Multi-material shape optimization via a level set method

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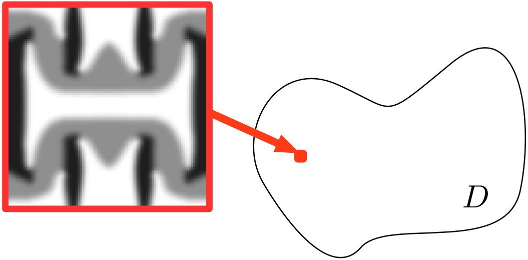
A foreword around multi-phase optimization

Multi-phase optimization is about finding the optimal repartition of two, or several, materials with conflicting properties within a fixed set. This problem has multiple applications in industrial design:

- *At the macroscopic level:* Repartition of several materials within a given structure to combine their respective assets.
- *At the microscopic level:* Mixture of several phases to achieve new materials with unique features (e.g. design of materials with negative Poisson's ratio, or negative coefficient of thermal expansion...).



Design of a material with negative CTE [Mi].

Preliminaries: the usual linear elasticity setting (I)

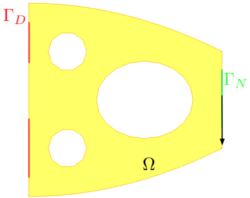
A shape is a bounded domain $\Omega \subset \mathbb{R}^d$, which is

- fixed on a part Γ_D of its boundary,
- submitted to surface loads g, applied on $\Gamma_N \subset \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$.

The displacement vector field $u_\Omega : \Omega \to \mathbb{R}^d$ is governed by the linear elasticity system:

$$\begin{cases} -\mathrm{div}(Ae(u_\Omega)) = 0 & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \Gamma_D \\ Ae(u_\Omega)n = g & \text{on } \Gamma_N \\ Ae(u_\Omega)n = 0 & \text{on } \Gamma := \partial\Omega \setminus (\Gamma_D \cup \Gamma_N) \end{cases},$$

where $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the strain tensor, and A is the Hooke's law of the material.



A 'Cantilever'

The deformed cantilever

Preliminaries: the usual linear elasticity setting (II)

Goal: Starting from an initial structure Ω_0, find a new one Ω that minimizes a certain functional of the domain $J(\Omega)$.

Examples:

- The work of the external loads g or compliance $C(\Omega)$ of domain Ω:

$$C(\Omega) = \int_\Omega Ae(u_\Omega) : e(u_\Omega) dx = \int_{\Gamma_N} g.u_\Omega \, ds$$

- A least-square error between u_Ω and a target displacement $u_0 \in \; H^1(\Omega)^d$ (useful when designing micro-mechanisms):

$$D(\Omega) = \left(\int_\Omega k(x) |u_\Omega - u_0|^\alpha dx \right)^{\frac{1}{\alpha}},$$

  where α is a fixed parameter, and $k(x)$ is a weight factor.

A volume constraint may be enforced with a fixed penalty parameter ℓ:

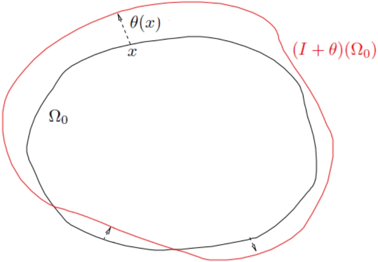
Minimize $J(\Omega) := C(\Omega) + \ell \, \mathrm{Vol}(\Omega)$, or $D(\Omega) + \ell \, \mathrm{Vol}(\Omega)$.

Differentiation with respect to the domain: Hadamard's method

Hadamard's boundary variation method describes variations of a reference, Lipschitz domain Ω of the form:

$$\Omega \to \Omega_\theta := (I+\theta)(\Omega),$$

for 'small' $\theta \in W^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right)$.



Definition 1.

Given a smooth domain Ω, a functional $F(\Omega)$ of the domain is shape differentiable at Ω if the function

$$W^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right) \ni \theta \mapsto F(\Omega_\theta)$$

is Fréchet-differentiable at 0, i.e. the following expansion holds around 0:

$$F(\Omega_\theta) = F(\Omega) + F'(\Omega)(\theta) + o\left(||\theta||_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)}\right).$$

Techniques close to optimal control theory make it possible to compute shape gradients; in the case of 'many' functionals of the domain $J(\Omega)$, the shape derivative has the particular structure:

$$J'(\Omega)(\theta) = \int_\Gamma v_\Omega \, \theta \cdot n \, ds,$$

where v_Ω is a scalar field depending on u_Ω, and possibly on an adjoint state p_Ω.

Example: If $J(\Omega) = C(\Omega) = \int_{\Gamma_N} g \cdot u_\Omega \, ds$ is the compliance, $v_\Omega = -Ae(u_\Omega) : e(u_\Omega)$.

The generic numerical algorithm

This shape gradient provides a natural descent direction for functional J: *for instance*, defining θ as

$$\theta = -v_\Omega n$$

yields, for $t > 0$ sufficiently small (*to be found numerically*):

$$J((I + t\theta)(\Omega)) = J(\Omega) - t\int_\Gamma v_\Omega^2 ds + o(t) < J(\Omega)$$

Gradient algorithm: For $n = 0, \ldots$ convergence,

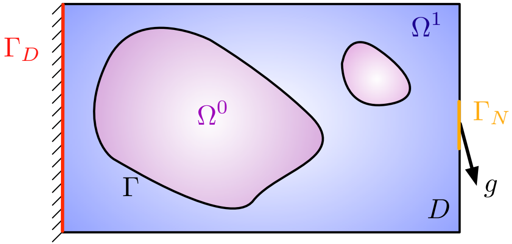
1. Compute the solution u_{Ω^n} (and p_{Ω^n}) of the elasticity system on Ω^n.
2. Compute the shape gradient $J'(\Omega^n)$ thanks to the previous formula, and infer a descent direction θ^n for the cost functional.
3. Advect the shape Ω^n according to θ^n, so as to get $\Omega^{n+1} := (I + \theta^n)(\Omega^n)$.

