A DETERMINISTIC APPROXIMATION METHOD IN SHAPE OPTIMIZATION UNDER RANDOM UNCERTAINTIES: SUPPLEMENTARY MATERIAL

G. ALLAIRE [1], C. DAPOGNY[2]

[1] *Centre de Mathématiques Appliquées (UMR 7641), École Polytechnique 91128 Palaiseau, France.*
[2] *Laboratoire Jean Kuntzmann, CNRS, Université Joseph Fourier, Grenoble INP, Université Pierre Mendès France, BP 53, 38041 Grenoble Cedex 9, France.*

Contents

This short note contains additional material associated to our main article [1]. Section 1 presents three numerical examples which are complementary to those of the main article. Section 2 recalls classical definitions and results around the notions of differentiation with respect to the domain and in the field of tangential calculus, which are consistently used in the main article.

1. Additional numerical examples

1.1. Minimization of the mean value of a least-square criterion under material uncertainties.

In this first additional example, we illustrate the proposed model for dealing with random uncertainties over the elastic material's parameters, as described in Section 3.3 of [1]. The setting is that of the minimization of the thickness h of a plate with given cross-section Ω. More precisely, let us consider the situation depicted in Figure 1, associated to the optimization of a gripping mechanism. The considered plate is fixed on a part $\Gamma_D \subset \partial\Omega$, and surface loads $g \in L^2(\Gamma_N)^2$ are applied on another part $\Gamma_N \subset \partial\Omega$; g equals $(0,-1)$ on the upper part of Γ_N, and $(0,1)$ on its lower part.

Uncertainties $\widehat{E}$ are expected over the Young's modulus E of the material, namely $E = E_0 + \widehat{E}$ with $E_0 = 100$. As for the uncertainties $\widehat{E}$, they are characterized by the datum of their correlation function:

$$\forall x, y \in \Omega,\ \ \mathrm{Cor}(\widehat{E})(x,y) = \beta^2 e^{-\frac{|x-y|}{d}},$$

where the *correlation length* d is set to $d = 0.1$, and β is a parameter quantifying the magnitude of $\widehat{E}$. A Karhunen-Loève expansion is then performed, then truncated, so that $\widehat{E}$ takes a form suitable for computations:

$$\widehat{E}(x, \omega) = \sum_{i=1}^{3} \sqrt{\lambda_i} E_i(x) \xi_i(\omega),$$

where (λ_i, E_i) are the first three eigenpairs associated to the Hilbert-Schmidt operator induced by $\mathrm{Cor}(\widehat{E})$.

The cost function under consideration $\mathcal{C}(h, E)$ is a least-square criterion measuring the discrepancy between the solution $u_{h,E}$ to the linear elasticity system (see (3.1) in [1]), and a target displacement u_0:

$$\forall h \in \mathcal{U}_{ad}, \ \mathcal{C}(h, E) = \int_{\Gamma_T} |u_{h,E} - u_0|^2 \, ds,$$

where Γ_T is a non optimizable subset of $\partial\Omega$, disjoint from Γ_D and Γ_N. The target displacement is $u_0 = (0, -1)$ on the upper part of Γ_T, and $u_0 = (0, 1)$ on its lower part. The objective function at stake is then the approximate mean value $\widetilde{\mathcal{M}}(h)$ of the cost function $\mathcal{C}$ (see formula (3.37) in [1]).

We perform several tests of optimization of $\widetilde{\mathcal{M}}(h)$, for different values of β, without imposing any constraint on the volume of structures (since the functional at stake does not vary monotonically with it). The results are displayed on Figure 2, and the convergence histories are those of Figure 3.

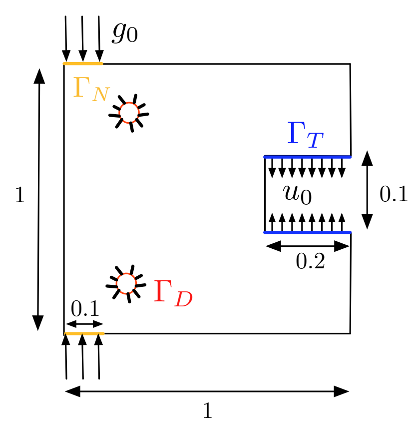

FIGURE 1. *Description of the gripping mechanism test case of Section* **??**.

