A new approach to topological ligaments in structural optimization

Charles Dapogny[1]

[1] Laboratoire Jean Kuntzmann, Université Grenoble Alpes, Grenoble, France

13th - 18th June, 2021

Introduction

- The treatment of an optimal design problem

$$\min_{\Omega} J(\Omega) \text{ s.t. } C(\Omega) \leq 0$$

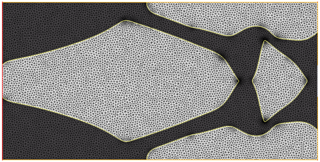
usually relies on the sensitivity of $J(\Omega)$, $C(\Omega)$ with respect to the shape Ω.

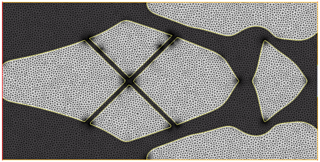
- We present a means to account for the sensitivity with respect to the addition of a bar to Ω.

- This method makes a connection with the topic of small inhomogeneities in asymptotic analysis.

- Three applications are proposed:

  1. Combined use with a more "classical" boundary variation algorithm;
  2. Erection of an "optimal" scaffold structure in additive manufacturing;
  3. "Suitable" initialization of the optimization of a truss structure.





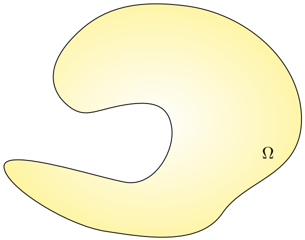
Different sensitivities with respect to the domain (I)

A model shape and topology optimization problem reads:

$$\min_{\Omega} J(\Omega) + \ell C(\Omega),$$

where

- Ω is a shape in $\mathbb{R}^d$, e.g. an elastic structure.
- $J(\Omega)$ measures the physical performance of Ω.
- $C(\Omega)$ is a constraint functional.
- ℓ is a (fixed) Lagrange multiplier.



- Most numerical algorithms rely on the "derivatives" of $\Omega \mapsto J(\Omega)$ and $\Omega \mapsto C(\Omega)$.
- Multiple notions of derivative with respect to the design are available, relying in turn on different descriptions of "small variations of shapes".

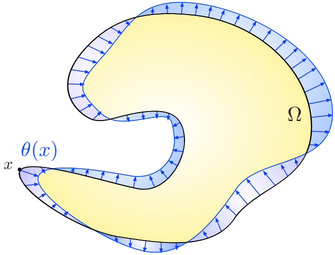
Different sensitivities with respect to the domain (II)

❶ *Hadamard's boundary variation method.*

Variations of a shape are considered under the form

$$\Omega_\theta := (\mathrm{Id} + \theta)(\Omega),$$

where $\theta : \mathbb{R}^d \to \mathbb{R}^d$ is a "small" vector field [HenPi].



This gives rise to the notion of shape derivative $J'(\Omega)(\theta)$ for a function $\Omega \mapsto J(\Omega)$:

$$J(\Omega_\theta) = J(\Omega) + J'(\Omega)(\theta) + \mathrm{o}(\theta).$$

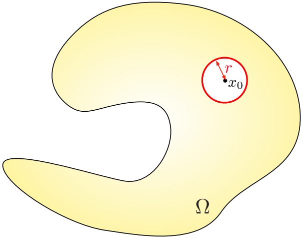
A. Henrot and M. Pierre, *Shape Variation and Optimization*, EMS Tracts in Mathematics, Vol. 28, 2018.

❷ *Nucleation of a small hole.*

Variations of Ω are considered under the form

$$\Omega_{x_0,r} = \Omega \setminus \overline{B(x_0, r)},$$

where $x_0 \in \Omega$ and $r \ll 1$ [NoSo].



This yields the notion of topological derivative $\mathrm{d}J_T(\Omega)(x_0)$ for a function $\Omega \mapsto J(\Omega)$:

$$J(\Omega_{x_0,r}) = J(\Omega) + r^d \mathrm{d}J_T(\Omega)(x_0) + \mathrm{o}(r^d).$$

A. A. Novotny and J. Sokołowski, *Topological derivatives in shape optimization*, Springer Science & Business Media, 2012.

