computed damage field.
Numerical simulation: 20 × 10^6 d.o.f., mesh size h = .01mm

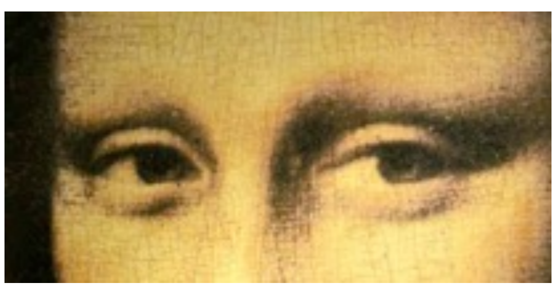
La question du trajet des fissures

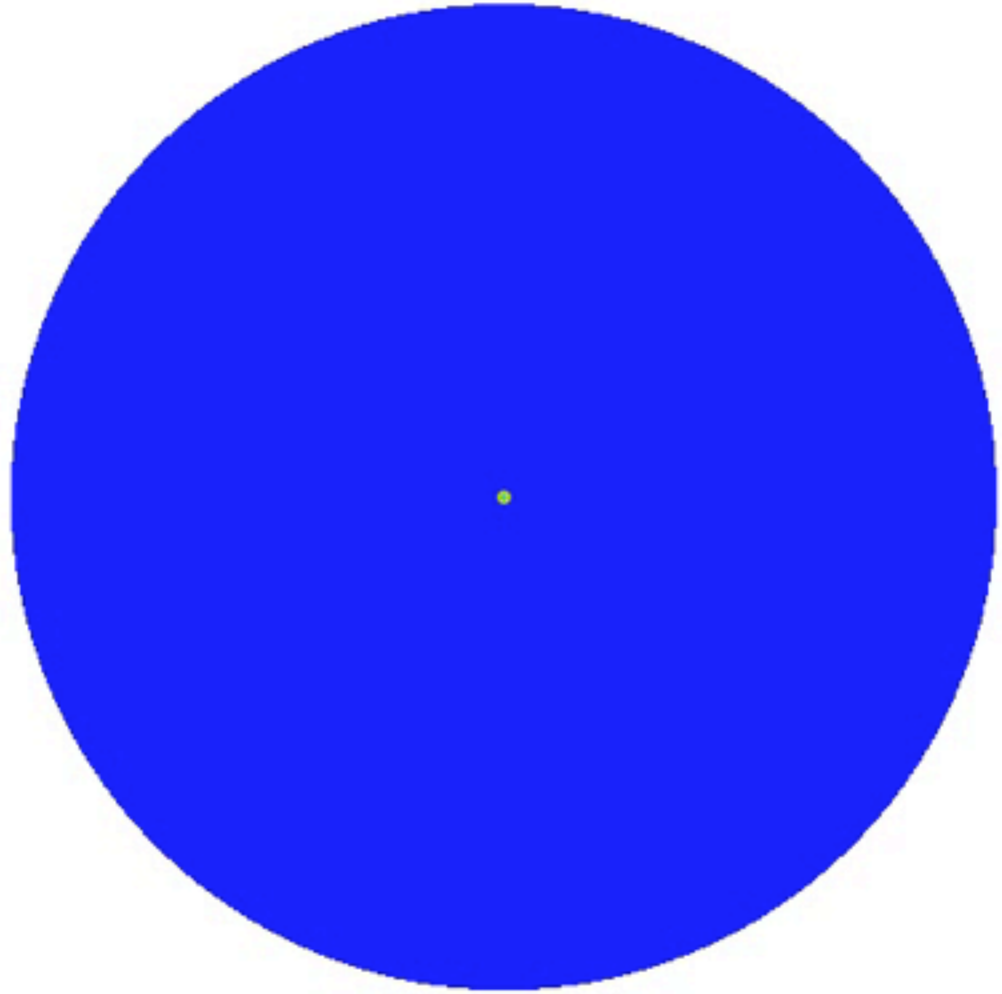


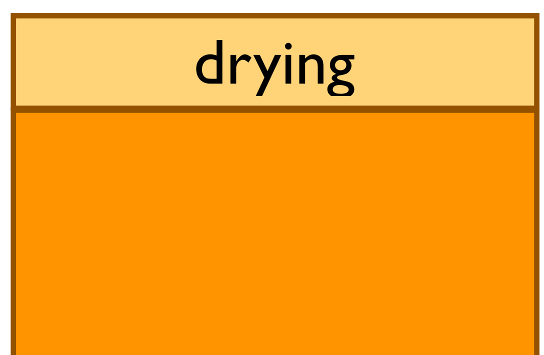


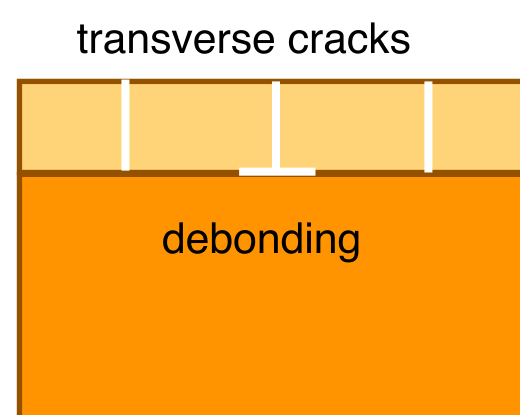