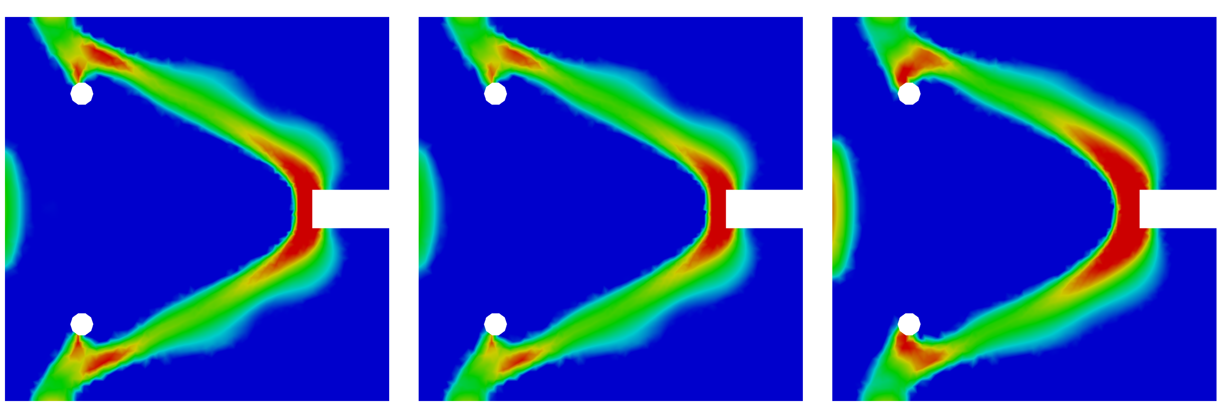

FIGURE 2. *(From left to right) Optimal shapes obtained in the gripping mechanism test case of Section 1.1, associated to the values* $\beta = 0, 100,$ *and* 10000.

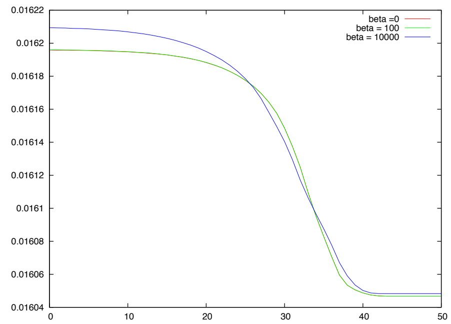

FIGURE 3. *Convergence history for the objective function in the gripping mechanism test case of Section 1.1 in* [1].

1.2. **Minimization of the mean value and of the variance of the compliance of a crane.**

Our second example deals with the geometric optimization of a crane - see Figure 4 for details - in the setting of geometric optimization. Body forces $f_0 = (0, -10)$ are applied on a region of the considered shapes Ω which is not subject to optimization (the red region on Figure 4), and perturbations $f_1, f_2 = (0, -m)$ are expected on two disjoint, non optimizable areas (the blue regions). For simplicity, surface loads are neglected ($g = 0$). The cost function of interest is the *compliance* $\mathcal{C}(\Omega, f)$ of shapes (see Formula (5.3) in [1]) , and we aim at minimizing the weighted sum

$$\mathcal{J}(\Omega) := \widetilde{\mathcal{M}}(\Omega) + \delta\sqrt{\widetilde{\mathcal{V}}(\Omega)} \tag{1.1}$$

of the approximate mean value $\widetilde{\mathcal{M}}(\Omega)$ and standard deviation $\sqrt{\widetilde{\mathcal{V}}(\Omega)}$ of $\mathcal{C}$ - see Section 5.2.2 in [1].

Several instances are performed, while imposing a target volume $V_T = 900$ on shapes. The corresponding results are represented in Figure 5.