Different sensitivities with respect to the domain (IV)

❸ *Graft of a thin ligament.*

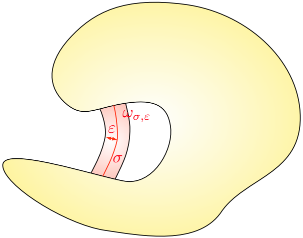
One third means to define "small variations" of Ω is:

$$\Omega_{\sigma,\varepsilon} := \Omega \cup \omega_{\sigma,\varepsilon},$$

where

$$\omega_{\sigma,\varepsilon} := \left\{ x \in \mathbb{R}^d, \ d(x,\sigma) < \varepsilon \right\},$$

is a tube with thickness $\varepsilon \ll 1$ around a curve σ [NaSo].



This type of variations has been seldom considered in the literature. It lends itself to a notion of topological ligament:

$$J(\Omega_{\sigma,\varepsilon}) = J(\Omega) + \underbrace{\varepsilon^{d-1}}_{\approx|\omega_{\sigma,\varepsilon}|} \mathrm{d}J_L(\Omega)(\sigma) + \mathrm{o}(\varepsilon^{d-1}).$$

S. Nazarov and J. Sokolowski, *The topological derivative of the dirichlet integral due to formation of a thin ligament*, Siberian Mathematical Journal, 45 (2004), pp. 341–355.

Purposes of this work

1. Account for the sensitivity of the elastic displacement $u_\Omega \rightsquigarrow u_{\sigma,\varepsilon}$ when a thin tube $\omega_{\sigma,\varepsilon}$ is added to Ω, as $\varepsilon \to 0$.

2. Infer the derivative of the functional

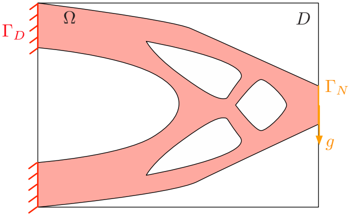
$$J_\sigma(\varepsilon) := J(\Omega_{\sigma,\varepsilon}),$$

under a form which allows for a fast evaluation of whether it is beneficial to add $\omega_{\sigma,\varepsilon}$ to Ω.

3. Use the insight provided by this new type of sensitivity in practical applications.

C. Dapogny, *The topological ligament in shape optimization: a connection with thin tubular inhomogeneities*, submitted, (2020).

Shape optimization of elastic structures



Shapes are bounded Lipschitz domains $\Omega \subset D$ in $\mathbb{R}^d$:

- They are clamped on a fixed subset $\Gamma_D \subset \partial D$.
- Traction loads $g : \mathbb{R}^d \to \mathbb{R}^d$ are applied on $\Gamma_N \subset \partial D$.
- The remaining part $\Gamma = \partial\Omega \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N})$ is traction-free.

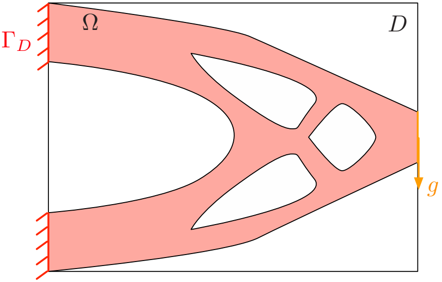
The elastic displacement $u_\Omega \in H^1(\Omega)^d$ is the unique solution to

$$\begin{cases} -\mathrm{div}(Ae(u_\Omega)) = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ Ae(u_\Omega)n = g & \text{on } \Gamma_N, \\ Ae(u_\Omega)n = 0 & \text{on } \Gamma, \end{cases}$$

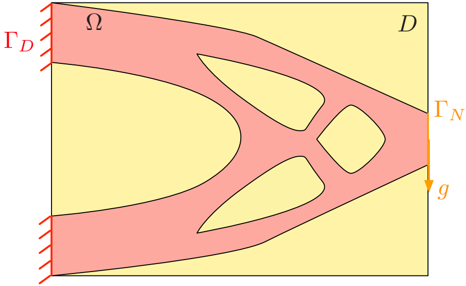
where A is the (homogeneous) Hooke's law of the material.

