Robust and distributionally robust shape and topology optimization

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Foreword: Shape and topology optimization

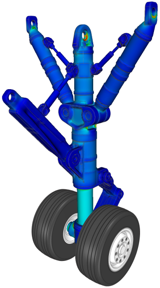
- Shape optimization aims to minimize a function of the domain.

- Such problems can be traced back to the early human history...

- ... The needs to realize energy savings and get free from fossile fuels have aroused much enthusiasm for the discipline.

- It finds applications in varied physical contexts, such as:
  - *Structure mechanics:* Industrial components, civil engineering (buildings, beams),...
  - *Fluid mechanics:* External aerodynamics (aircrafts), heat exchangers, ...
  - *Electromagnetism:* Electric machines (motors), nanooptics (photonic devices), ...

Hooke's principle: "As hangs the flexible chain, so but inverted stands the rigid arch".



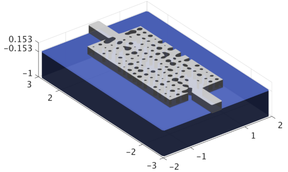
Optimized design of a landing gear (courtesy of Ansys).

Foreword: optimal design and robustness

- The mathematical description of realistic systems involves physical parameters, e.g.
  - *In structure mechanics:* loads, elastic moduli.
  - *In fluid mechanics:* viscosity, density of the fluid.
- These are often known imperfectly, because either
  - They are measured or estimated,
  - They become altered through use, or due to wear.
- The optimal character of a design is very sensitive to the parameters describing its environment,

  $\Rightarrow$ Need for "robust" optimal design formulations.
- All the formulations of this requirement suffer from technical, or conceptual flaws.
- We present three robust optimal design paradigms, depending on the knowledge about uncertainties.

Turbine blades operate under uncertain load and temperature conditions.



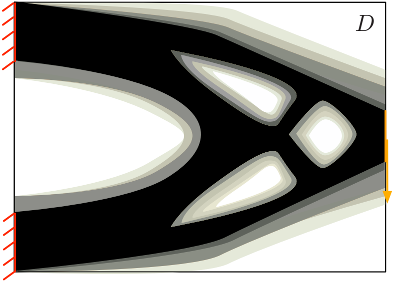
The wavelength of the light injected in a nanophotonic device is uncertain.

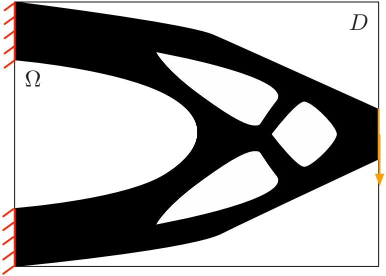
Contents

A generic, abstract optimal design setting (I)

- The design h is sought within a set $\mathcal{U}_{\rm ad}$:
  - $h : D \to [0, 1]$ may be a "grayscale" density function, defined on a large "hold-all" domain $D \subset \mathbb{R}^d$;
  - h may be a "black-and-white" shape $\Omega \subset D$.
- The physical parameters are aggregated into an element ξ in a set Ξ,

When h is an elastic structure, ξ may represent the loads applied on h, or the material parameters (Young's modulus, Poisson's ratio).





A generic, abstract optimal design setting (II)

- The cost of a design h when the parameters ξ are at play is $\mathcal{C}(h, \xi)$.

  When h is a structure, $\mathcal{C}(h, \xi)$ may be the compliance of h.

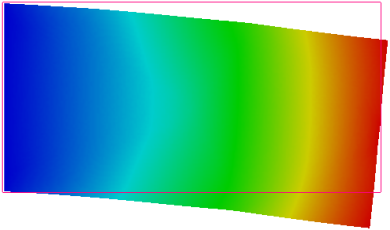
- In applications, this cost is often of the form:

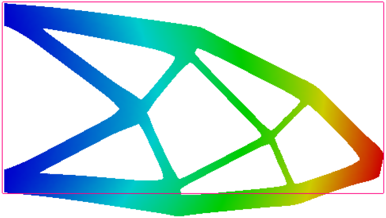
$$\mathcal{C}(h, \xi) = S(\xi, u_{h,\xi}),$$

  where the state $u_{h,\xi} \in V$ is the solution to a ξ-dependent system, say, for simplicity:

$$\mathcal{A}(h)u_{h,\xi} = b(\xi).$$

  When h is an elastic structure, $\mathcal{A}(h)$ is the linear elasticity operator and $u_{h,\xi}$ is the displacement of h.