The multi-material shape optimization setting (I)

- A fixed working domain $D \subset \mathbb{R}^d$ is occupied by two complementary phases Ω^0 and Ω^1, filled with elastic materials with Hooke's laws A_0, A_1.
- The structure D is clamped on a region $\Gamma_D \subset \partial D$, surface loads are applied on $\Gamma_N \subset \partial D$, as well as body forces f.
- The total, discontinuous Hooke's law in D is:

$$A_{\Omega^0} := A_0 \chi_0 + A_1 \chi_1,$$

where χ_i is the characteristic function of the phase Ω^i.



The multi-material shape optimization setting (II)

- The displacement

$$u_{\Omega^0} \in H^1_{\Gamma_D}(D)^d := \left\{ u \in H^1(D)^d, \ u = 0 \text{ on } \Gamma_D \right\}$$

of the total structure D satisfies:

$$\begin{cases} -\mathrm{div}(A_{\Omega^0} e(u)) = f & \text{in } D \\ u = 0 & \text{on } \Gamma_D \\ A_1 e(u) n = g & \text{on } \Gamma_N \end{cases}.$$

- **Goal:** Minimize a functional of the mixture of the form:

$$J(\Omega^0) = \int_D j(x, u_{\Omega^0}) \, dx + \int_{\Gamma_N} k(x, u_{\Omega^0}) \, ds,$$

under constraints (e.g. on the volume of one of the phases).

- **Example:** The compliance of the total structure D:

$$C(\Omega^0) = \int_D A_{\Omega^0} e(u_{\Omega^0}) : e(u_{\Omega^0}) \, dx = \int_D f \cdot u_{\Omega^0} \, dx + \int_{\Gamma_N} g \cdot u_{\Omega^0} \, ds.$$

The multi-material shape optimization setting (III)

- The material properties are different from either side of $\Gamma \Rightarrow$ some quantities are discontinuous across Γ.
- If α is a discontinuous quantity, with values α^0, α^1 in $\overline{\Omega^0}$, $\overline{\Omega^1}$ respectively, $[\alpha] := \alpha^1 - \alpha^0$ is the jump of α across Γ.
- If $\mathcal{M}$ is a tensor-valued function, denote as:

$$\forall x \in \Gamma, \ \mathcal{M} = \begin{pmatrix} \mathcal{M}_{\tau\tau}(x) & \mathcal{M}_{\tau n}(x) \\ \mathcal{M}_{n\tau}(x) & \mathcal{M}_{nn}(x) \end{pmatrix}$$

  its representation in a local basis (τ, n) of $\mathbb{R}^d$.
- *Difficulty:* The strain tensor $e \equiv e(u_{\Omega^0})$ has continuous components $e_{\tau\tau}$, but discontinuous components $e_{\tau n}$, $e_{n\tau}$, e_{nn}. The stress tensor $\sigma \equiv \sigma(u_{\Omega^0})$ has continuous components $\sigma_{n\tau}$, $\sigma_{\tau n}$ and σ_{nn}, but $\sigma_{\tau\tau}$ is discontinuous.

Theorem 1 ([AlJouVG]).

The functional $J(\Omega^0)$ is shape differentiable, and its derivative reads:

$$\forall \theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \;\; J'(\Omega^0)(\theta) = -\int_\Gamma \mathcal{D}(x, u_{\Omega^0}, p_{\Omega^0})\, \theta \cdot n \, ds,$$

where the integrand factor $\mathcal{D}(x,u,p)$ is defined as:

$$\mathcal{D}(x,u,p) = -\sigma(p)_{nn} : [e(u)_{nn}] - 2\sigma(u)_{n\tau} : [e(p)_{n\tau}] + [\sigma(u)_{\tau\tau}] : e(p)_{\tau\tau},$$

and $p_{\Omega^0} \in H^1_{\Gamma_D}(D)^d$ is an adjoint state, defined as the solution to:

$$\begin{cases} -\mathrm{div}\,(A_{\Omega^0}\, e(p)) & = & -j'(x, u_{\Omega^0}) & \text{in } D, \\ p & = & 0 & \text{on } \Gamma_D, \\ (A_1\, e(p))\, n & = & -k'(x, u_{\Omega^0}) & \text{on } \Gamma_N, \end{cases}$$

Shape derivative in the sharp-interface context (II)

This formula is difficult to use in numerical practice, since it involves the jumps of discontinuous quantities across Γ.

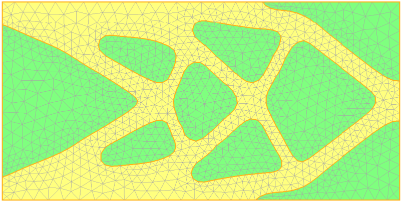
Potential remedies:

- *Discrete approach:* [AlDaDelMi]

  Consider the shape derivative of the discretization $J_h(\Omega^0)$ of $J(\Omega^0)$ on the actual mesh, which features the numerical solution u_h (resp. p_h) of the state (resp. adjoint) elasticity system.

- *Body-fitted approach:* [AlDaFr]

  The interface Γ is explicitely discretized at each step of the process.



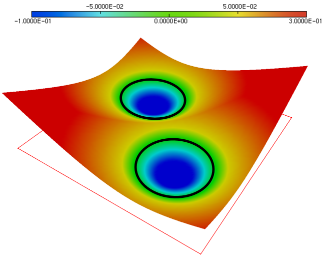
The signed distance function

Definition 2.