The ersatz material approximation (I)

We approximate this setting by "filling the void" $D \setminus \overline{\Omega}$ with a soft material ηA, $\eta \ll 1$.



$\approx$



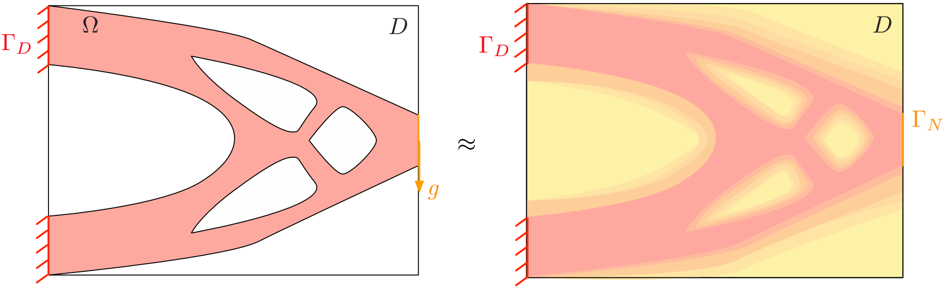
$$\begin{cases} -\mathrm{div}(Ae(u_\Omega)) = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ Ae(u_\Omega)n = g & \text{on } \Gamma_N, \\ Ae(u_\Omega)n = 0 & \text{on } \Gamma. \end{cases}$$

$$\begin{cases} -\mathrm{div}(A_\eta e(u_\eta)) = 0 & \text{in } D, \\ u_\eta = 0 & \text{on } \Gamma_D, \\ A_\eta e(u_\eta)n = g & \text{on } \Gamma_N, \\ A_\eta e(u_\eta)n = 0 & \text{on } \partial D \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N}), \end{cases}$$

$$A_\eta(x) = \begin{cases} A & \text{if } x \in \Omega, \\ \eta A & \text{otherwise.} \end{cases}$$

The ersatz material approximation (II)

We may as well use a smoothed version $\widetilde{A_\eta}$ of A_η.



$$\begin{cases} -\mathrm{div}(Ae(u_\Omega)) = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ Ae(u_\Omega)n = g & \text{on } \Gamma_N, \\ Ae(u_\Omega)n = 0 & \text{on } \Gamma. \end{cases}$$

$$\begin{cases} -\mathrm{div}(\widetilde{A_\eta} e(u_\eta)) = 0 & \text{in } D, \\ u_\eta = 0 & \text{on } \Gamma_D, \\ \widetilde{A_\eta} e(u_\eta)n = g & \text{on } \Gamma_N, \\ \widetilde{A_\eta} e(u_\eta)n = 0 & \text{on } \partial D \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N}), \end{cases}$$

$$\widetilde{A_\eta}(x) = (\text{smoothed}) \begin{cases} A & \text{if } x \in \Omega, \\ \eta A & \text{otherwise.} \end{cases}$$

The ersatz material approximation (III)

- A quantity of interest $J(\Omega)$, depending on Ω via u_Ω can be given an approximate counterpart by the same token.

  Example: The shape functional

$$J(\Omega) = \int_\Omega j(u_\Omega)\,\mathrm{d}x \text{ where } j : \mathbb{R}^d \to \mathbb{R} \text{ is smooth,}$$

  can be approximated as

$$J(\Omega) \approx \int_D j(x, u_\eta(x))\,\mathrm{d}x, \text{ up to modifying } j.$$

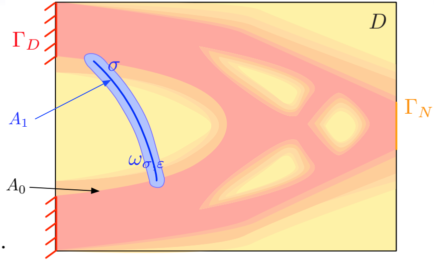
- In the same spirit, we aim to construct an ersatz material approximation

$$J_\sigma(\varepsilon) \approx J(\Omega_{\sigma,\varepsilon}), \text{ where } \Omega_{\sigma,\varepsilon} := \Omega \cup \omega_{\sigma,\varepsilon}.$$