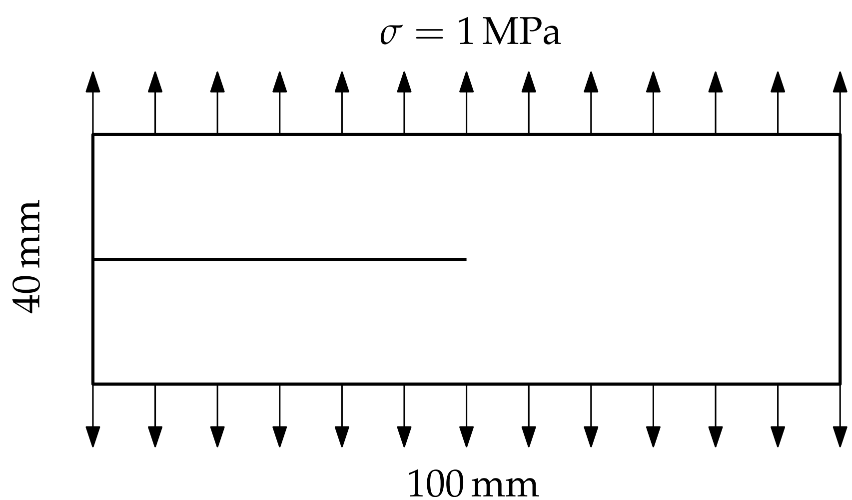





drying

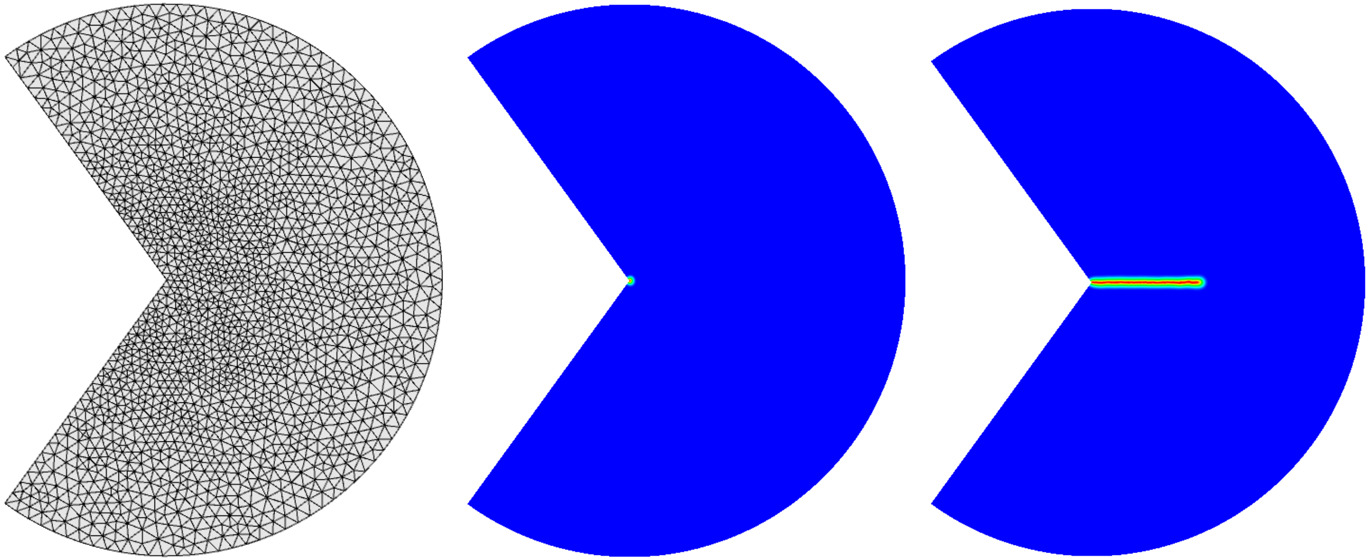

transverse cracks
debonding

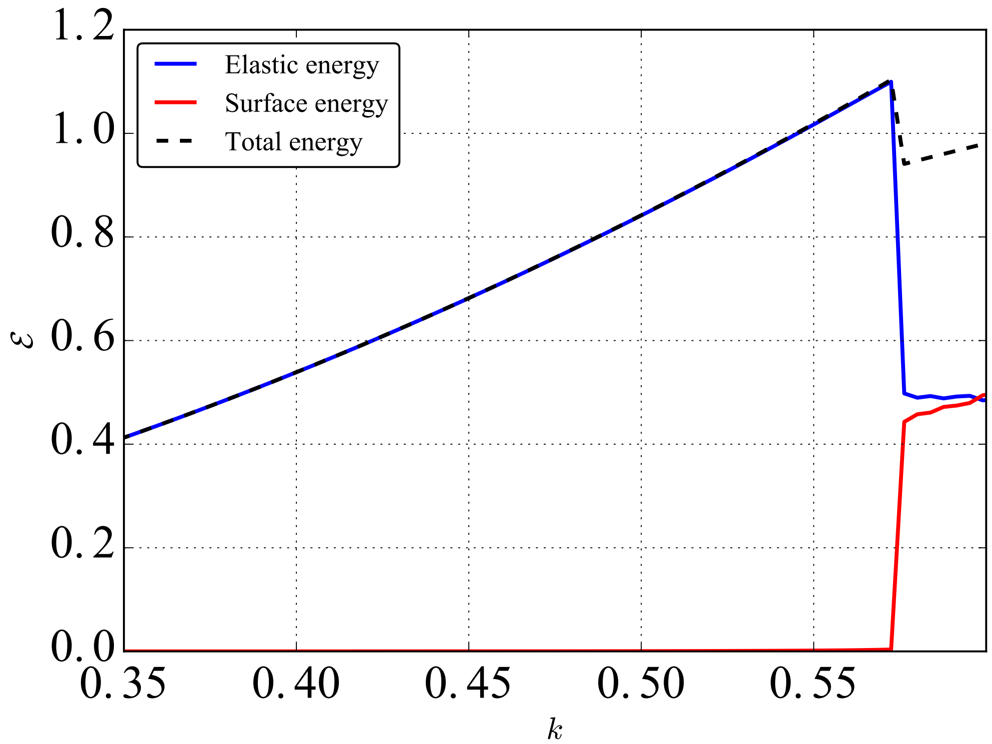

p
6
4
2
0
-2


$\sigma = 1\,\mathrm{MPa}$
40 mm
100 mm

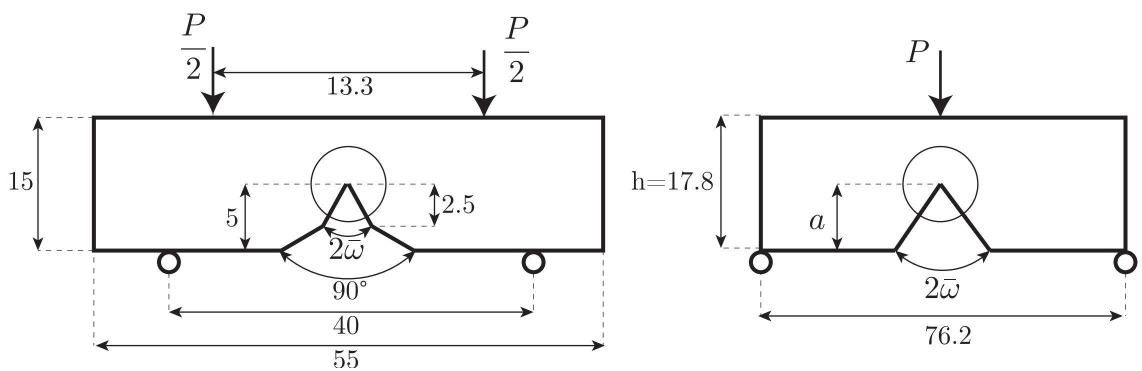
La question de la nucléation des fissures