The *signed distance function* d_Ω *to a bounded domain* $\Omega \subset \mathbb{R}^d$ *is defined as:*

$$\forall x \in \mathbb{R}^d, \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \partial\Omega \\ d(x, \partial\Omega) & \text{if } x \in {}^c\overline{\Omega} \end{cases},$$

where $d(\cdot, \partial\Omega)$ *stands for the usual Euclidean distance function to* $\partial\Omega$.



Graph of the signed distance function to a union of two disks (in black)

The smoothed-interface setting (I)

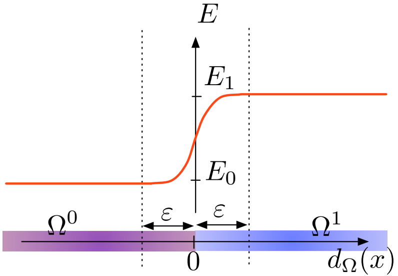
- The discontinuous tensor A_{Ω^0} is approximated by:

$$\forall x \in D, \;\; A_{\Omega^0,\varepsilon}(x) := A_0 + h_\varepsilon(d_{\Omega^0}(x))(A_1 - A_0),$$

where h_ε is a smooth approximation of the Heaviside function:

$$h_\varepsilon(t) = \begin{cases} 0 & \text{if } t < -\varepsilon \\ \frac{1}{2}\left(1 + \frac{t}{\varepsilon} + \frac{1}{\pi}\sin\left(\frac{\pi t}{\varepsilon}\right)\right) & \text{if } -\varepsilon \le t \le \varepsilon \\ 1 & \text{if } t > \varepsilon \end{cases}.$$

- This accounts for a smooth interpolation of the material properties between the two phases over a tubular neighborhood of Γ of fixed width 2ε.



The smoothed-interface setting (II)

- The smoothed-interface problem is then that of minimizing:

$$J_\varepsilon(\Omega^0) = \int_D j(x, u_{\Omega^0,\varepsilon})\, dx + \int_{\Gamma_N} k(x, u_{\Omega^0,\varepsilon})\, ds$$

(under constraints), where $u_{\Omega^0,\varepsilon}$ arises as the solution to:

$$\begin{cases} -\mathrm{div}(A_{\Omega^0,\varepsilon} e(u)) = f & \text{in } D \\ u = 0 & \text{on } \Gamma_D \\ A_1 e(u) n = g & \text{on } \Gamma_N \end{cases}.$$

- It is worth considering for at least two reasons:
  - It is an approximation of the sharp-interface problem, and is easier to handle numerically.
  - It has some interest on its own, especially when it comes to modelling interfaces: interfaces may involve complex and ill-understood processes, which are better described e.g. by non monotone transition regions.

Signed distance function and geometry (I)

Definition 3.

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz, bounded open set;

- *Let $x \in \mathbb{R}^d$; the set of projections $\Pi_{\partial\Omega}(x)$ of x onto $\partial\Omega$ is:*

$$\Pi_{\partial\Omega}(x) = \{y \in \partial\Omega, \ d(x, \partial\Omega) = |x - y|\}.$$

- *When this set is a singleton, $p_{\partial\Omega}(x)$ is the projection of x onto $\partial\Omega$.*

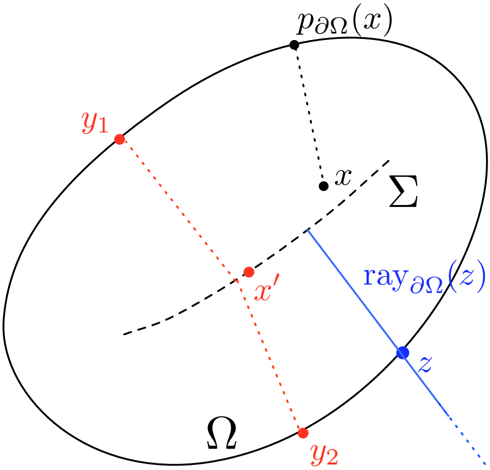
- *The skeleton Σ of $\partial\Omega$ is:*

$$\Sigma := \left\{x \in \mathbb{R}^d, \ d_\Omega^2 \text{ is not differentiable at } x\right\}.$$

- *For $x \in \partial\Omega$, the ray emerging from x is:*

$$ray_{\partial\Omega}(x) := p_{\partial\Omega}^{-1}(x).$$

Signed distance function and geometry (II)



x has a unique projection over $\partial\Omega$, whereas x' has two such points y_1, y_2.

Signed distance function and geometry (III)

Proposition 2.

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz, bounded open set;

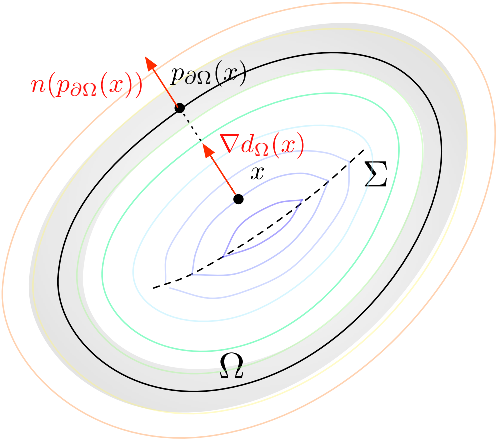
- *A point $x \in \mathbb{R}^d$ has a unique projection point $p_{\partial\Omega}(x)$ iff $x \notin \Sigma$. In such a case, d_Ω is differentiable at x, and its gradient reads:*

$$\nabla d_\Omega(x) = \frac{x - p_{\partial\Omega}(x)}{d_\Omega(x)}.$$

  In particular, $|\nabla d_\Omega(x)| = 1$ wherever it makes sense.