The ersatz material approximation: perturbed setting (I)

- Let $A_0(x)$ be a smooth Hooke's law in D.
- The "background" displacement u_0 is the $H^1(D)^d$ solution to:

$$\begin{cases} -\mathrm{div}(A_0 e(u_0)) = 0 & \text{in } D, \\ u_0 = 0 & \text{on } \Gamma_D, \\ A_0 e(u_0) n = g & \text{on } \Gamma_N, \\ A_0 e(u_0) n = 0 & \text{on } \partial D \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N}). \end{cases}$$



- In a perturbed situation, the properties $A_0(x)$ are traded for $A_1(x)$ in a tube

$$\omega_{\sigma,\varepsilon} := \{x \in D, \ d(x,\sigma) < \varepsilon\}$$

with "small" thickness $\varepsilon \ll 1$ around a curve σ.

- The perturbed elastic displacement u_ε is the solution to:

$$\begin{cases} -\mathrm{div}(A_\varepsilon e(u_\varepsilon)) = 0 & \text{in } D, \\ u_\varepsilon = 0 & \text{on } \Gamma_D, \\ A_\varepsilon e(u_\varepsilon) n = g & \text{on } \Gamma_N, \\ A_\varepsilon e(u_\varepsilon) n = 0 & \text{on } \partial D \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N}), \end{cases} \quad \text{where } A_\varepsilon(x) = \begin{cases} A_1(x) & \text{if } x \in \omega_{\sigma,\varepsilon}, \\ A_0(x) & \text{otherwise.} \end{cases}$$

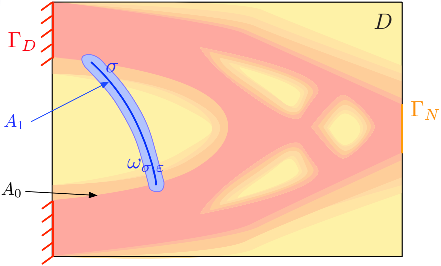
The ersatz material approximation: perturbed setting (II)

The perturbed version of a quantity

$$J_\sigma(0) := \int_D j(u_0)\,\mathrm{d}x$$

reads

$$J_\sigma(\varepsilon) = \int_D j(u_\varepsilon)\,\mathrm{d}x.$$



- Intuitively, the asymptotic expansion of $J'_\sigma(0)$,

$$J_\sigma(\varepsilon) = J_\sigma(0) + \varepsilon^{d-1} J'_\sigma(0) + \mathrm{o}(\varepsilon^{d-1})$$

  measures the sensitivity of J_σ with respect to changing material properties from A_0 to A_1 into the thin tube $\omega_{\sigma,\varepsilon}$.

- When A_0 is obtained from $\Omega \subset D$ by the ersatz material approximation, i.e.

$$A_0(x) = \begin{cases} A & \text{if } x \in \Omega, \\ \eta A & \text{if } x \in D \setminus \Omega, \end{cases}$$

  $J'_\sigma(0)$ is an approximate sensitivity of $J(\Omega)$ with respect to the addition of $\omega_{\sigma,\varepsilon}$.

1 Foreword

2 Methodology

- From thin tubular inhomogeneities to topological ligaments
- Thin tubular asymptotic expansion

3 Applications

- Inserting a bar in the course of shape evolution
- Optimization of the scaffold structure in additive manufacturing
- A "clever" initialization for truss structures

Proposition 1.

The function $J_\sigma(\varepsilon) := \int_D j(u_\varepsilon)\,\mathrm{d}x$ *is differentiable at* 0, *and its derivative reads:*

$$J_\sigma'(0) = \int_\sigma \mathcal{M}(y)e(p_0) : e(u_0)\,\mathrm{d}\ell(y),$$

where the polarization tensor $\mathcal{M}(y)$ *is defined for* $y \in \sigma$ *by*

$$\mathcal{M}(y)e = \alpha_T(y)\mathrm{tr}(e)\mathrm{I} + \beta_T(y)e + \gamma_T(y)(e\tau\cdot\tau)\tau\otimes\tau + \delta_T(y)(en\cdot n)n\otimes n.$$