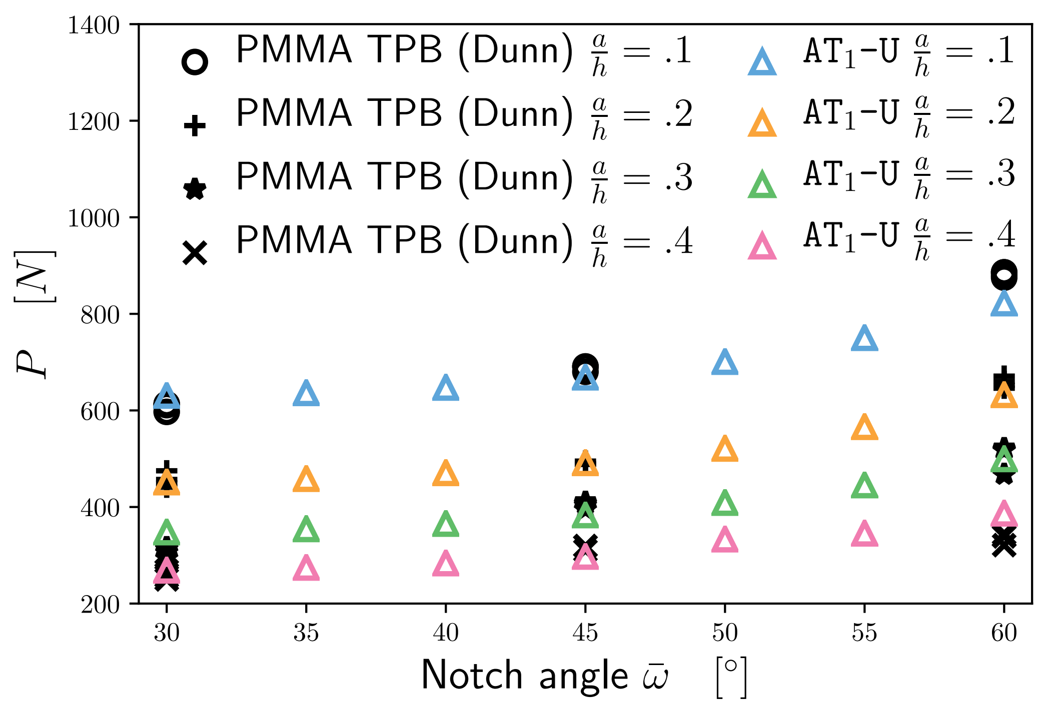
Nucléation en fond d'entaille





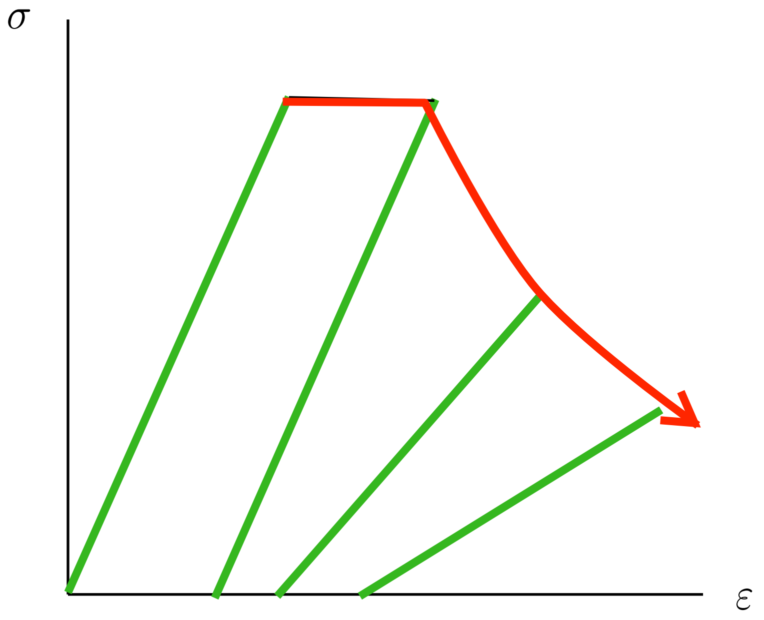
Material	E [MPa]	ν	K_{Ic} [MPa$\sqrt{\text{m}}$]	σ_c [MPa]	source
$Al_2O_3 - 7\%ZrO_2$	350,000	0.24	4.1	290	[93]
PMMA	2,300	0.36	1.03	124	[39, 93]
Plexiglass	3,000	0.36	1.86	104.9	[86]
PVC H80	85	0.32	0.32	2.51	[48, 51]
PVC H100	125	0.32	0.26	4.02	[48, 51]
PVC H130	175	0.32	0.34	5.70	[48, 51]
PVC H200	310	0.32	0.57	9.38	[48, 51]
Steel	205,000	0.3	52	1170	[48, 89]
Duraluminium	70,000	0.3	50.6	705	[86]





Endommagement avec plasticité

rupture ductile



perte de rigidité et déformations résiduelles

Damage alone

$$\mathsf{W}_D = \frac{1}{2}\,\mathsf{E}(\alpha)\varepsilon\cdot\varepsilon + \mathsf{w}(\alpha) + \mathsf{w}_1\ell^2\nabla\alpha\cdot\nabla\alpha$$

Plasticity alone

$$\mathsf{W}_P = \frac{1}{2}\,\mathsf{E}(\varepsilon-\varepsilon^p)\cdot(\varepsilon-\varepsilon^p) + \sigma_Y\ p$$

$$\dot{p} = \sqrt{\frac{2}{3}\dot{\varepsilon}^p\cdot\dot{\varepsilon}^p}$$

Damage with Plasticity

$$\mathsf{W} = \frac{1}{2}\,\mathsf{E}(\alpha)(\varepsilon-\varepsilon^p)\cdot(\varepsilon-\varepsilon^p) + \mathsf{w}(\alpha) + \boxed{\sigma_Y(\alpha)p} + \mathsf{w}_1\ell^2\nabla\alpha\cdot\nabla\alpha$$

$\sigma_Y(\alpha)$ decreasing from σ_Y^0 to 0

Evolution law (variational approach)

- ✓ Stress-strain relation $\sigma = \mathsf{E}(\alpha)(\varepsilon - \varepsilon^p)$

- ✓ Plasticity criterion

$$\sqrt{\frac{3}{2}\sigma^D \cdot \sigma^D} \leq \sigma_Y(\alpha)$$

$$\text{Flow rule : } \dot{\varepsilon}^p = \dot{p}\,\frac{\sigma^D}{\sigma_Y(\alpha)}$$

- ✓ Damage criterion

$$\tfrac{1}{2}\,\mathsf{S}'(\alpha)\sigma \cdot \sigma + 2\mathsf{w}_1\ell^2\Delta\alpha \leq \mathsf{w}'(\alpha) + \sigma'_Y(\alpha)p$$