- *If Ω is of class $\mathcal{C}^1$, this last quantity equals $\nabla d_\Omega(x) = n(p_{\partial\Omega}(x))$.*
- *If Ω is of class $\mathcal{C}^k$, $k \geq 2$, then d_Ω is also of class $\mathcal{C}^k$ on a neighborhood of $\partial\Omega$.*

Signed distance function and geometry (IV)



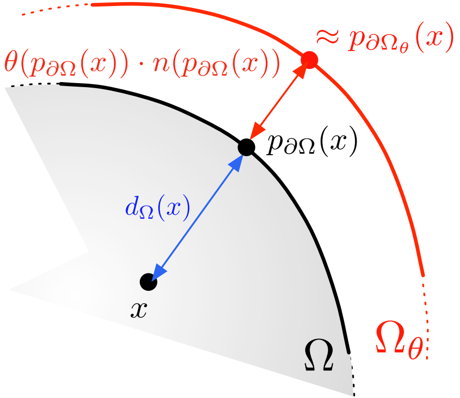
Some level sets of d_Ω are depicted in color; d_Ω is as smooth as the boundary $\partial\Omega$ on the shaded area (at least).

Shape differentiability of the signed distance function (I)

Lemma 3.

Let $\Omega \subset \mathbb{R}^d$ be a $\mathcal{C}^1$ bounded domain, and $x \notin \Sigma$. The function $\theta \mapsto d_{\Omega_\theta}(x)$, from $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ into $\mathbb{R}$ is Gâteaux-differentiable at $\theta = 0$, with derivative:

$$d'_\Omega(\theta)(x) = -\theta(p_{\partial\Omega}(x)) \cdot n(p_{\partial\Omega}(x)).$$

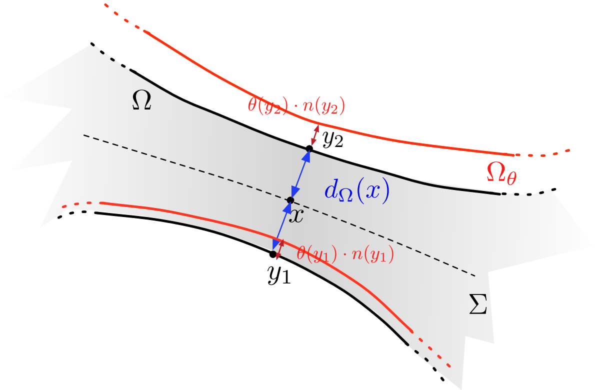


Shape differentiability of the signed distance function (II)

Remark: A more general formula holds, which encompasses the case $x \in \Sigma$:

$$\text{If } x \in \Omega, \;\; d_\Omega'(\theta)(x) = -\inf_{y \in \Pi_{\partial\Omega}(x)} \theta(y) \cdot n(y),$$

$$\text{If } x \in {}^c\overline{\Omega}, \;\; d_\Omega'(\theta)(x) = -\sup_{y \in \Pi_{\partial\Omega}(x)} \theta(y) \cdot n(y).$$



Shape differentiability of the signed distance function (II)

- **Formal clue:** Taking the shape derivative in

$$|\nabla d_\Omega(x)|^2 = 1$$

yields:

$$\nabla d'_\Omega(\theta)(x) \cdot \nabla d_\Omega(x) = 0.$$

$\Rightarrow$ The shape derivative of d_Ω is constant along the rays.

- **Rigorous proof:** Use of the definition:

$$d_\Omega^2(x) = \min_{y \in \partial\Omega} |x - y|^2$$

in combination to a theorem for differentiating a minimum value with respect to a parameter.

Shape differentiability of the signed distance function (III)

Lemma 4.

Let Ω be a $\mathcal{C}^1$ bounded domain, enclosed in a large computational domain D, and $j : \mathbb{R}_x \times \mathbb{R}_s \to \mathbb{R}$ be of class $\mathcal{C}^1$; define the functional:

$$J(\Omega) = \int_D j(x, d_\Omega(x))\, dx.$$

Then $\theta \mapsto J(\Omega_\theta)$ is Gâteaux-differentiable at $\theta = 0$ with derivative:

$$J'(\Omega)(\theta) = -\int_D \frac{\partial j}{\partial s}(x, d_\Omega(x))\, \theta(p_{\partial\Omega}(x)) \cdot n(p_{\partial\Omega}(x))\, dx.$$

This formula is awkward insofar it is not easily put under the form:

$$J'(\Omega)(\theta) = \int_\Gamma v\, \theta \cdot n\, ds,$$

and does not lend itself to the inference of a 'natural' descent direction for J.

A coarea formula

Proposition 5.

Let $\Omega \subset D$ be a bounded domain of class $\mathcal{C}^2$, and let $\varphi \in L^1(D)$. Then,

$$\int_D \varphi(x)dx = \int_{\partial\Omega} \left(\int_{\mathrm{ray}_{\partial\Omega}(y)\cap D} \varphi(z) \prod_{i=1}^{d-1} (1 + d_\Omega(z)\kappa_i(y))dz \right) dy,$$

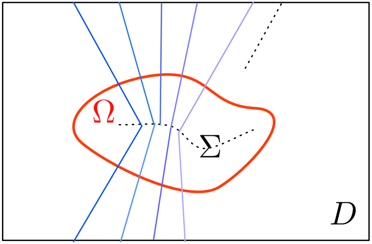
where z denotes a point in the ray emerging from $y \in \partial\Omega$ and dz is the line integration along that ray.

Hint of proof:

Apply the coarea formula to the mapping:

$$p_{\partial\Omega} : D \setminus \Sigma \to \partial\Omega$$

to recast the integration over $D \approx D \setminus \Sigma$ as an integration over $\partial\Omega$ composed with an integration over the pre-images $p_{\partial\Omega}^{-1}(x) = \mathrm{ray}_{\partial\Omega}(x)$, $x \in \partial\Omega$.



Theorem 6.