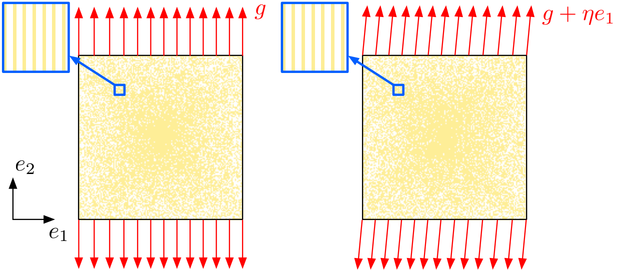
A generic, abstract optimal design setting (III)

- The ideal optimal design problem, when $\xi \equiv \xi^0$ is known perfectly, reads:

$$\min_{h \in \mathcal{U}_{\text{ad}}} \mathcal{C}(h, \xi^0),$$

where constraints are omitted for simplicity.

- The optimal character of a design is strongly dependent on the actual value of ξ.



The elastic microstructure withstanding a vertical traction load ξ^0 with minimum compliance $\mathcal{C}(h, \xi^0)$ is also the worst one when an arbitrarily small horizontal component is added to ξ^0: $\mathcal{C}(h, \xi^0 + \eta e_1) = \infty$ [CheChe].

$\Rightarrow$ Need for a means to incorporate "robustness" into the optimal design problem.

The worst-case design approach (I)

Joint work with G. Allaire

- We assume minimal knowledge about the uncertain parameters ξ.
  - The value ξ^0 of ξ under ideal conditions.
  - A maximum bound m on the difference between ξ and ξ^0.

- Robustness with respect to ξ is then enforced via a worst-case formulation:

$$\min_{h \in \mathcal{U}_{\text{ad}}} J_{wc}(h), \text{ where } J_{wc}(h) = \sup_{||\xi - \xi^0||_{\Xi} \leq m} \mathcal{C}(h, \xi).$$

- This difficult min-max bilevel program is intractable, except in a few, very particular situations [AmCi, DeGAlJou].

- Assuming the amplitude m of the uncertainties to be "small", it can be given formal, approximate counterparts.

The worst-case design approach (II)

Idea: Linearize the cost function around the ideal value ξ^0 of the data:

$$\text{For } \xi = \xi^0 + \widehat{\xi}, \text{ with ``small'' } \widehat{\xi}, \quad \mathcal{C}(h,\xi) \approx \mathcal{C}(h,\xi^0) + \frac{\partial \mathcal{C}}{\partial \xi}(h,\xi^0)(\widehat{\xi}).$$

- We then define a (formal) approximate worst-case functional $\widetilde{J_{\text{wc}}}(h)$ by:

$$\widetilde{J_{\text{wc}}}(h) = \sup_{||\widehat{\xi}||_{\Xi} \leq m} \left(\mathcal{C}(h,\xi^0) + \frac{\partial \mathcal{C}}{\partial \xi}(h,\xi^0)(\widehat{\xi}) \right).$$

- The above supremum of an affine mapping over a ball rewrites:

$$\widetilde{J_{\text{wc}}}(h) = \mathcal{C}(h,\xi^0) + m \left|\left| \frac{\partial \mathcal{C}}{\partial \xi}(h,\xi^0) \right|\right|_{\Xi^*},$$

  where Ξ^* is the dual space Ξ.

- This expression involves the awkward derivative of the mapping

$$\xi \longmapsto \mathcal{C}(h,\xi) = S(\xi, u_{h,\xi}).$$

- This derivative can be calculated thanks to the adjoint method.

The worst-case design approach (III)

- The chain rule yields:

$$\frac{\partial \mathcal{C}}{\partial \xi}(h,\xi^0)(\widehat{\xi}) = \frac{\partial S}{\partial \xi}(\xi^0, u_{h,\xi^0})(\widehat{\xi}) + \left\langle \frac{\partial S}{\partial u}(\xi^0, u_{h,\xi^0}), u^1_{h,\xi^0}(\widehat{\xi}) \right\rangle_{V^*,V},$$

where the derivative $u^1_{h,\xi^0}(\widehat{\xi}) := \frac{\partial u_{h,\xi}}{\partial \xi}\Big|_{\xi=\xi^0}(\widehat{\xi})$ is the solution to:

$$\mathcal{A}(h)u^1_{h,\xi^0}(\widehat{\xi}) = \frac{\partial b}{\partial \xi}(\xi^0)(\widehat{\xi}).$$

- We now define the adjoint state $p_{h,\xi^0} \in V$ as the solution to the problem:

$$\mathcal{A}(h)^T p_{h,\xi^0} = -\frac{\partial S}{\partial u}(\xi^0, u_{h,\xi^0}).$$