2 critical stress

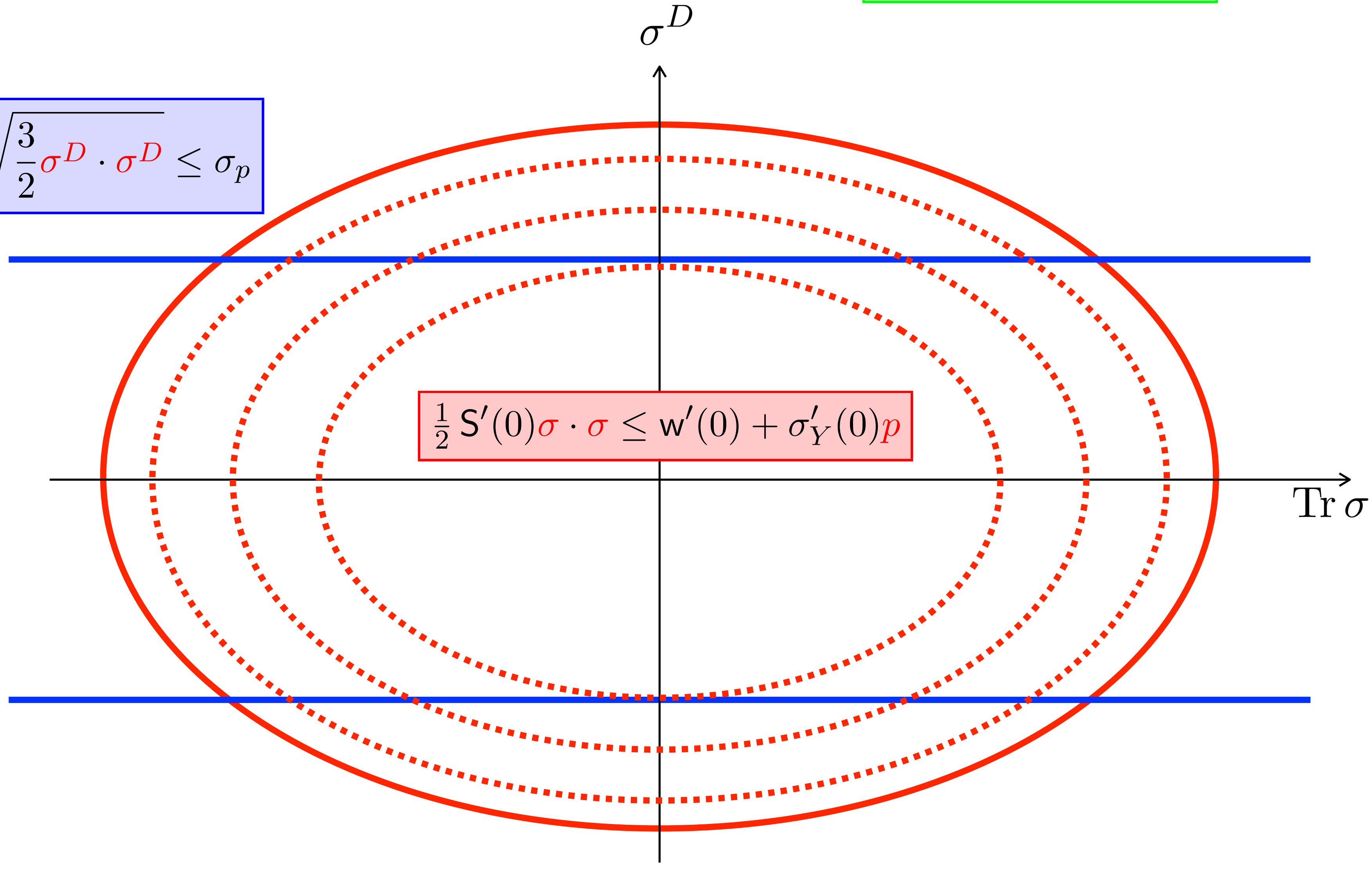
$$\sigma_p := \sigma_Y(0)$$

$$\sigma_c := \sqrt{\frac{2\mathsf{w}'(0)}{\mathsf{S}'(0)}}$$

✓ Onset of damage

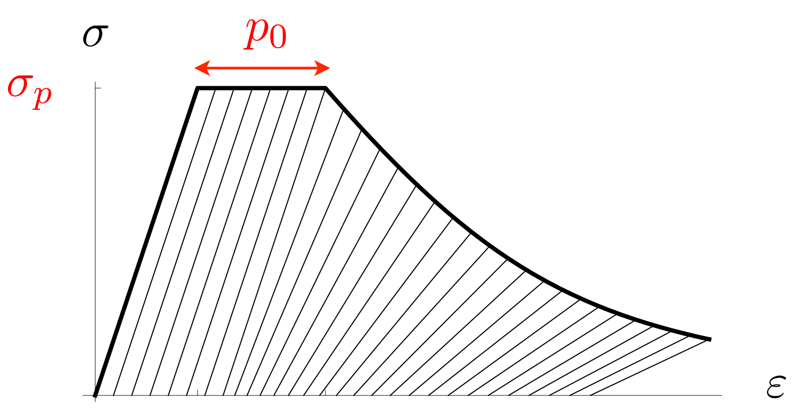
$$\sigma_p < \sigma_c := \sqrt{\frac{2\mathsf{w}'(0)}{\mathsf{S}'(0)}}$$

$$\sqrt{\frac{3}{2}\sigma^D \cdot \sigma^D} \leq \sigma_p$$

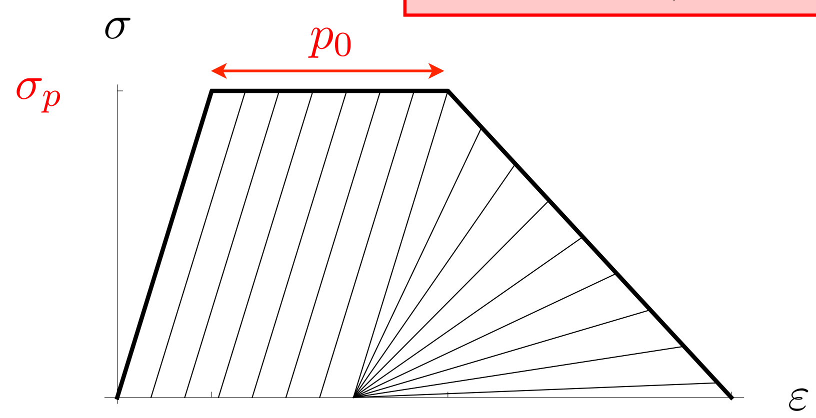


Uniaxial local response

✓ Case where plasticity occurs before damage $\sigma_p < \sigma_c := \sqrt{\frac{2\mathsf{w}'(0)}{\mathsf{S}'(0)}}$