The coefficients $\alpha_T, \beta_T, \gamma_T$ *and* δ_T *are given by:*

$$\alpha_T = 2(\lambda_1-\lambda_0)\frac{\lambda_0+2\mu_0}{\lambda_1+2\mu_1},\ \ \beta_T = 4(\mu_1-\mu_0)\frac{\mu_0}{\mu_1},$$

$$\gamma_T = 4(\mu_1-\mu_0)\left(\frac{2\lambda_1+2\mu_1-\lambda_0}{\lambda_1+2\mu_1} - \frac{\mu_0}{\mu_1}\right),\ \ \delta_T = 4(\mu_1-\mu_0)\frac{\mu_1\lambda_0-\mu_0\lambda_1}{\mu_1(\lambda_1+2\mu_1)}.$$

and the adjoint state p_0 *is the unique solution in* $H^1(D)^2$ *to the system:*

$$\begin{cases} -\mathrm{div}(A_0 e(p_0)) = -j'(u_0) & \text{in } D, \\ p_0 = 0 & \text{on } \Gamma_D, \\ A_0 e(p_0)n = 0 & \text{on } \partial D \setminus \overline{\Gamma_D}. \end{cases}$$

The derivative formula

- A rigorous proof of this 2d formula is provided in [BeFran].
- A similar, albeit much more intricate formula holds in 3d, without rigorous proof to the best of our knowledge.
- In [Da2], we propose a heuristic energy method to achieve these calculations, even in 3d.

E. Beretta and E. Francini, *An asymptotic formula for the displacement field in the presence of thin elastic inhomogeneities*, SIAM J. Math. Anal., 38 (2006), pp. 1249–1261.

C. Dapogny, *The topological ligament in shape optimization: a connection with thin tubular inhomogeneities*, submitted, (2020).

The general strategy to add a tube to a shape

In order to graft a tube $\omega_{\sigma,\varepsilon}$ to a fixed shape $\Omega \subset D$,

1. We convert the elasticity problem on Ω into a two-phase elasticity problem in D thanks to the ersatz material method.
2. We calculate the ersatz material approximations u_η, p_η of u_Ω, p_Ω.
3. For "many" curve configurations σ (segments), we calculate the quantity

$$J'_\sigma(0) = \int_\sigma \mathcal{M}e(u_\Omega) : e(p_\Omega)\,\mathrm{d}\ell,$$

   measuring the sensitivity of adding a tube (a bar) with direction σ to Ω.

4. The curve σ realizing the largest negative value of $J'_\sigma(0)$ yields the "optimal" tube (bar) to be added to Ω.

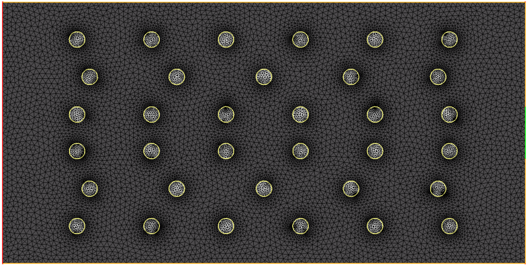
Inserting a bar in the course of shape evolution (I)

- We minimize the compliance of a shape Ω under a volume constraint:

$$\min_{\Omega} C(\Omega) \text{ s.t. } \mathrm{Vol}(\Omega) \leq V_T,$$

$$\text{where } C(\Omega) := \int_\Omega Ae(u_\Omega) : e(u_\Omega)\,\mathrm{d}x, \text{ and } \mathrm{Vol}(\Omega) = \int_\Omega \mathrm{d}x.$$

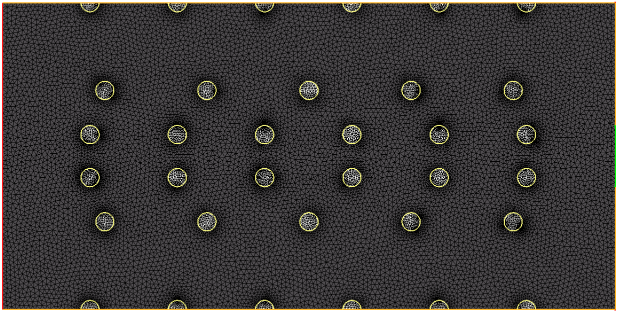
- We use the level set based mesh evolution method [AdDaFre].
- Like with any boundary variation algorithm, the optimized shape may fall into local minima with trivial topologies.