- A classical adjoint-based computation yields:

$$\begin{aligned}\frac{\partial}{\partial \xi}\left(S(\xi, u_{h,\xi})\right)\Big|_{\xi=\xi^0}(\widehat{\xi}) &= \frac{\partial S}{\partial \xi}(\xi^0, u_{h,\xi^0})(\widehat{\xi}) - \left\langle \mathcal{A}(h)^T p_{h,\xi^0}, u^1_{h,\xi^0}(\widehat{\xi})\right\rangle_{V^*,V},\\ &= \frac{\partial S}{\partial \xi}(\xi^0, u_{h,\xi^0})(\widehat{\xi}) - \left\langle \frac{\partial b}{\partial \xi}(\xi^0)^T p_{h,\xi^0}, \widehat{\xi}\right\rangle_{\Xi^*,\Xi},\end{aligned}$$

- Finally:

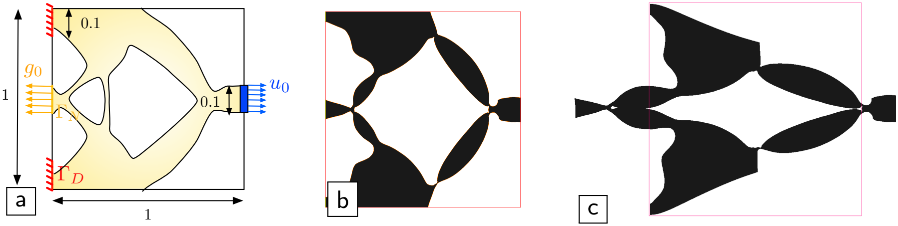
$$\boxed{\widetilde{J_{\text{wc}}}(h) = \mathcal{C}(h,\xi^0) + m\left|\left|\frac{\partial S}{\partial \xi}(\xi^0, u_{h,\xi^0}) - \frac{\partial b}{\partial \xi}(\xi^0)^T p_{h,\xi^0}\right|\right|_{\Xi^*}.}$$

A numerical example (I)

- We consider the optimization of the shape of a force inverter:

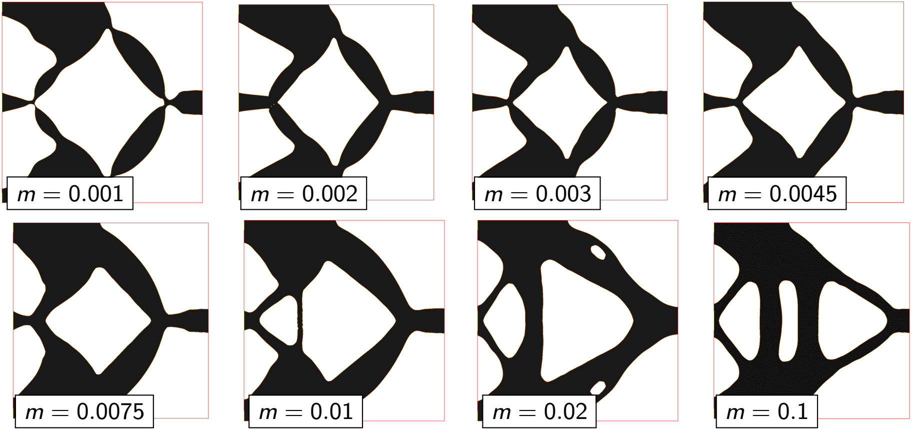
$$\min_{\Omega} \mathcal{C}(\Omega,\xi), \text{ where } \mathcal{C}(\Omega,\xi) = \int_{\Omega} \chi(x) |u_{\Omega,\xi} - u_T|^2 \, dx.$$

- The uncertain parameters ξ are the Lamé coefficients of the elastic material.
- The state $u_{\Omega,\xi}$ is the elastic displacement of Ω.



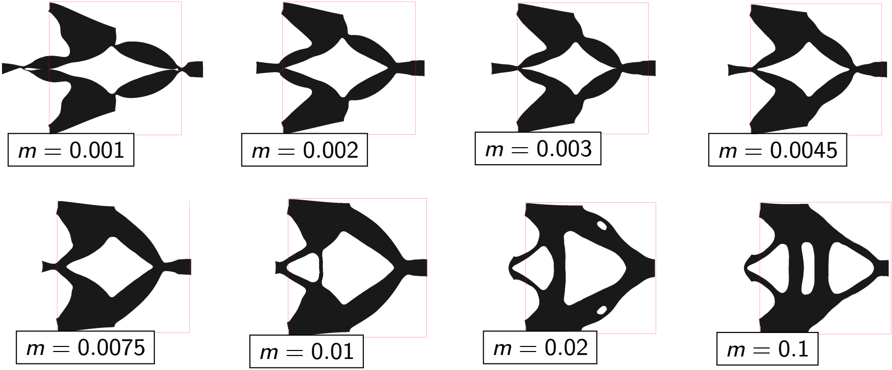
(a) Setting of the problem; (b) Optimized shape without perturbations; (c) Displacement of this design.