E–P–DP



E–P–D–R

Evolution of the damage criterion during the P stage

$$\frac{1}{2}\,\mathsf{S}'(0){\sigma_Y^0}^2 \leq \mathsf{w}'(0) - \left|\sigma_Y'(0)\right| p$$

Onset of damage : $p_0 = \dfrac{\mathsf{S}'(0)}{2\left|\sigma_Y'(0)\right|}(\sigma_c^2 - {\sigma_p}^2)$

Then damage alone or damage with plasticity according to $\mathsf{w}(\alpha)$, $\mathsf{S}(\alpha)$, $\sigma_Y(\alpha)$ properties

E_0 ν σ_p

Example

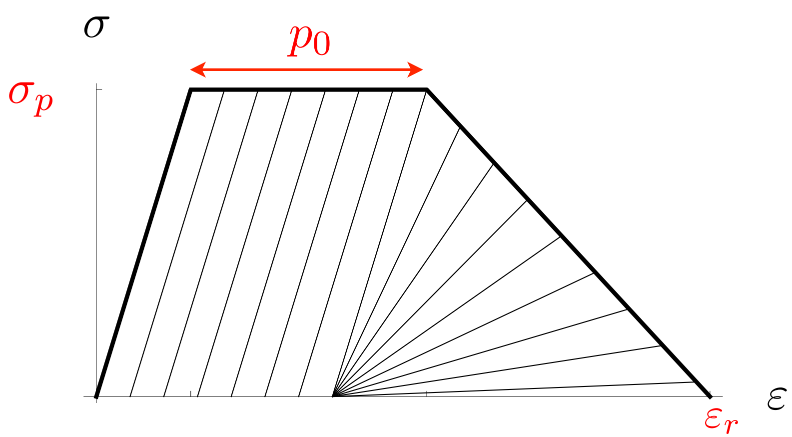
$\mathsf{k} > 1$ $\theta = \dfrac{\sigma_p}{\sigma_c} < 1$

$$\mathsf{E}(\alpha) = \frac{(1-\alpha)^2}{\mathsf{k} - (\mathsf{k}-1)(1-\alpha)^2}\mathsf{E}_0$$

$$\mathsf{w}(\alpha) = \frac{\mathsf{k}\sigma_c^2}{2\mathsf{E}_0}(1-(1-\alpha)^2)$$

$$\sigma_Y(\alpha) = (1-\alpha)^2\sigma_p$$

Homogeneous response



$$p_0 = \frac{\mathsf{k}(1-\theta^2)\sigma_p}{2\theta\mathsf{E}_0}$$

$$\varepsilon_r = \frac{\mathsf{k}(1+\theta^2)\sigma_p}{2\theta\mathsf{E}_0}$$

$\mathsf{E}_0 \qquad \nu \qquad \sigma_p$

Example

$\mathsf{k} > 1 \qquad \theta = \dfrac{\sigma_p}{\sigma_c} < 1$

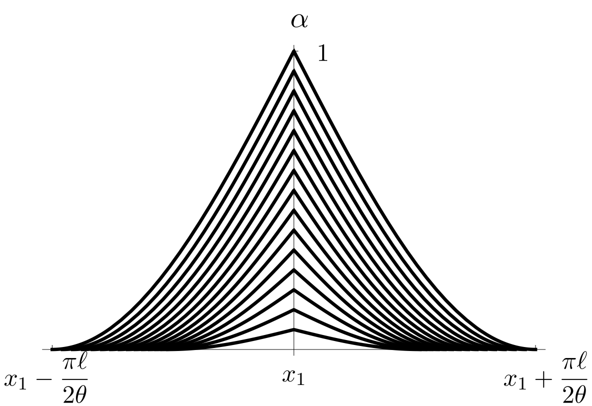
$$\mathsf{E}(\alpha) = \frac{(1-\alpha)^2}{\mathsf{k} - (\mathsf{k}-1)(1-\alpha)^2}\mathsf{E}_0$$

$$\mathsf{w}(\alpha) = \frac{\mathsf{k}\sigma_c^2}{2\mathsf{E}_0}(1-(1-\alpha)^2)$$

$$\sigma_Y(\alpha) = (1-\alpha)^2\sigma_p$$

Response with damage localization

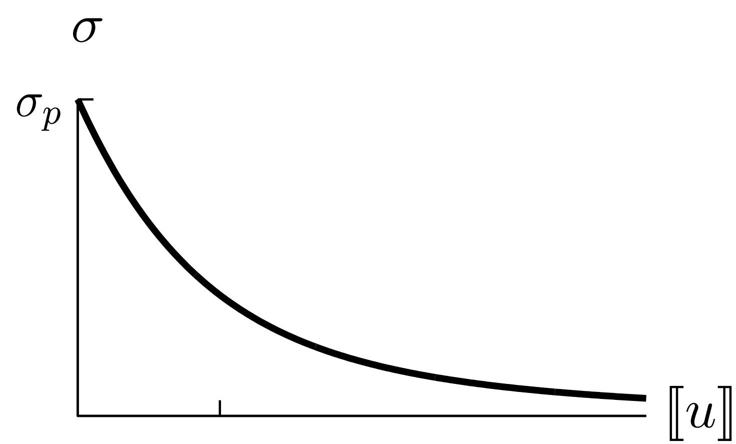
Nucleation of a cohesive crack at the center of the damage zone



damage profile

$$G_c = \frac{\pi\mathsf{k}\theta}{2}\frac{\sigma_c^2\ell}{\mathsf{E}_0}$$

dissipated energy to create a crack



$$\frac{[\![u]\!]}{\ell} = \frac{\mathsf{k}\sigma_c}{\mathsf{E}_0}\left(\sqrt{\frac{\sigma_p}{\sigma}} - \sqrt{\frac{\sigma}{\sigma_p}}\right)$$

cohesive law

✓ numerical method

– time discretization

$$\mathcal{E}_i(u,\alpha,\varepsilon^p) = \int_\Omega \Big(\tfrac{1}{2}\,\mathsf{E}(\alpha)(\varepsilon(u)-\varepsilon^p)\cdot(\varepsilon(u)-\varepsilon^p + \mathsf{w}(\alpha) + \mathsf{w}_1\ell^2\nabla\alpha\cdot\nabla\alpha\Big)dx$$
$$+ \int_\Omega \sigma_Y(\alpha)\Big(p_{i-1} + \|\varepsilon^p - \varepsilon^p_{i-1}\|\Big)dx - f_i(u)$$