The objective function

$$J_\varepsilon(\Omega^0) = \int_D j(x, u_{\Omega^0,\varepsilon})\, dx + \int_{\Gamma_N} k(x, u_{\Omega^0,\varepsilon})\, ds,$$

is s.t. $\theta \mapsto J_\varepsilon(\Omega^0_\theta)$ *admits a Gâteaux-derivative at* $\theta = 0$*, which is*

$$\forall\, \theta \in W^{1,\infty}(D, \mathbb{R}^d), \quad J'_\varepsilon(\Omega^0)(\theta) = -\int_\Gamma j(x)\, \theta(x) \cdot n(x) ds(x).$$

Here, n *is the outer unit normal to* Ω^0 *and* j *is the scalar function defined by*

$$j(x) = \int_{\mathrm{ray}_\Gamma(x) \cap D} h'_\varepsilon\,(d_{\Omega^0}(z))\,(A_1 - A_0) e(u)(z) : e(p)(z) \prod_{i=1}^{d-1} (1 + d_{\Omega^0}(z)\kappa_i(x)) dz,$$

where $u \equiv u_{\Omega^0,\varepsilon}$ *and the adjoint state* $p \equiv p_{\Omega^0,\varepsilon}$ *is the solution to:*

$$\begin{cases} -\mathrm{div}\,(A_{\Omega^0,\varepsilon}\, e(p)) &= & -j'(x, u_{\Omega^0,\varepsilon}) & \text{in } D, \\ p &= & 0 & \text{on } \Gamma_D, \\ (A_1\, e(p))\, n &= & -k'(x, u_{\Omega^0,\varepsilon}) & \text{on } \Gamma_N, \end{cases}$$

Shape derivative of the smoothed-interface functional (II)

Sketch of (formal) proof: For functions $v, q \in H^1_{\Gamma_D}(D)^d$, define the Lagrangian functional $\mathcal{L}(\Omega^0, v, q)$ as:

$$\mathcal{L}(\Omega^0, v, q) = \int_D j(x, v)\, dx + \int_{\Gamma_N} k(x, v)\, ds + \int_D A_{\Omega^0,\varepsilon}(x) e(v) : e(q)\, dx - \int_D f \cdot q\, dx - \int_{\Gamma_N} g \cdot q\, ds.$$

By definition,

$$\forall q \in H^1_{\Gamma_D}(D)^d, \;\; J_\varepsilon(\Omega^0) = \mathcal{L}(\Omega^0, u_{\Omega^0,\varepsilon}, q).$$

Let us search for the critical points (u, p) of $\mathcal{L}(\Omega^0, \cdot, \cdot)$.

- Expressing $\frac{\partial \mathcal{L}}{\partial p}(\Omega^0, u, p) = 0$ yields $u = u_{\Omega^0,\varepsilon}$.
- Expressing $\frac{\partial \mathcal{L}}{\partial u}(\Omega^0, u, p) = 0$ yields $p = p_{\Omega^0,\varepsilon}$.

Shape derivative of the smoothed-interface functional (III)

Thus, for any $q \in H^1_{\Gamma_D}(D)^d$, assuming that $u_{\Omega^0,\varepsilon}$ is differentiable with respect to the domain,

$$J'_\varepsilon(\Omega^0)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega^0, u_{\Omega^0,\varepsilon}, q) + \frac{\partial \mathcal{L}}{\partial u}(\Omega^0, u_{\Omega^0,\varepsilon}, q)(u'_{\Omega^0,\varepsilon}(\theta)).$$

Now choosing $q = p_{\Omega^0,\varepsilon}$, and using $\frac{\partial \mathcal{L}}{\partial u}(\Omega^0, u, p) = 0$ yield:

$$J'_\varepsilon(\Omega^0)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega^0, u_{\Omega^0,\varepsilon}, p_{\Omega^0,\varepsilon}),$$

which can be calculated thanks to Lemma 4.

□

Approximate formulae

The formula of Theorem 6 can be given consistent and convenient approximations in two important limits in applications:

- *Jacobian-free formula:* If the interface Γ is approximately plane, that is $d_{\Omega^0}\kappa_i \approx 0$, we obtain:

  $$J'_\varepsilon(\Omega^0)(\theta) = -\int_\Gamma j(x)\,\theta(x)\cdot n(x)ds(x),$$

  with

  $$j(x) \approx \int_{\mathrm{ray}_\Gamma(x)\cap D} h'_\varepsilon\left(d_{\Omega^0}(z)\right)(A_1 - A_0)e(u)(z) : e(p)(z)dz.$$

- *Thin-interface formula:* If the transition layer is very thin, i.e. ε is very small,

  $$J'_\varepsilon(\Omega^0)(\theta) \approx -\int_\Gamma (A_1 - A_0)e(u)(x) : e(p)(x)\,\theta(x)\cdot n(x)ds(x).$$

Consistency of the smoothed-interface approach

Theorem 7.

Assume that Ω^0 is of class $\mathcal{C}^2$. Then the smoothed-interface problem *converges* *to its sharp-interface counterpart in the sense that:*

$$J_\varepsilon(\Omega^0) \xrightarrow{\varepsilon \to 0} J(\Omega^0),$$

and, for any deformation field $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$,

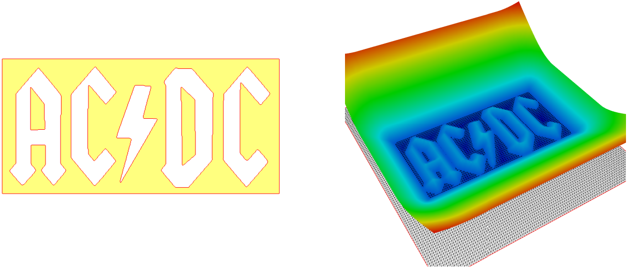
$$J'_\varepsilon(\Omega^0)(\theta) \xrightarrow{\varepsilon \to 0} J'(\Omega^0)(\theta).$$

The Level Set Method

A paradigm: [OSe] *the motion of an evolving domain is best described in an implicit way.*

A bounded domain $\Omega \subset \mathbb{R}^d$ is equivalently defined by a function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that:

$$\phi(x) < 0 \quad \text{if } x \in \Omega \quad ; \quad \phi(x) = 0 \quad \text{if } x \in \partial\Omega \quad ; \quad \phi(x) > 0 \quad \text{if } x \in {}^c\overline{\Omega}$$



A bounded domain $\Omega \subset \mathbb{R}^2$ (left); graph of an associated level set function (right).