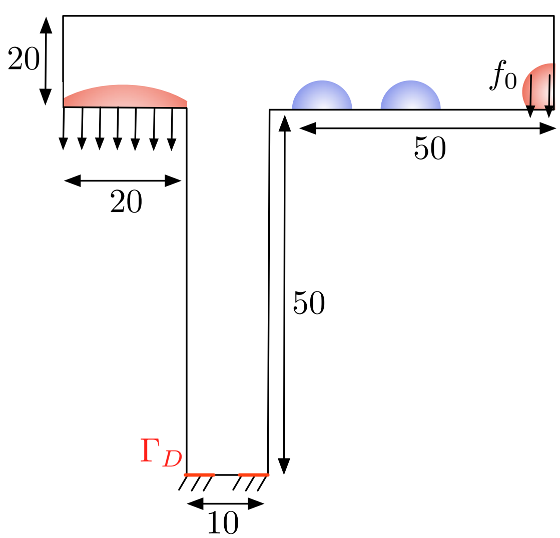

FIGURE 4. *Description of the crane test case of Section 1.2.*

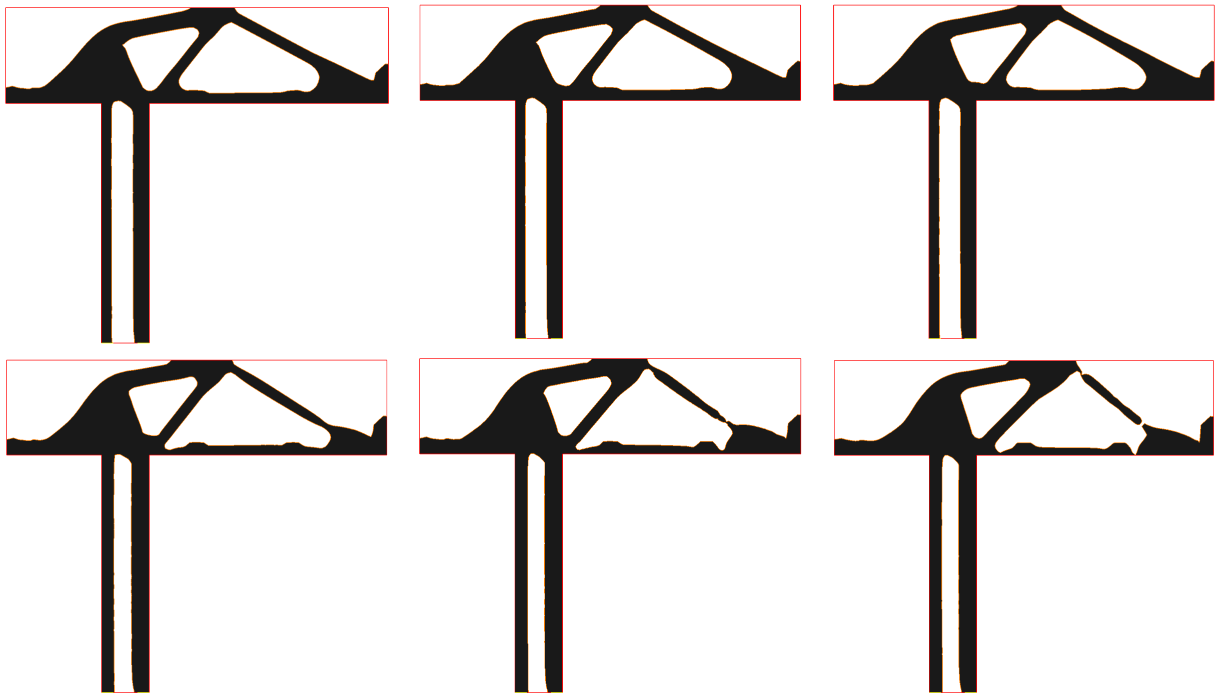

FIGURE 5. *Minimization of the objective function (1.1) for $m = 1$ (left column), $m = 2$ (middle column), and $m = 3$ (right column), and, from top to bottom, $\delta = 0, 5$.*

1.3. **Minimization of the stress of an L-shaped beam under uncertainties over its geometry.**

In this section, we complete the test case treated in Section 5.2.5 of [1]. The situation is identical to that discussed in the main article: in the context of geometric optimization, we aim at minimizing the following cost function, depending on the stress tensor of shapes:

$$\mathcal{C}(\Omega) = \int_\Omega ||\sigma(u_\Omega)||^5 \, dx. \tag{1.2}$$

Random perturbations are expected on the geometry of shapes, which are still of the form given by Formula (4.14) of [1], with scalar field $v(x,\omega)$ characterized by the correlation function (5.7), except that they now apply on the whole L-shaped box D, and not only on its inferior part D_p. Again, a Karhunen-Loève expansion of v is performed, and truncated after its first five terms, so that it is approximated as:

$$v(x,\omega) \approx \sum_{i=1}^{5} v_i(x)\xi_i(\omega),$$

where the functions v_i are represented on Figure 6, and the ξ_i are centered, normalized, and uncorrelated random variables. The weighted sum

$$\mathcal{J}(\Omega) = \mathcal{C}(\Omega) + \delta\sqrt{\widetilde{\mathcal{V}}(\Omega)} \tag{1.3}$$

of the unperturbed cost functional $\mathcal{C}(\Omega)$ and the approximate standard deviation $\sqrt{\widetilde{\mathcal{V}}(\Omega)}$ is considered as an objective function. Several examples of optimal shapes, associated to the target volume $V_T = 0.75$ and different values of the parameter δ, are represented on Figure 7.

Understandably enough, when compared to the corresponding shapes obtained in the main article (that is, in the situation where perturbations apply only on the inferior part of D), the present shapes show thicker bars on their superior parts.