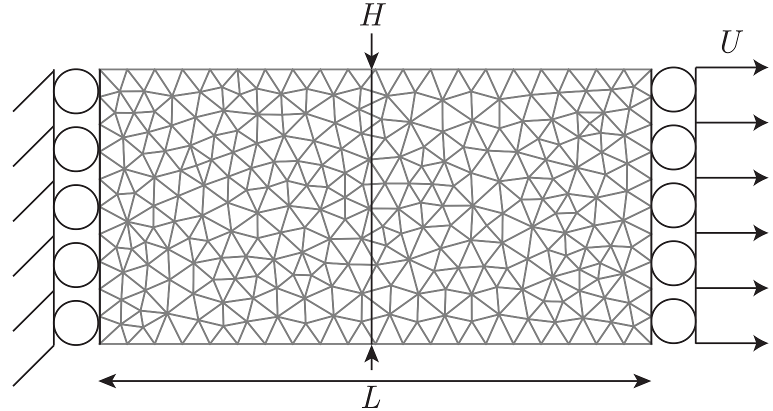
– alternate minimization algorithm:

$$u_i^n = \mathrm{argmin}_u \mathcal{E}_i(u, \alpha_i^n, (\varepsilon^p)_i^n)$$

$$\alpha_i^{n+1} = \mathrm{argmin}_{\alpha\geq\alpha_{i-1}} \mathcal{E}_i(u_i^n, \alpha, (\varepsilon^p)_i^n)$$

$$(\varepsilon^p)_i^{n+1} = \mathrm{argmin}_{\varepsilon^p} \mathcal{E}_i(u_i^n, \alpha_i^{n+1}, \varepsilon^p)$$

local problem=projection



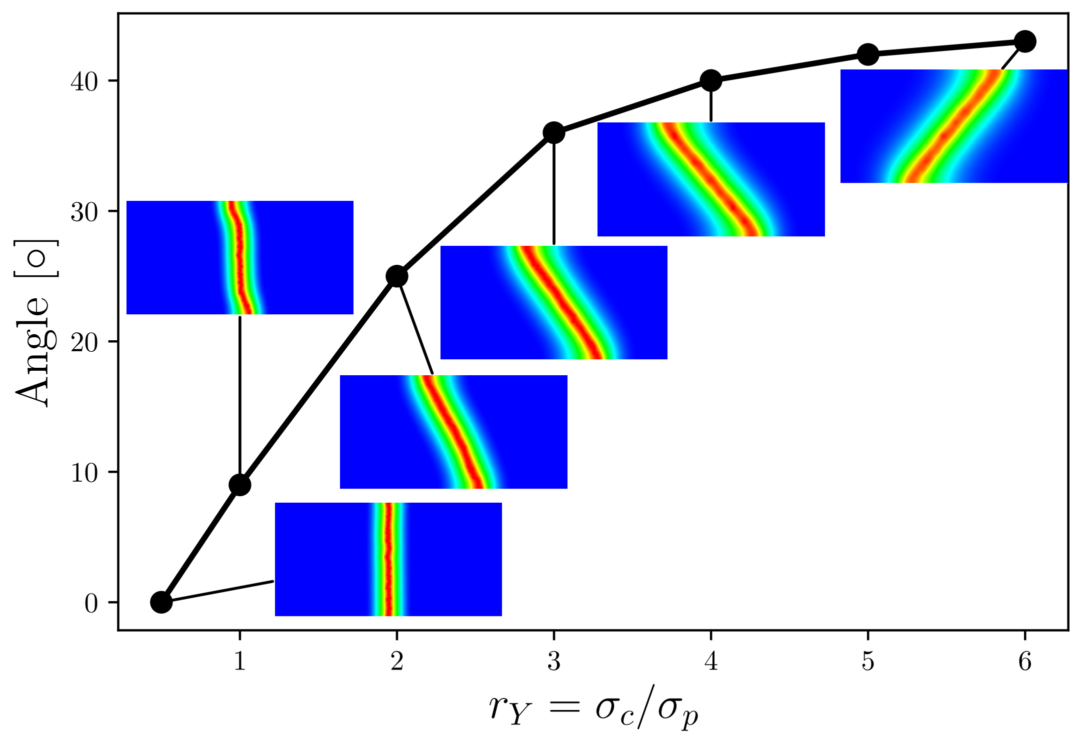
$$\mathsf{W} = \frac{1}{2}\,\mathsf{E}(\alpha)(\varepsilon - \varepsilon^p)\cdot(\varepsilon - \varepsilon^p) + \mathsf{w}(\alpha) + \sigma_Y(\alpha)p + \mathsf{w}_1\ell^2\nabla\alpha\cdot\nabla\alpha$$

$$\mathsf{E}(\alpha) = (1-\alpha)^2\mathsf{E}_0$$

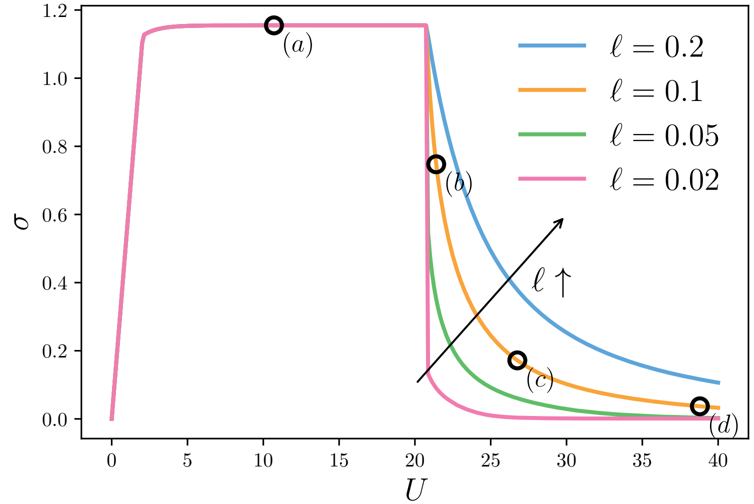
$$\mathsf{w}(\alpha) = \frac{\sigma_c^2}{\mathsf{E}_0}\alpha$$