A numerical example (II)



Optimized shapes of the inverter.

A numerical example (III)



Deformed configurations of the optimized inverter designs.

The worst-case design approach: conclusions

Assets of this approach:

- It rests on minimal assumptions about the uncertain parameters ξ.

Drawbacks of this approach:

- The general bi-level optimization formulation is difficult and costly; it can often be addressed only via a (coarse) approximation.

- Worst-case formulations often lead to "pessimistic" designs, showing poor nominal performance for the sake of anticipating an unlikely worst-case scenario.

- (For simplicity) the uncertain parameter ξ lies in a finite-dimensional set $\Xi \subset \mathbb{R}^k$.
- Probabilistic approaches rest on the law of ξ, as a probability measure $\mathbb{P} \in \mathcal{P}(\Xi)$:

$$\forall A \subset \Xi, \text{ the probability that } \xi \text{ belong to } A \text{ is } \mathbb{P}\Big\{\xi \in A\Big\} = \int_A \mathrm{d}\mathbb{P}(\xi).$$

- The mean value $\xi^0 \in \mathbb{R}^k$ of ξ is:

$$\xi^0 = \int_\Xi \xi \, \mathrm{d}\mathbb{P}(\xi).$$

- The covariance matrix $\Sigma^0 \in \mathbb{S}^k_+$ of ξ is:

$$\Sigma^0 = \int_\Xi (\xi - \xi^0) \otimes (\xi - \xi^0) \, \mathrm{d}\mathbb{P}(\xi).$$

- **Assumption:** The components of ξ are uncorrelated random variables:

$$\Sigma^0_{ij} = 0 \text{ whenever } i \neq j.$$

Probabilistic optimal design (II)

- Probabilistic optimal design problems feature a statistical quantity of $\mathcal{C}(h,\xi)$, e.g.
  - The mean value of the cost $\mathcal{C}(h,\xi)$:
$$J_{\text{mean}}(h) := \int_{\Xi} \mathcal{C}(h,\xi)\, \mathrm{d}\mathbb{P}(\xi).$$
  - Other statistical quantities of $\mathcal{C}(h,\xi)$ such as its variance:
$$J_{\text{var}}(h) = \int_{\Xi} \left(\mathcal{C}(h,\xi) - J_{\text{mean}}(h)\right)^2 \, \mathrm{d}\mathbb{P}_{\textbf{true}}(\xi).$$
  - A probability of failure:
$$J_{\text{fail}}(h) := \mathbb{P}\Big\{\xi \in \Xi \text{ s.t. } \mathcal{C}(h,\xi) > C_T\Big\},$$
    where C_T is a safety threshold.
- These quantities and their derivatives are typically evaluated by very costly Monte-Carlo methods, stochastic collocation algorithms, ...

Probabilistic optimal design (III)

- **Idea:** Assuming that the discrepancy $\xi - \xi^0$ is "small", approximate quantities can be obtained by linearization of $\xi \mapsto \mathcal{C}(h, \xi)$:

$$\mathcal{C}(h,\xi) \approx \mathcal{C}(h,\xi^0) + \frac{\partial \mathcal{C}}{\partial \xi}(h,\xi^0)(\xi - \xi^0) + \frac{1}{2}\frac{\partial^2 \mathcal{C}}{\partial \xi^2}(h,\xi^0)(h, \xi - \xi^0, \xi - \xi^0).$$

- For instance, $J_{\mathrm{mean}}(h)$ is approximated by:

$$\widetilde{J_{\mathrm{mean}}}(h) \approx \mathcal{C}(h,\xi^0) + \sum_{i=1}^{k} \frac{\partial \mathcal{C}}{\partial \xi_i}(\xi^0) \underbrace{\int_{\Xi} (\xi_i - \xi_i^0)\, \mathrm{d}\mathbb{P}(\xi)}_{=0}$$

$$+ \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 \mathcal{C}}{\partial \xi_i \partial \xi_j}(\xi^0) \underbrace{\int_{\Xi} (\xi_i - \xi_i^0)(\xi_j - \xi_j^0)\, \mathrm{d}\mathbb{P}(\xi)}_{=\Sigma^0_{ij}}.$$

- Such quantities can be handled by "standard" optimal design algorithms.