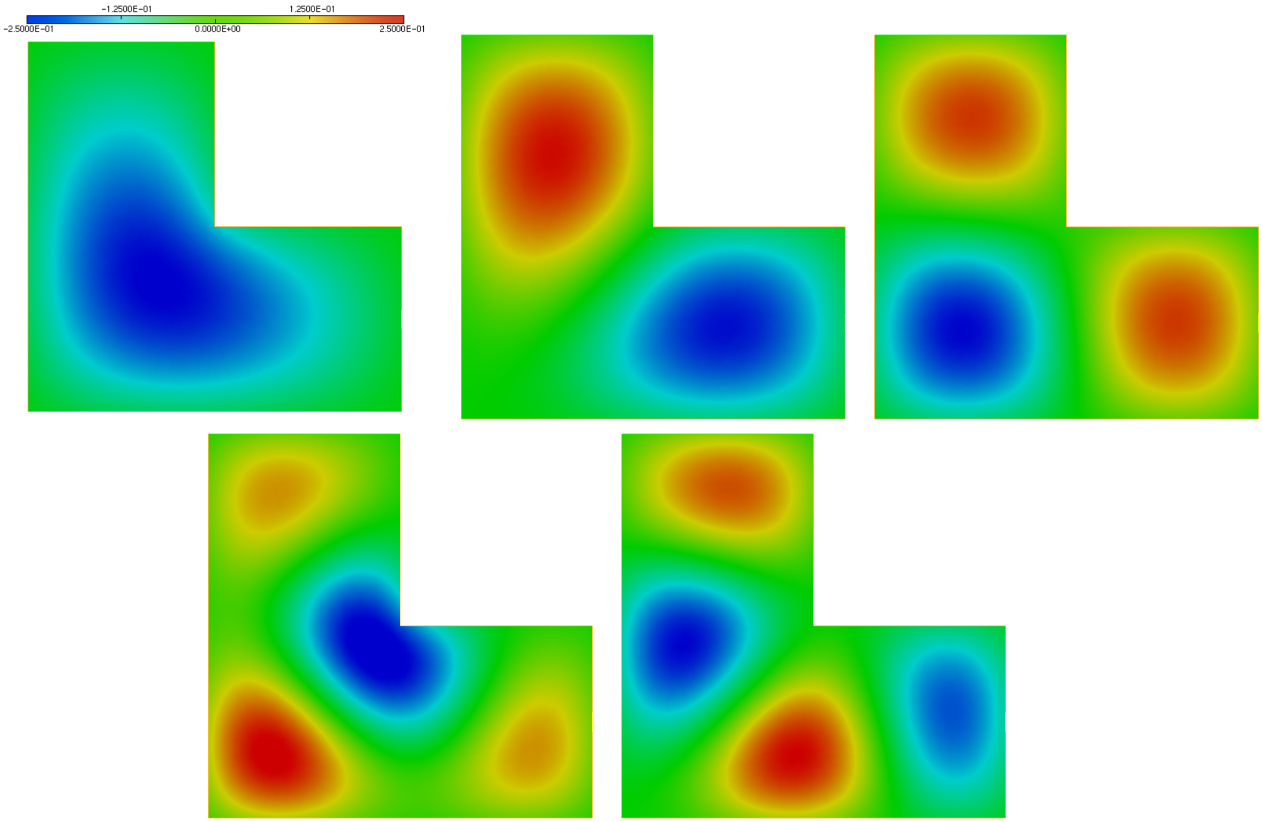

Figure 6. *Plots of the first five eigenfunctions v_i of the correlation (5.7) (in* [1]*), retained in the approximation of the random perturbation field $v(x, \omega)$.*

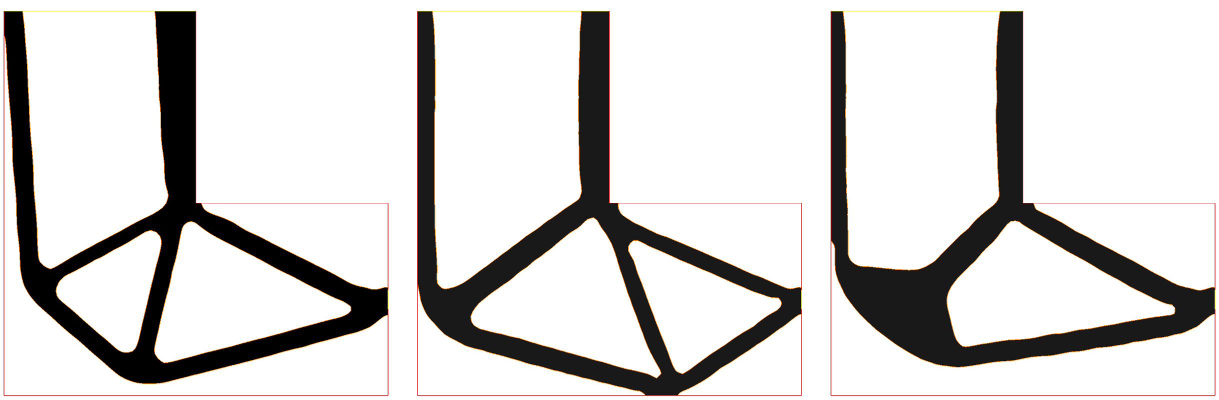

Figure 7. *Optimized L-shaped beams with respect to the stress criterion (1.2)-(1.3), where the parameter δ equals (from the left to the right) $0, 0.5, 2$.*

2. Some useful facts around shape derivatives and tangential calculus

In this section, we collect some material around the notion of differentiation with respect to the domain, and functions defined on codimension 1 submanifolds of $\mathbb{R}^d$. The case we have in mind is that of a (subset Σ of the) boundary Γ of a bounded domain $\Omega \subset \mathbb{R}^d$.