$$\sigma_Y(\alpha) = (1-\alpha)^2\sigma_p$$

$\ell/L = 0.01$

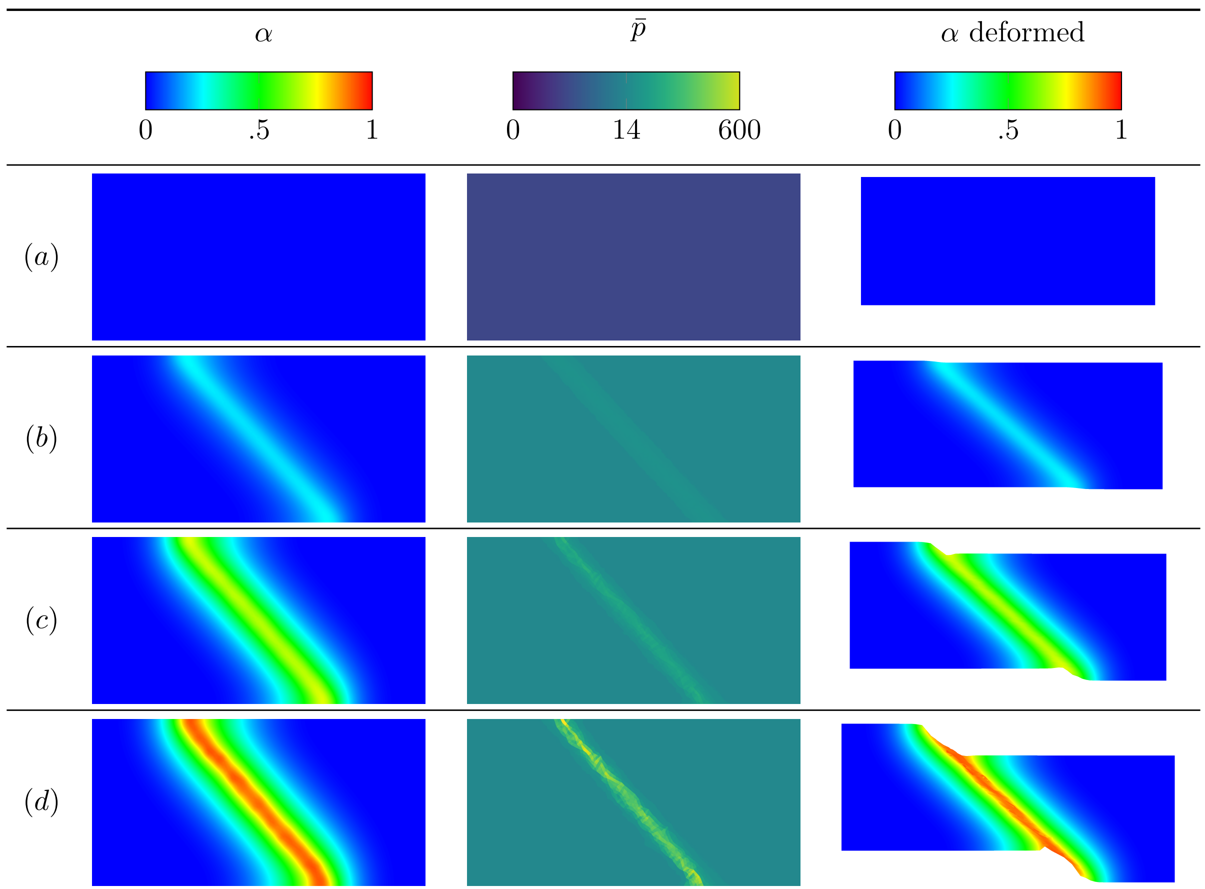


$\sigma_c = 5\sigma_p$



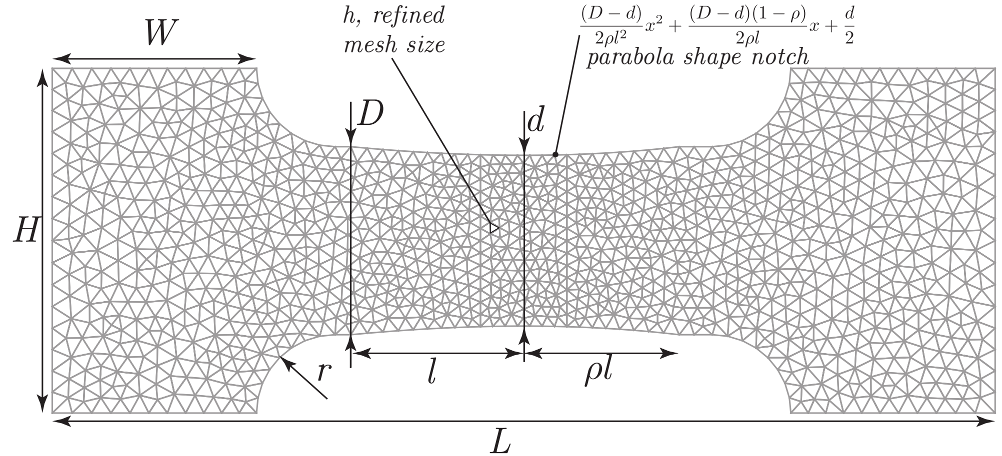
L=2

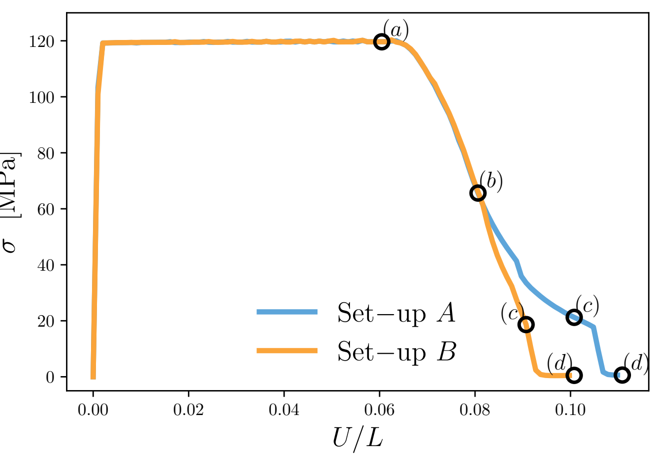
H=1