Surface evolution equations in the level set framework

The motion of an evolving domain $\Omega(t) \subset \mathbb{R}^d$ along a velocity field $v(t,x) \in \mathbb{R}^d$ translates in terms of an associated 'level set function' $\phi(t,.)$ into the level set advection equation:

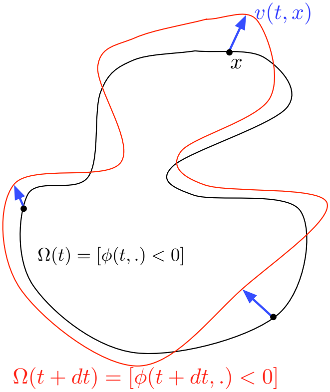
$$\forall t,\ \forall x \in \mathbb{R}^d,\ \frac{\partial \phi}{\partial t}(t,x) + v(t,x).\nabla\phi(t,x) = 0$$

In many applications, the velocity $v(t,x)$ is normal to the boundary $\partial\Omega(t)$:

$$v(t,x) := V(t,x)\frac{\nabla\phi(t,x)}{|\nabla\phi(t,x)|}.$$

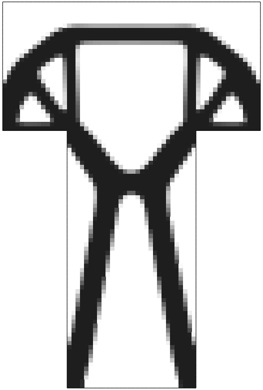
Then the evolution equation rewrites as a Hamilton-Jacobi equation:

$$\forall t,\ \forall x \in \mathbb{R}^d,\ \frac{\partial \phi}{\partial t}(t,x) + V(t,x)|\nabla\phi(t,x)| = 0$$



The level set method for shape optimization [AlJouToa]

- The shapes Ω^n under evolution are embedded in a working domain D equipped with a fixed mesh.
- The successive shapes Ω^n are accounted for in the level set framework, i.e. via a function $\phi^n : D \to \mathbb{R}$ which implicitly defines them.
- This approach is very versatile and does not require a mesh of the shapes at each iteration.

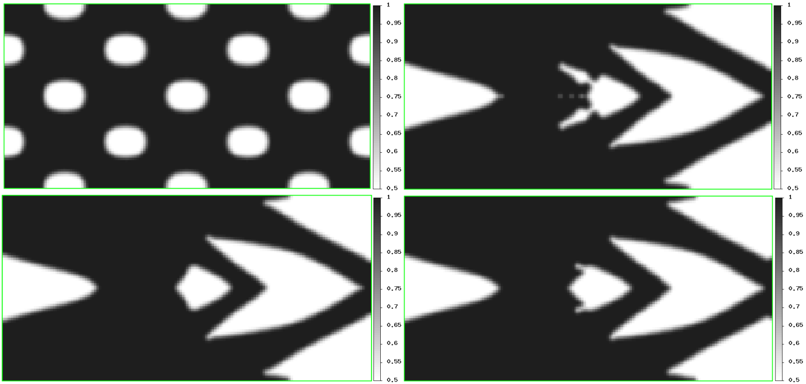


Shape accounted for with a level set description

Two-phase long cantilever (I)

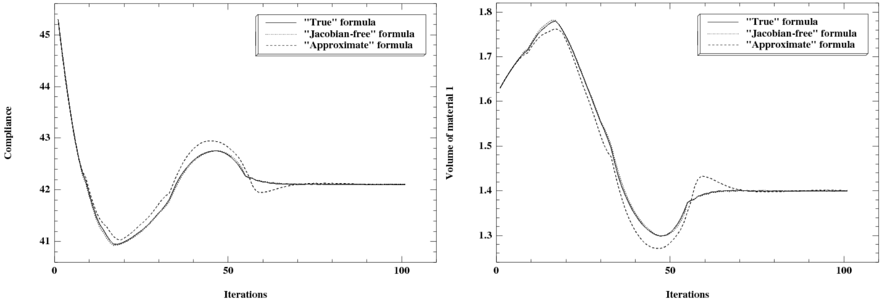
- Optimization of the repartition of two materials with the same Poisson ratio $\nu_0 = \nu_1 = 0.3$, but different Young's moduli $E_0 = 0.5$, $E_1 = 1$.
- The thickness parameter ε is set to $2\Delta_x$.
- The compliance of the total structure D is minimized.
- A constraint is imposed on the volume of the stronger phase: $V_T = 0.7|D|$, owing to an augmented Lagrangian algorithm.

Two-phase long cantilever (II)



Initial shape, optimized shape using the 'true' formula, optimized shape using the 'Jacobian-free' formula, optimized shape using the 'thin-interface' formula.

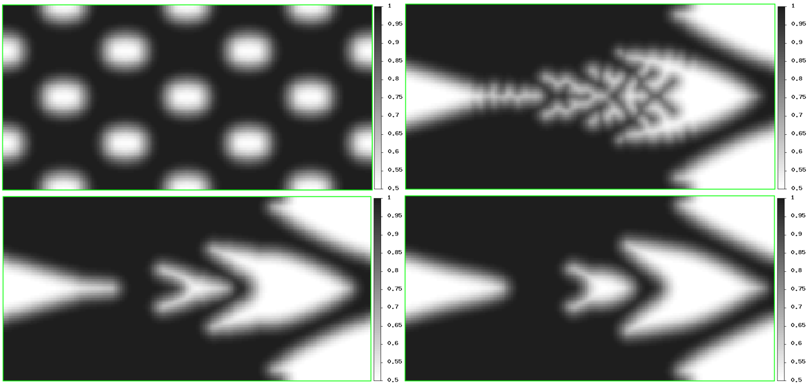
Two-phase long cantilever (III)



Convergence histories in the three cases of interest.