- The cost function is the compliance of shapes:

$$\mathcal{C}(\Omega, \xi) = \int_\Omega \xi \cdot u_{\Omega,\xi} \, dx = \int_\Omega Ae(u_{\Omega,\xi}) : e(u_{\Omega,\xi}) \, dx.$$

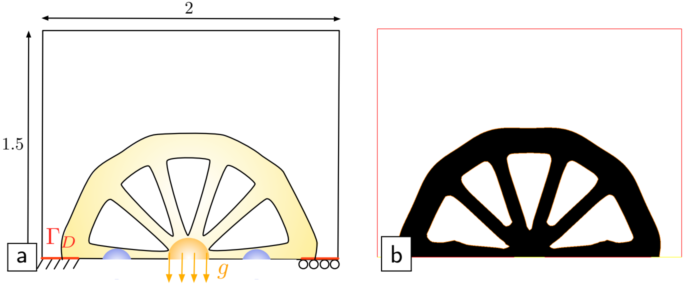
- The law $\mathbb{P}$ of the uncertain loads reads:

$$\mathbb{P} = \delta_{\xi^1} + \delta_{\xi^2},$$

where the two load scenarii $\xi^1, \xi^2 = (0, -m)$ are supported in the blue spots.

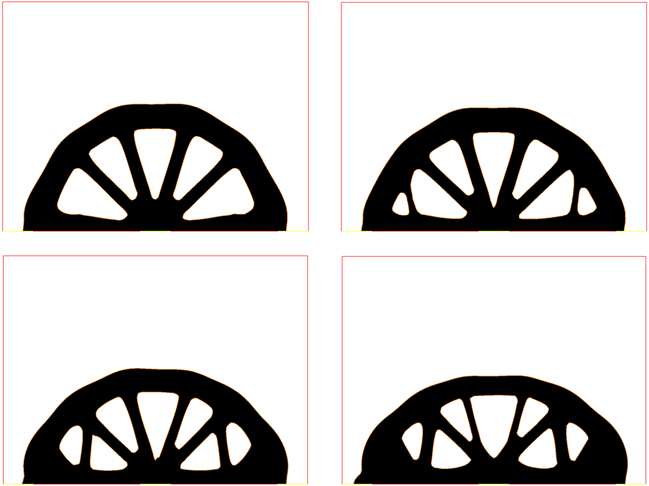
- We solve the problem:

$$\min_\Omega \left(\widetilde{J_{\text{mean}}}(h) + \delta \widetilde{J_{\text{var}}}(h) \right) \text{ s.t. } \text{Vol}(\Omega) = V_T.$$



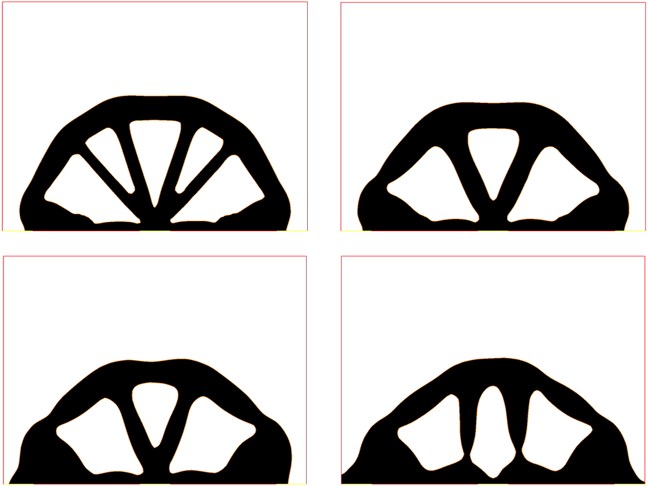
(a) The bridge test case; (b) optimal shape in the unperturbed situation.

Optimization of a bridge under random loads (II)



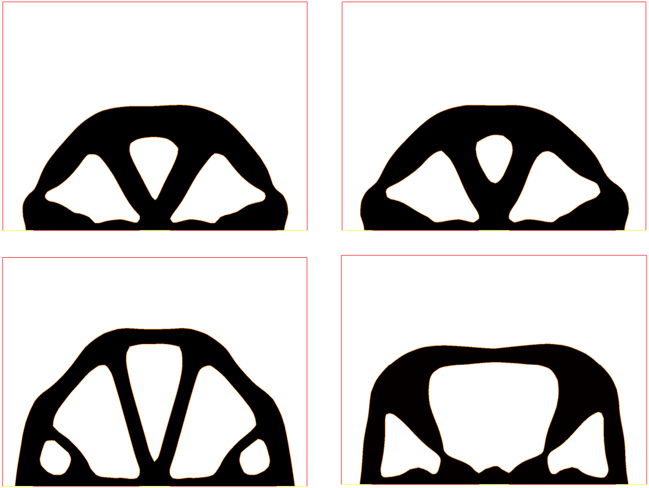
Optimized shapes for $\delta = 0$ *and* $m = 1, 2, 5, 10$.

Optimization of a bridge under random loads (III)



Optimized shapes for $\delta = 3$ *and* $m = 1, 2, 5, 10$.

Comparison with the worst-case approach



Optimized shapes for the linearized worst-case design approach with $m = 1, 2, 5, 10$.