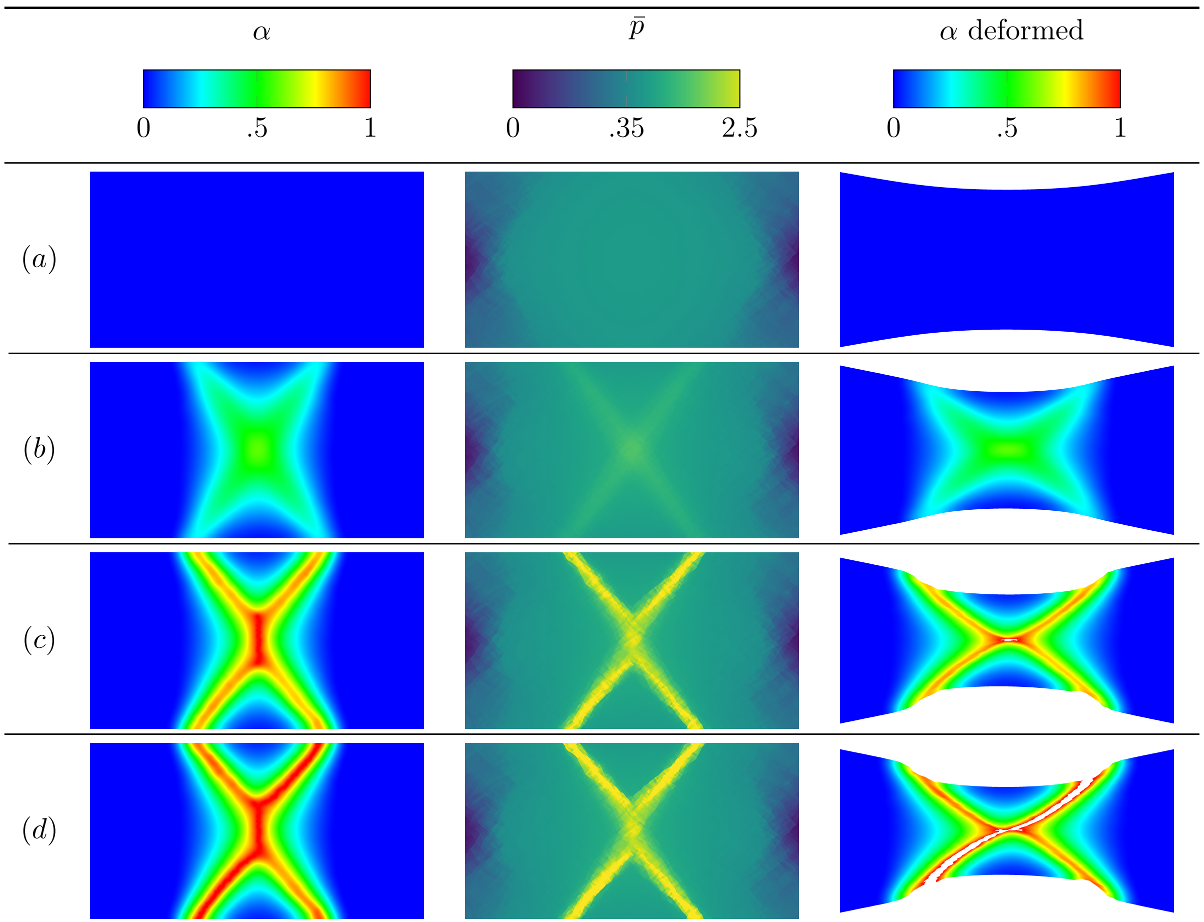


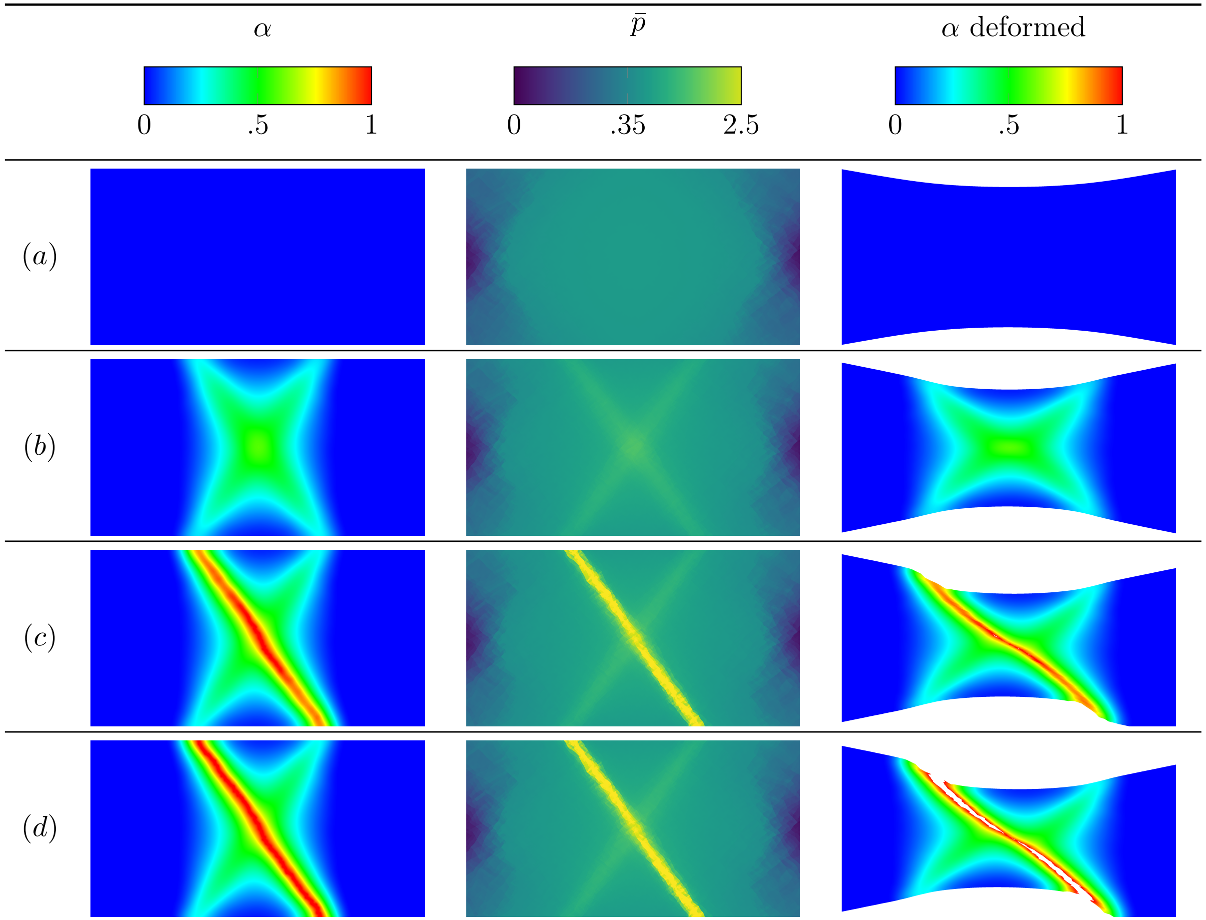
E [GPa]	ν	σ_p [MPa]	σ_c [GPa]	ℓ [µm]
70	.33	100	2	400

L	H	W	r	D	d	l	h	ρ
6	2.2	1.3	.5	1.2	1.09	1.1	$\ell/3$	1








α
$\bar{p}$
α deformed
0 .5 1
0 .35 2.5
0 .5 1
(a)
(b)
(c)
(d)