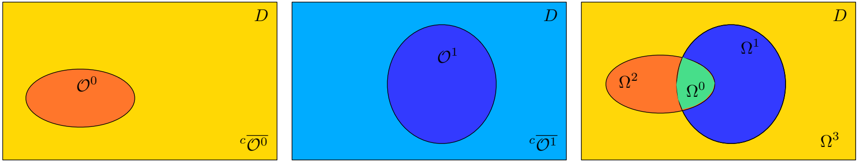
Two-phase long cantilever (IV)

Use of a larger thickness parameter $\varepsilon = 8\Delta x$ for the transition zone.



Initial shape, optimized shape using the 'true' formula, optimized shape using the 'Jacobian-free' formula, optimized shape using the 'thin-interface' formula.

Extension to more than 2 (e.g. 3, 4) phases



Two subdomains $\mathcal{O}^0, \mathcal{O}^1 \subset D$, and the 4 phases derived by combining them.

- One subdomain $\mathcal{O}_0 \subset D$ accounts for two phases $\Omega^0 = \mathcal{O}^0$, $\Omega^1 = {}^c\overline{\mathcal{O}^0}$.
- Combining 2 subdomains $\mathcal{O}^0, \mathcal{O}^1 \subset D$, one can represent up to 4 phases:

$$\Omega^0 = \mathcal{O}^0 \cap \mathcal{O}^1, \;\; \Omega^1 = {}^c\overline{\mathcal{O}^0} \cap \mathcal{O}^1, \;\; \Omega^2 = \ldots$$

- The previous framework can be easily extended to deal with multiple phases:

$\Rightarrow$ Using m different level set functions allows to account for up to 2^m distinct phases.

Multiple-phase short cantilever

- *Two phases and void:*

  The Young's moduli of the different phases are:

  $$E_0 = 0.5,\ E_2 = 1,\ E_1 = E_3 = 1e^{-3}.$$

  (Phases 1 and 3 mimick void).
  Volume constraint:

  $$V_0 = 0.2|D|,\ V_2 = 0.1|D|.$$

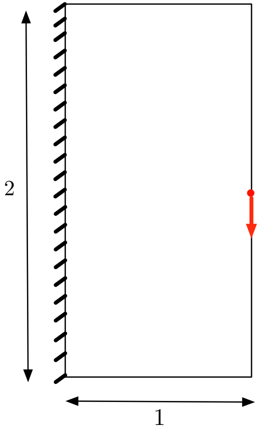
- *Three phases and void:*

  The Young's moduli are:

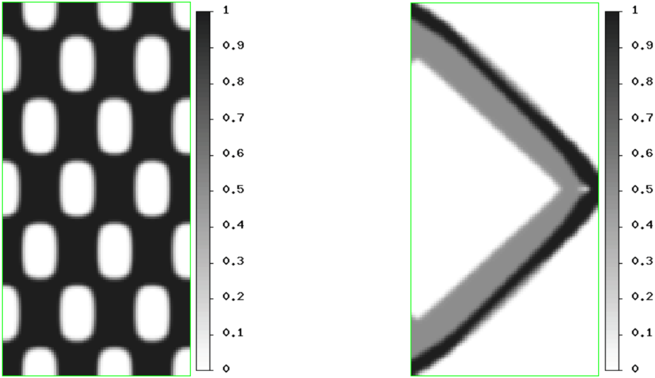
  $$E_0 = 0.5,\ E_1 = 0.25,\ E_2 = 1,\ E_3 = 1e^{-3}.$$

  Volume constraint:

  $$V_0 = V_1 = V_2 = 0.1|D|.$$

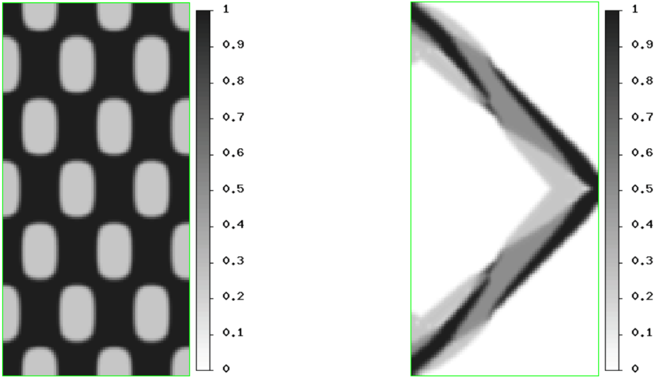


Two-phase short cantilever



Short cantilever using two phases and void; (left) initialization, (right) optimal shape.

Three-phase short cantilever

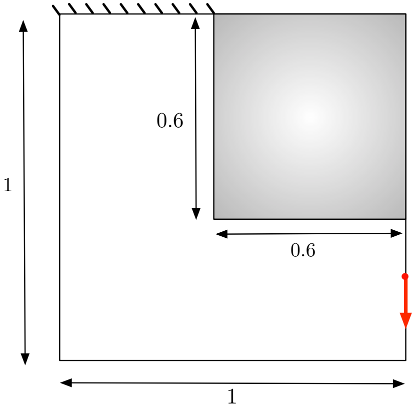


Short cantilever with three phases and void; (left) initialization, (right) optimal shape.