2.1. Extension of the normal vector field to a bounded domain.

Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain, and let $n_\Omega : \partial\Omega \to \mathbb{S}^{d-1}$ be the normal vector field to $\partial\Omega$, pointing outward Ω. The following result provides a 'natural' way to extend n_Ω to a neighborhood of $\partial\Omega$ (see e.g. [3], chapter 7, Thm. 3.1, 3.3 and [5], §5.4.4 for a proof).

Proposition 1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let d_Ω be its signed distance function; that is, d_Ω is defined as:*

$$\forall x \in \mathbb{R}^d,\ \ d_\Omega(x) = \begin{cases} d(x, \partial\Omega) & \text{if } x \in {}^c\overline{\Omega} \\ 0 & \text{if } x \in \partial\Omega \\ -d(x, \partial\Omega) & \text{if } x \in \Omega \end{cases},$$

where $d(\cdot, \partial\Omega)$ stands for the usual Euclidean distance function to $\partial\Omega$. Then:

- *d_Ω is differentiable at almost every point $x \in \mathbb{R}^d$, and its gradient at such a point x reads:*

  $$\nabla d_\Omega(x) = \frac{x - p_{\partial\Omega}(x)}{d_\Omega(x)},$$

  where $p_{\partial\Omega}(x)$ is the unique point $y \in \partial\Omega$ such that $|d_\Omega(x)| = |x - y|$. In particular,

  $$|\nabla d_\Omega(x)| = 1.$$

- *If Ω is additionnally of class $\mathcal{C}^k$, for some $k \geq 2$, then d_Ω is of class $\mathcal{C}^k$ on a neighborhood V of $\partial\Omega$. Its gradient then reads:*

  $$\forall x \in V,\ \ \nabla d_\Omega(x) = n_\Omega(p_{\partial\Omega}(x)).$$

  In particular, ∇d_Ω is an extension of the normal vector field n_Ω to V which is of unit norm.

2.2. Differential operations on functions defined on codimension 1 manifolds.

Definition 1. *Let $\Gamma \subset \mathbb{R}^d$ be an oriented $\mathcal{C}^2$ submanifold of $\mathbb{R}^d$, of dimension $(d-1)$.*

- *Let $f \in \mathcal{C}^1(\Gamma, \mathbb{R})$ be a function, and, for an arbitrary point $x \in \Gamma$, let $df(x) : T_x\Gamma \to \mathbb{R}$ be its differential. The tangential gradient $\nabla_\Gamma f$ of f is the (unique) vector field on Γ defined by the following identity:*

  $$\forall x \in \Gamma,\ \ \forall v \in T_x\Gamma,\ \ df(x)(v) = \langle \nabla_\Gamma f(x), v \rangle,$$

  where $\langle \cdot, \cdot \rangle$ stands for the Euclidean scalar product on $\mathbb{R}^d$. Alternatively, $\nabla_\Gamma f$ may be defined as:

  $$\nabla_\Gamma f = \left[\nabla \widetilde{f}\right]_\Gamma := \nabla \widetilde{f} - (\nabla \widetilde{f} \cdot n)n,$$

  where $\widetilde{f}$ is any $\mathcal{C}^1$ extension of f to a neighborhood of Γ.
- *The tangential divergence* $\text{div}_\Gamma(V) : \Gamma \to \mathbb{R}$ *of a vector field $V \in \mathcal{C}^1(\Gamma, \mathbb{R}^d)$ is the function:*

  $$\text{div}_\Gamma(V) = \text{div}(\widetilde{V}) - \nabla \widetilde{V} n \cdot n,$$

  where $\widetilde{V}$ is any $\mathcal{C}^1$ extension of V to a neighborhood of Γ.
- *Let $\sigma : \Gamma \to \mathcal{S}(\mathbb{R}^d)$ be a tensor field defined on Γ. The tangential part $\sigma_{\tau\tau}$ of σ is the symmetric bilinear form on the tangent bundle $T\Gamma$ satisfying:*

  $$\forall x \in \Gamma,\ \ \forall v, w \in T_x\Gamma,\ \ \sigma_{\tau\tau}(x)(v, w) = \sigma(x)(v, w).$$