Probabilistic optimal design: conclusions

Assets of this approach:

- The optimization of probabilistic functionals tends to produce optimized designs with better nominal performances than their worst-case counterparts.

Drawbacks of this approach:

- The probability law $\mathbb{P}_{\text{true}}$ of ξ is assumed to be known, while often, this law is only accessible through a set of samples ξ^i, $i = 1, \ldots, N$.

- The recent idea of distributionally robustness alleviates the need for the knowledge of the "true" law $\mathbb{P}_{\text{true}}$ of ξ;

- It solely relies on an "estimate" $\mathbb{P}$ of the latter.

Minimize the worst expectation of $\mathcal{C}(h,\xi)$ when the law $\mathbb{Q}$ of ξ is "close" to $\mathbb{P}$.

$$\sup_{\substack{\mathbb{Q}\in\mathcal{P}(\Xi),\\ \mathbb{Q} \text{ ``close'' to } \mathbb{P}}} \int_{\Xi} \mathcal{C}(h,\xi)\,\mathrm{d}\mathbb{Q}(\xi).$$

D. Kuhn, S. Shafiee, and W. Wiesemann, *Distributionally robust optimization*, Acta Numerica, 34, (2025), pp. 579–804.

F. Lin, X. Fang, and Z. Gao, *Distributionally robust optimization: A review on theory and applications*, Numerical Algebra, Control & Optimization, 12 (2022), p. 159.

The distributionally robust approach (II)

- (For simplicity) the uncertain parameter ξ lies in a finite-dimensional set $\Xi \subset \mathbb{R}^k$.

- The only available information about $\xi \in \Xi$ is a (reconstructed) nominal law $\mathbb{P}$.

  Example: $\mathbb{P}$ is the empirical mean of observed samples ξ^i, $i = 1, \dots, N$:

$$\mathbb{P} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi^i}.$$

- The distributionally robust optimal design problem reads:

$$\boxed{\min_{h \in \mathcal{U}_{\text{ad}}} J_{\text{dr}}(h), \text{ where } J_{\text{dr}}(h) = \sup_{\mathbb{Q} \in \mathcal{A}} \int_{\Xi} \mathcal{C}(h, \xi) \, \mathrm{d}\mathbb{Q}(\xi),}$$

  where the ambiguity set $\mathcal{A} \subset \mathcal{P}(\Xi)$ contains the laws $\mathbb{Q}$ that are "close" to $\mathbb{P}$.

The distributionally robust approach (III)

We consider three types of distributionally robust problems:

1. $\mathcal{A} \equiv \mathcal{A}_{\mathrm{W}}$ is the set of probability measures which are "close" to $\mathbb{P}$,

$$\mathcal{A}_{\mathrm{W}} = \Big\{ \mathbb{Q} \in \mathcal{P}(\Xi), \;\; d(\mathbb{P}, \mathbb{Q}) \leq m \Big\},$$

in terms of a certain "distance" $d(\cdot, \cdot)$ on $\mathcal{P}(\Xi)$: the Wasserstein distance.

2. $\mathcal{A} \equiv \mathcal{A}_{\mathrm{M}}$ is the set of probability measures on Ξ whose first- and second-order moments are "close" to those of $\mathbb{P}$:

$$\mathcal{A}_{\mathrm{M}} = \left\{ \mathbb{Q} \in \mathcal{P}(\Xi) \text{ s.t. } \sup_{|\alpha| \leq 2} \left| \int_{\Xi} \xi^{\alpha} \, \mathrm{d}\mathbb{P}(\xi) - \int_{\Xi} \xi^{\alpha} \, \mathrm{d}\mathbb{Q}(\xi) \right| \leq m \right\}.$$

3. Another statistical quantity of the cost than its expectation is made robust: its conditional value at risk.

The Wasserstein distance (I)

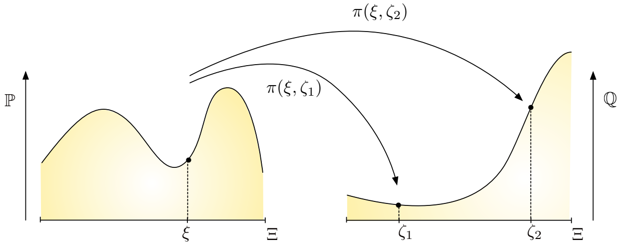
- A coupling is a probability measure $\pi \in \mathcal{P}(\Xi \times \Xi)$.
- The first and second marginals $\pi_1, \pi_2 \in \mathcal{P}(\Xi)$ of $\pi \in \mathcal{P}(\Xi \times \Xi)$ are defined by:

$$\forall \varphi \in \mathcal{C}(\Xi), \quad \int_{\Xi \times \Xi} \varphi(\xi)\, \mathrm{d}\pi(\xi, \zeta) = \int_{\Xi} \varphi(\xi)\, \mathrm{d}\pi_1(\xi), \text{ and}$$

$$\int_{\Xi \times \Xi} \varphi(\zeta)\, \mathrm{d}\pi(\xi, \zeta) = \int_{\Xi} \varphi(\zeta)\, \mathrm{d}\pi_2(\zeta).$$