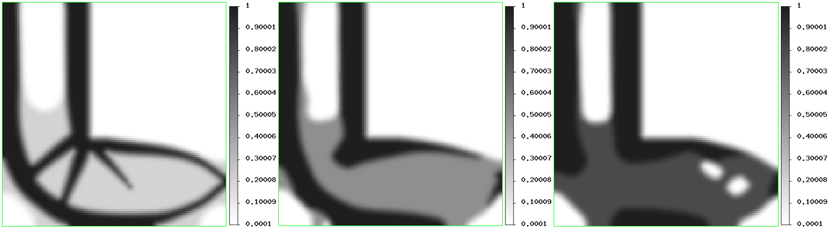
Two-phase L-Beam

- Phase 0 has Young's modulus $E_0 = 1$.
- Phases 1 and 3 mimick void ($E_1 = E_3 = 1e^{-3}$).
- Phase 2 has different Young's moduli depending on the considered example.
- A constraint on the volumes of phases 0 and 2 is imposed:

$$V_T^0 = V_T^2 = 0.25|D|.$$

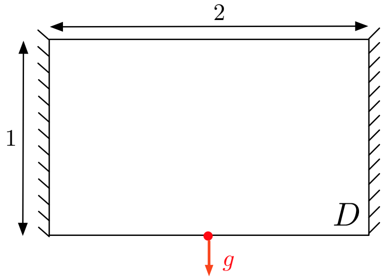


Two-phase L-Beam



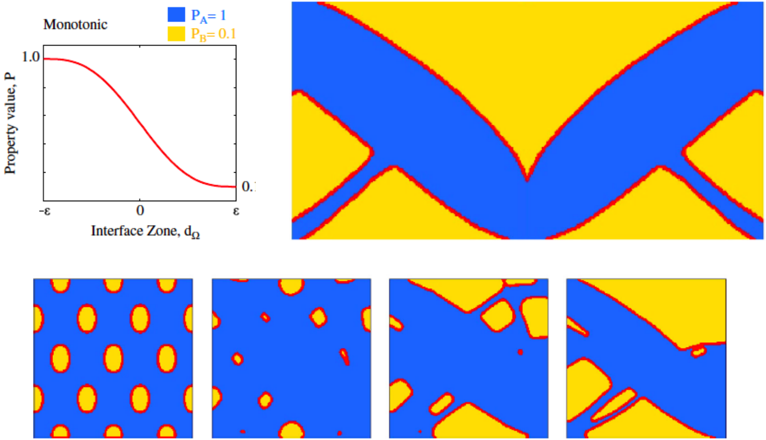
Optimal designs for the two-phase L-Beam problem with (from left to right) $E_2 = 0.2, 0.5, 0.8$.

An example using non monotone interfaces



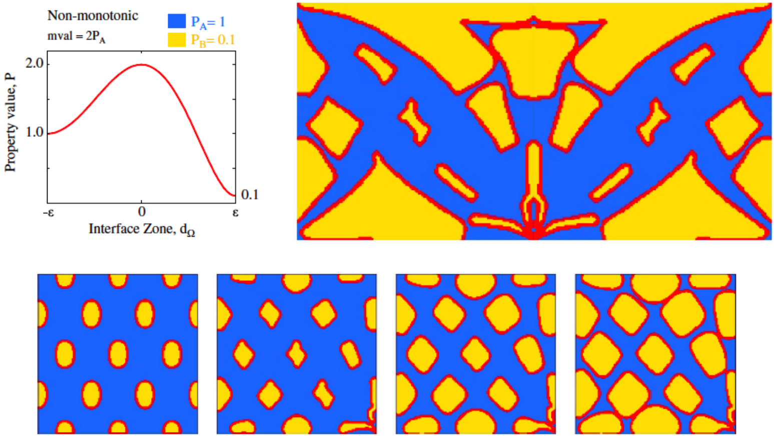
- Work carried out by G. Allaire, Y. Bréchet, R. Estevez, G. Michailidis, G. Parry and N. Vermaak [VerMi].
- Optimization of the repartition of two materials with the same Poisson ratio $\nu_0 = \nu_1 = 0.3$, but different Young's moduli $E_0 = 0.1$, $E_1 = 1$.
- The compliance of the total structure D is minimized, under a constraint $V_T^1 = 0.5|D|$ on the volume of the stronger phase.
- The properties of the material inside the transition layer are non monotone.

An example using non monotone interfaces



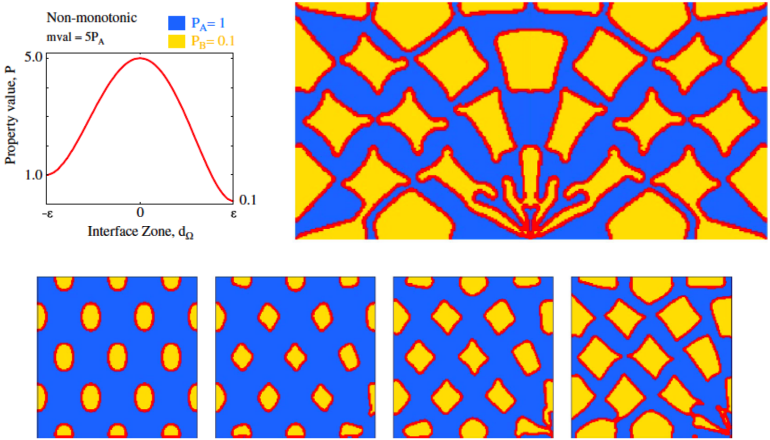
(Top-left) Profile of the Young's modulus in the transition layer, (top-right) final design, (bottom) iterations $1, 10, 25, 40$.

An example using non monotone interfaces



(Top-left) Profile of the Young's modulus in the transition layer, (top-right) final design, (bottom) iterations $1, 50, 75, 90$.

An example using non monotone interfaces



(Top-left) Profile of the Young's modulus in the transition layer, (top-right) final design, (bottom) iterations $1, 5, 50, 110$.

Thank you for your attention!

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