  In a local orthonormal basis (τ, n) of $\mathbb{R}^d$ obtained by gathering $(d-1)$ unit tangent vectors to Γ (collectively denoted by τ) and the normal vector n, σ may be expressed as:

  $$\sigma = \begin{pmatrix} \sigma_{\tau\tau} & \sigma_{\tau n} \\ \sigma_{n\tau} & \sigma_{nn} \end{pmatrix}$$

  Under the additional assumption that $\sigma \in \mathcal{C}^1(\Gamma, \mathcal{S}(\mathbb{R}^d))$, the tangential divergence $\text{div}_\Gamma(\sigma) : \Gamma \to \mathbb{R}^d$ *of σ is the vector field whose coordinates read:*

  $$(\text{div}_\Gamma(\sigma))_i = \left[\text{div}\left((\sigma_{i,j})_{j=1,\dots,d}\right)\right]_\Gamma,\ \ i = 1, ..., d.$$

2.3. **Green's formulae on codimension 1 submanifolds.**

The following Green's formulae are variants of the integration by parts formula on boundaries in [5] (Prop. 5.4.9). Their proofs lie e.g. in [2] (see Prop. 5.3 and 5.4):

Proposition 2. *Let $\Gamma \subset \mathbb{R}^d$ be a, compact, oriented $\mathcal{C}^2$ submanifold of dimension $(d-1)$ with possibly empty boundary Σ; let $n : \Gamma \to \mathbb{S}^{d-1}$ be the unit normal vector field to Γ corresponding to the prescribed orientation of Γ, and denote as $n_\Sigma : \Sigma \to \mathbb{S}^{d-1}$ the unit normal vector field to Σ, pointing outward Γ. Let ds (resp. $d\ell$) be the volume form on Γ (resp. on Σ), and κ be the mean curvature of Γ.*

(1) Let $f \in W^{2,1}(\Gamma, \mathbb{R})$ and $V \in W^{2,1}(\Gamma, \mathbb{R}^d)$; the following identity holds:

$$\int_\Gamma V \cdot \nabla_\Gamma f \, ds = \int_\Sigma f \, V \cdot n_\Sigma \, d\ell + \int_\Gamma \left(-f \mathrm{div}_\Gamma(V) + \kappa f V \cdot n\right) \, ds.$$

(2) Let $u \in W^{2,1}(\Gamma, \mathbb{R}^d)$ and $\sigma \in W^{2,1}(\Gamma, \mathcal{S}(\mathbb{R}^d))$; the following identity holds:

$$\int_\Gamma \sigma_{\tau\tau} : e(u)_{\tau\tau} \, ds = \int_\Sigma [u]_\Gamma \cdot (\sigma_{\tau\tau} \cdot n_\Sigma) \, d\ell - \int_\Gamma [u]_\Gamma \cdot \mathrm{div}_\Gamma(\sigma) \, ds.$$

2.4. **Lagrangian and Eulerian derivatives.**

We presently collect some classical definitions and notations around the notions of *Lagrangian and Eulerian* derivatives of an application $\Omega \mapsto u(\Omega)$ which, to a domain Ω, associates a function defined on Ω; see [3, 8] for further details.

Definition 2. *Let $k \geq 1$, $m, p \in \mathbb{R}$ be fixed, and let $u : \Omega \mapsto u(\Omega)$ be a mapping which, to a bounded domain $\Omega \subset \mathbb{R}^d$ of class $\mathcal{C}^k$ associates a function $u(\Omega) \in W^{m,p}(\Omega)$.*

- *The function $\Omega \mapsto u(\Omega)$ has a Lagrangian (or material) derivative $\dot{u}(\Omega)$ at a given domain Ω provided the transported application:*

$$\mathcal{C}^{k,\infty}(\mathbb{R}^d, \mathbb{R}^d) \ni \theta \mapsto u(\Omega_\theta) \circ (I + \theta) \in W^{m,p}(\Omega),$$

is Fréchet differentiable at $\theta = 0$; $\theta \mapsto \dot{u}(\Omega)(\theta)$ is then defined as the corresponding Fréchet derivative.
- *The function u has a Eulerian derivative $u'(\Omega)(\theta)$ at a given domain Ω in the direction $\theta \in \mathcal{C}^{k,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ if it has a Lagrangian derivative at Ω, and $\nabla u(\Omega) \cdot \theta \in W^{m,p}(\Omega)$. One defines then:*

$$u'(\Omega)(\theta) = \dot{u}(\Omega)(\theta) - \nabla u(\Omega) \cdot \theta \in W^{m,p}(\Omega).$$

The definition is similar when it comes to mappings $\Gamma \mapsto u(\Gamma)$ which, to the boundary Γ of a domain associate a function $u(\Gamma)$ defined on Γ itself.