- *Interpretation:* If $\pi \in \mathcal{P}(\Xi \times \Xi)$ is a coupling with marginals $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\Xi)$,

$$\pi(\xi, \zeta) \approx \text{ amount of mass transferred from } \xi \text{ to } \zeta.$$



The Wasserstein distance (II)

Definition 1.

Let Ξ be a compact subset of $\mathbb{R}^k$; the Wasserstein distance $W(\mathbb{P}, \mathbb{Q})$ between two probability measures $\mathbb{P}$, $\mathbb{Q} \in \mathcal{P}(\Xi)$ is

$$W(\mathbb{P}, \mathbb{Q}) = \inf_{\substack{\pi \in \mathcal{P}(\Xi \times \Xi) \\ \pi_1 = \mathbb{P},\ \pi_2 = \mathbb{Q}}} \int_{\Xi \times \Xi} c(\xi, \zeta)\, \mathrm{d}\pi(\xi, \zeta),$$

where the ground cost $c(\xi, \zeta)$ on $\mathbb{R}^k$ is chosen to be quadratic $c(\xi, \zeta) = |\xi - \zeta|^2$.

- The Wasserstein distance "lifts" the ground cost $c(\xi, \zeta)$ on Ξ into a distance over probability measures on Ξ.
- It is a flexible means to evaluate the distance between $\mathbb{P}$ and $\mathbb{Q} \in \mathcal{P}(\Xi)$, which smoothly appraises differences (e.g. translations) between the supports of $\mathbb{P}$ and $\mathbb{Q}$.

G. Peyré and M. Cuturi, *Computational optimal transport: With applications to data science*, Foundations and Trends in Machine Learning, 11 (2019), pp. 355–607.

F. Santambrogio, *Optimal transport for applied mathematicians*, Birkäuser, 2015.

Entropy-regularization of the Wasserstein distance

- For a variety of reasons, the Wasserstein distance is often regularized [Cu]:

$$W_\varepsilon(\mathbb{P},\mathbb{Q}) = \inf_{\substack{\pi\in\mathcal{P}(\Xi\times\Xi)\\ \pi_1=\mathbb{P},\ \pi_2=\mathbb{Q}}} \left(\int_{\Xi\times\Xi} c(\xi,\zeta)\,\mathrm{d}\pi(\xi,\zeta) + \varepsilon H(\pi) \right),$$

where the entropy $H(\pi)$ of a coupling π is:

$$H(\pi) = \begin{cases} \int_{\Xi\times\Xi} \log \frac{\mathrm{d}\pi}{\mathrm{d}\pi_0}\,\mathrm{d}\pi & \text{if } \pi \text{ is absolutely continuous w.r.t. } \pi_0, \\ \infty & \text{otherwise.} \end{cases}$$

- The fixed reference coupling $\pi_0 \in \mathcal{P}(\Xi\times\Xi)$ plays the role of a "prior".
- A judicious choice about π_0, with nice statistical guarantees, is

$$\pi_0(\xi,\zeta) = \mathbb{P}(\xi)\mathrm{d}\nu_\xi(\zeta), \quad \text{with} \quad \mathrm{d}\nu_\xi(\zeta) := \alpha_\xi e^{-\frac{c(\xi,\zeta)}{2\sigma^2}} \mathbb{1}_\Xi(\zeta)\mathrm{d}\zeta,$$

for some $\sigma > 0$ and a normalization factor α_ξ [AlMa]:

$$\text{For all } \varphi\in\mathcal{C}(\Xi\times\Xi), \quad \int_{\Xi\times\Xi} \varphi(\xi,\zeta)\,\mathrm{d}\pi_0(\xi,\zeta) = \int_\Xi \left(\int_\Xi \varphi(\xi,\zeta)\,\mathrm{d}\nu_\xi(\zeta) \right) \mathrm{d}\mathbb{P}(\xi).$$

Intuition:

- π_0 "spreads" the mass of $\mathbb{P}$ at ξ over a characteristic length scale σ.
- σ accounts for a "degree of confidence" in the nominal law $\mathbb{P}$.