- To remedy this, we periodically interrupt the optimization process to insert bars.

Inserting a bar in the course of shape evolution (II)

The “‘benchmark” 2d cantilever test case is considered.



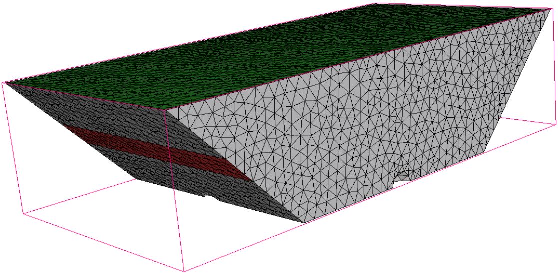
- The compliance of the shape Ω is minimized:

$$C(\Omega) = \int_{\Omega} Ae(u_{\Omega}) : e(u_{\Omega}) \, \mathrm{d}x.$$

- A constraint is imposed on its volume $\mathrm{Vol}(\Omega)$.
- Every now and then, a bar is added to Ω at an “optimal location”.

Inserting a bar in the course of shape evolution (III)

The optimization of a 3d bridge Ω is considered.



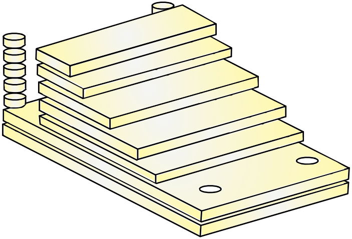
- We minimize the compliance of Ω

$$C(\Omega) = \int_\Omega Ae(u_\Omega) : e(u_\Omega)\,\mathrm{d}x.$$

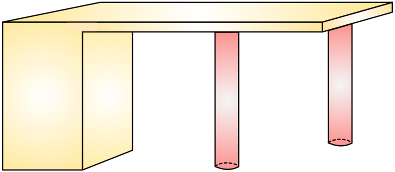
- A volume constraint is enforced.
- Every now and then, a bar is added to Ω at an “optimal location”.

Optimization of supports in additive manufacturing (I)

- Additive manufacturing processes feature a layer by layer assembly of the shape Ω.
- Most of these technologies experience difficulties dealing with overhang regions.
- One remedy is to erect a scaffold structure S with Ω, such that:
  - The compliance of the total structure $\Omega \cup S$ has minimum value.
  - S has minimum volume and... it does not itself present overhangs!
- To achieve this, we propose to
  1. Incrementally add vertical pillars to Ω, made of a different material.
  2. (Optionally) Optimize S further via a more "classical" algorithm.



Layer by layer construction of a structure by additive manufacturing



Supporting pillars for an overhang feature.

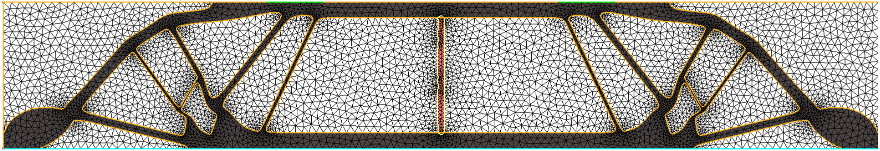
Optimization of supports in additive manufacturing (II)

The scaffold structure S of a fixed MBB beam Ω is optimized.

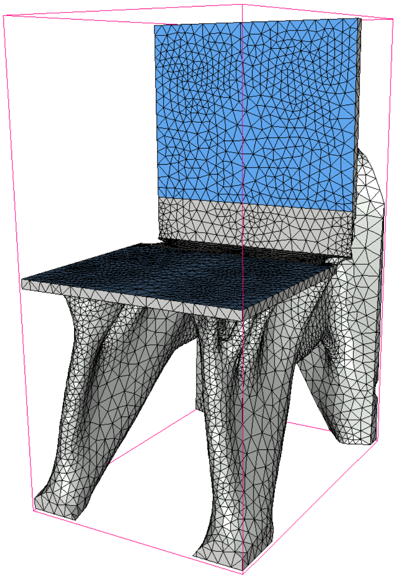
- We minimize the compliance of the total structure $\Omega \cup S$

$$C(S) := \int_{\Omega \cup S} Ae(u_{\Omega \cup S}) : e(u_{\Omega \cup S})\, \mathrm{d}x.$$

- A constraint is imposed on the volume $\mathrm{Vol}(S)$ of supports.