Definition 3. *Let $k \geq 1$, $m, p \in \mathbb{R}$ be fixed, and let $u : \Gamma \mapsto u(\Gamma)$ be a mapping which, to the boundary Γ of a bounded domain $\Omega \subset \mathbb{R}^d$ of class $\mathcal{C}^k$ associates $u(\Gamma) \in W^{m,p}(\Gamma)$.*

- *u has a Lagrangian (or material) derivative $\dot{u}(\Gamma)$ at Γ provided the transported application:*

$$\mathcal{C}^{k,\infty}(\mathbb{R}^d, \mathbb{R}^d) \ni \theta \mapsto u(\Gamma_\theta) \circ (I + \theta) \in W^{m,p}(\Gamma),$$

is Fréchet differentiable at $\theta = 0$.
- *u has a Eulerian derivative $u'(\Gamma)(\theta)$ at Γ in the direction $\theta \in \mathcal{C}^{k,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ if it has a Lagrangian derivative at Γ, and $\nabla_\Gamma u(\Gamma) \cdot \theta \in W^{m,p}(\Gamma)$. One defines then:*

$$u'(\Gamma)(\theta) = \dot{u}(\Gamma)(\theta) - \nabla_\Gamma u(\Gamma) \cdot \theta \in W^{m,p}(\Gamma).$$

2.5. **Shape derivatives of some geometric quantities depending on the domain.**

The first result of interest in this section is the following formula for the Lagrangian derivative of the normal vector field n_{Ω_θ} (see e.g. [7]), whose proof is outlined for the sake of convenience.

Lemma 3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $\mathcal{C}^2$. The Lagrangian derivative of the normal vector field $\mathcal{C}^{2,\infty}(\mathbb{R}^d, \mathbb{R}^d) \ni \theta \mapsto n_{\Omega_\theta}$ reads:*

$$\forall y \in \partial\Omega, \quad \frac{d}{d\theta}(n_{\Omega_\theta}((I+\theta)(y)))\bigg|_{\theta=0} = \nabla n(y)^T \cdot \theta(y) - \nabla_\Gamma(\theta \cdot n)(y).$$

What's more, the asymptotic expansion around $\theta = 0$ corresponding to the above Fréchet derivative holds uniformly with respect to $y \in \partial\Omega$.

Proof. The proof relies on the following formula, for $\theta \in \mathcal{C}^{2,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ small enough:

$$n_{\Omega_\theta}((I+\theta)(y)) = \frac{\mathrm{com}(I+\nabla\theta(y))n(y)}{|\mathrm{com}(I+\nabla\theta(y))n(y)|}. \tag{2.1}$$

Taking the derivative at $\theta = 0$ in the well-known matrix identity

$$(I+\nabla\theta(y))^T \mathrm{com}(I+\nabla\theta(y)) = \det(I+\nabla\theta(y))I,$$

(which makes sense since all the terms involved are polynomial expressions in θ), we obtain:

$$\frac{d}{d\theta}\mathrm{com}(I+\nabla\theta(y))\bigg|_{\theta=0} = \mathrm{div}(\theta)(y)I - \nabla\theta(y)^T.$$

Now, straightforward calculations lead to:

$$\begin{array}{rcl} \frac{d}{d\theta}(n_{\Omega_\theta}((I+\theta)(y)))\big|_{\theta=0} & = & (\nabla\theta(y)^T n(y) \cdot n(y))n(y) - \nabla\theta(y)^T n(y) \\ & = & \nabla n(y)^T \cdot \theta(y) - \nabla_\Gamma(\theta \cdot n)(y) \end{array}.$$

□

Let us eventually state and prove the following result, which is a slight variation of Thm. in [5]:

Lemma 4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $\mathcal{C}^2$, and $\theta \in \mathcal{C}^{2,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Let $V \in W^{2,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ be a vector field, and let $J(\Omega)$ be the function of the domain defined by:*

$$J(\Omega) = \int_\Gamma V \cdot n_\Omega \, ds.$$

Then, $J(\Omega)$ is shape differentiable at Ω with shape derivative:

$$J'(\Omega)(\theta) = \int_\Gamma (\nabla(V \cdot n) \cdot \theta - [V]_\Gamma \cdot \nabla_\Gamma(\theta \cdot n) - [\theta]_\Gamma \cdot \nabla_\Gamma(V \cdot n) + \kappa(V \cdot n)(\theta \cdot n)) \, ds.$$

Proof. Performing a change of variables in the surface integral defining $J(\Omega)$ yields (see e.g. [5], Prop. 5.4.3):

$$J(\Omega) = \int_\Gamma V((I+\theta)(y)) \cdot n_{\Omega_\theta}((I+\theta)(y)) |\mathrm{com}(I+\nabla\theta(y))n(y)| \, ds(y).$$

Now, we know that the mapping $f : \theta \mapsto V \circ (I+\theta) \in L^2(\Gamma)^d$ is Fréchet-differentiable at $\theta = 0$, with derivative:

$$f'(0)(\theta) = \nabla V \theta,$$

that $g : \theta \mapsto n_{\Omega_\theta}(y + \theta(y)) \in \mathcal{C}(\Gamma)$ is Fréchet-differentiable at $\theta = 0$ with derivative:

$$g'(0)(\theta) = \nabla n^T \cdot \theta - \nabla_\Gamma(\theta \cdot n),$$

and that $h : \theta \mapsto |\mathrm{com}(I+\nabla\theta)n| \in \mathcal{C}(\Gamma)$ is Fréchet-differentiable at $\theta = 0$ with derivative:

$$h'(0)(\theta) = \mathrm{div}_\Gamma(\theta).$$

Hence, J is shape differentiable at Ω, and its shape derivative reads:

$$J'(\Omega)(\theta) = \int_\Gamma ((\nabla V \theta) \cdot n + V \cdot (\nabla n^T \cdot \theta - \nabla_\Gamma(\theta \cdot n)) + V \cdot n \, \mathrm{div}_\Gamma(\theta)) \, ds.$$

Eventually, using Proposition 2, (1), and rearranging the last expression yield the desired formula. □

Let us end this section with the following result around the Lagrangian and Eulerian derivatives of the mean curvature κ. This formula was observed in [6] (with a different proof than the one presented here), and in [4] in a more general context.

Lemma 5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $\mathcal{C}^3$. For any $y \in \partial\Omega$, the Lagrangian derivative of the mean curvature $\mathcal{C}^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d) \ni \theta \mapsto \kappa_{\Omega_\theta}$ reads:*

$$\forall y \in \partial\Omega, \quad \frac{d}{d\theta}\left(\kappa_{\Omega_\theta}((I+\theta)(y))\right)\Big|_{\theta=0} = -\Delta_\Gamma(\theta\cdot n) + \nabla_\Gamma\kappa\cdot\theta,$$

and the associated first-order expansion in the neighborhood of $\theta = 0$ holds uniformly in $y \in \partial\Omega$. Consequently, the Eulerian derivative is:

$$\forall y \in \partial\Omega, \quad \frac{d}{d\theta}\left(\kappa_{\Omega_\theta}(y)\right)\Big|_{\theta=0} = -\Delta_\Gamma(\theta\cdot n).$$

Proof. The second fundamental form of the surface $\partial\Omega_\theta$ at point $(I+\theta)(y)$ is by definition $\nabla n_{\Omega_\theta}((I+\theta)(y))$, and satisfies the identity:

$$\nabla n_{\Omega_\theta}((I+\theta)(y))(I+\nabla\theta(y)) = \nabla\left(n_{\Omega_\theta}((I+\theta)(y))\right).$$

Now differentiating the above identity with respect to θ, at $\theta = 0$, and taking the trace of the resulting identity, we obtain:

$$\begin{aligned}\tfrac{d}{d\theta}\left(\kappa_{\Omega_\theta}((I+\theta)(y))\right)\big|_{\theta=0} + \text{tr}\left(\nabla n(y)\nabla\theta(y)\right) &= \text{tr}\left(\tfrac{d}{d\theta}\left(\nabla(n_{\Omega_\theta}((I+\theta)(y)))\right)\big|_{\theta=0}\right)\\ &= \text{tr}\left(\nabla\left(\tfrac{d}{d\theta}(n_{\Omega_\theta}((I+\theta)(y)))\big|_{\theta=0}\right)\right),\end{aligned}$$

where the last line stems from the commutation of the derivatives with respect to θ and y of n (because of the above formula (2.1)). Now using Lemma 3 produces:

$$\frac{d}{d\theta}\left(\kappa_{\Omega_\theta}((I+\theta)(y))\right)\Big|_{\theta=0} = -\nabla n(y) : \nabla\theta(y) + \text{div}\left(\nabla n(y)^T\cdot\theta(y) - \nabla_\Gamma(\theta\cdot n)(y)\right).$$

Now remarking that, since the vector field $\nabla_\Gamma(\theta\cdot n)$ is tangential,

$$\text{div}\left(\nabla_\Gamma(\theta\cdot n)(y)\right) = \text{div}_\Gamma\left(\nabla_\Gamma(\theta\cdot n)(y)\right),$$

and the identity:

$$\begin{aligned}\text{div}\left(\nabla n(y)^T\cdot\theta(y)\right) &= \text{div}\left(\nabla n(y)^T\right)\cdot\theta(y) + \nabla n(y) : \nabla\theta(y)\\ &= \nabla_\Gamma(y)\cdot\theta(y) + \nabla n(y) : \nabla\theta(y),\end{aligned}$$

the desired result follows. ☐

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