Optimization of supports in additive manufacturing (III)



The scaffold structure S of a fixed 3d chair Ω is optimized.

- The compliance of the total structure $\Omega \cup S$ is minimized

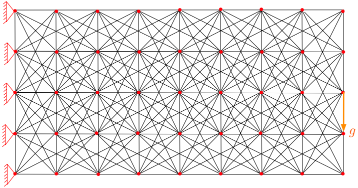
$$C(S) = \int_{\Omega \cup S} Ae(u_{\Omega \cup S}) : e(u_{\Omega \cup S}) \, dx.$$

- A constraint on the volume $\text{Vol}(S)$ of supports is enforced.

A "clever" initialization for truss structures (I)

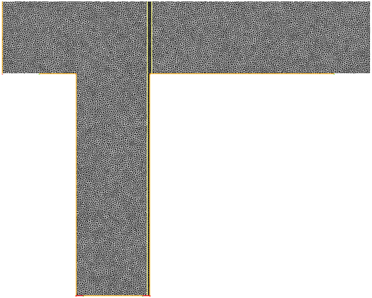
- Truss structures are collections of bars.
- Many truss optimization methods rely on the ground structure approach: an initial, dense network of bars is iteratively decimated.
- We propose instead to start from void and
  1. Incrementally add bars to the structure.
  2. (Optionally) Take on the optimization with a more "classical" boundary-variation algorithm.

Example of a truss structure.



Initialization of a truss algorithm by the ground structure approach.

A "clever" initialization for truss structures (II)



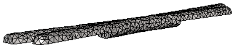
We consider the optimization of the shape of a 2d crane Ω.

- The compliance is minimized

$$C(\Omega) = \int_\Omega Ae(u_\Omega) : e(u_\Omega)\,\mathrm{d}x.$$

- A volume constraint is enforced.

A "clever" initialization for truss structures (III)



The shape of a 3d mast Ω is optimized.

- The compliance is minimized

$$C(\Omega) = \int_{\Omega} Ae(u_{\Omega}) : e(u_{\Omega}) \, \mathrm{d}x.$$

- A constraint is imposed on the volume of Ω.

Thank you for your attention!

References I

[AdDaFre] G. Allaire, C. Dapogny, and P. Frey, *Shape optimization with a level set based mesh evolution method*, Comput Methods Appl Mech Eng, 282 (2014), pp. 22–53.

[BeFran] E. Beretta and E. Francini, *An asymptotic formula for the displacement field in the presence of thin elastic inhomogeneities*, SIAM journal on mathematical analysis, 38 (2006), pp. 1249–1261.

[CapVo] Y. Capdeboscq and M.S. Vogelius, *A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction*, ESAIM: Mathematical Modelling and Numerical Analysis, 37(1), (2003), pp. 159–173.

[Da1] C. Dapogny, *A connection between topological ligaments in shape optimization and thin tubular inhomogeneities*, Comptes Rendus. Mathématique, 358(2), (2020), pp. 119–127.

[Da2] C. Dapogny, *The topological ligament in shape optimization: a connection with thin tubular inhomogeneities*, submitted, (2020), `Hal` preprint: `https://hal.archives-ouvertes.fr/hal-02924929/`.

References II

[HenPi] A. Henrot and M. Pierre, *Shape Variation and Optimization*, EMS Tracts in Mathematics Vol. 28, 2018.

[NaSo] S. Nazarov and J. Sokolowski, *The topological derivative of the dirichlet integral due to formation of a thin ligament*, Siberian Mathematical Journal, 45 (2004), pp. 341–355.

[NoSo] A. A. Novotny and J. Sokołowski, *Topological derivatives in shape optimization*, Springer Science & Business Media, 2012.