The distributionally robust optimal design problem

The entropy-regularized distributionally robust optimal design problem reads:

$$\min_{h \in \mathcal{U}_{\mathrm{ad}}} J_{\mathsf{dr},\varepsilon}(h), \text{ where } J_{\mathsf{dr},\varepsilon}(h) = \sup_{\substack{\mathbb{Q} \in \mathcal{P}(\Xi),\\ W_\varepsilon(\mathbb{P},\mathbb{Q}) \leq m}} \int_\Xi \mathcal{C}(h,\xi)\, \mathrm{d}\mathbb{Q}(\xi).$$

This bilevel min-max problem looks very difficult at first glance... but it can be given a tractable reformulation up to the use of convex duality.

A convex duality result (I)

Proposition 1 ([AIMa]).

Besides "mild" assumptions, suppose that:

- Ξ *is a convex and compact subset of* $\mathbb{R}^k$,
- $f : \Xi \to \mathbb{R}$ *is a continuous function,*
- $\mathbb{P} \in \mathcal{P}(\Xi)$ *is a probability measure.*

For any $m > 0$, *and for a sufficiently small value of* σ, *the following equality holds:*

$$\sup_{W_\varepsilon(\mathbb{P},\mathbb{Q})\leq m} \int_\Xi f(\zeta)\,\mathrm{d}\mathbb{Q}(\zeta) = \inf_{\lambda\geq 0} \left\{ \lambda m + \lambda\varepsilon \int_\Xi \log\left(\int_\Xi e^{\frac{f(\zeta)-\lambda c(\xi,\zeta)}{\lambda\varepsilon}} \mathrm{d}\nu_\xi(\zeta) \right) \mathrm{d}\mathbb{P}(\xi) \right\}.$$

Hint of proof: We introduce a Lagrange multiplier λ for the constraint on $W_\varepsilon(\mathbb{P},\mathbb{Q})$:

$$\begin{aligned} \sup_{W_\varepsilon(\mathbb{P},\mathbb{Q})\leq m} \int_\Xi f(\zeta)\,\mathrm{d}\mathbb{Q}(\zeta) &= \sup_{\mathbb{Q}\in\mathcal{P}(\Xi)} \inf_{\lambda\geq 0} \left(\int_\Xi f(\zeta)\,\mathrm{d}\mathbb{Q}(\zeta) + \lambda(m - W_\varepsilon(\mathbb{P},\mathbb{Q})) \right), \\ &= \inf_{\lambda\geq 0} \sup_{\mathbb{Q}\in\mathcal{P}(\Xi)} \left(\int_\Xi f(\zeta)\,\mathrm{d}\mathbb{Q}(\zeta) + \lambda(m - W_\varepsilon(\mathbb{P},\mathbb{Q})) \right), \end{aligned}$$

where the exchange of the infimum and supremum proceeds from convex duality.

A convex duality result (II)

Inserting the definition of $W_\varepsilon(\mathbb{P},\mathbb{Q})$, it follows:

$$\sup_{W_\varepsilon(\mathbb{P},\mathbb{Q})\le m}\int_\Xi f(\zeta)\mathrm{d}\mathbb{Q}(\zeta) = \inf_{\lambda\ge 0}\sup_{\substack{\pi\in\mathcal{P}(\Xi\times\Xi)\\ \pi_1=\mathbb{P}}}\left\{\lambda m + \int_\Xi \Big(f(\zeta)-\lambda c(\xi,\zeta)-\lambda\varepsilon H(\pi)\Big)\,\mathrm{d}\pi(\xi,\zeta)\right\},$$

Given the definition of $H(\pi)$, the maximization holds over couplings π of the form

$$\pi(\xi,\zeta) = a(\xi,\zeta)\pi_0(\xi,\zeta), \text{ for some function } a\in L^1(\Xi\times\Xi;\mathrm{d}\pi_0),$$

and so:

$$\sup_{W_\varepsilon(\mathbb{P},\mathbb{Q})\le m}\int_\Xi f(\zeta)\,\mathrm{d}\mathbb{Q}(\zeta) = \inf_{\lambda\ge 0}\sup_{\substack{\alpha\in L^1(\Xi\times\Xi;\mathrm{d}\pi_0)\\ \int_\Xi \alpha(\xi,\zeta)\,\mathrm{d}\nu_\xi(\zeta)=1}}\Bigg\{\lambda m$$

$$+\int_\Xi\Big(f(\zeta)-\lambda c(\xi,\zeta)-\lambda\varepsilon\log\alpha(\xi,\zeta)\Big)\alpha(\xi,\zeta)\,\mathrm{d}\pi_0(\xi,\zeta)\Bigg\}.$$

Exploiting the Euler-Lagrange equation for the inner maximization, we obtain:

$$\alpha(\xi,\zeta) = \left(\int_\Xi e^{\frac{f(\zeta)-\lambda c(\xi,\zeta)}{\lambda\varepsilon}}\,\mathrm{d}\nu_\xi(\zeta)\right)^{-1} e^{\frac{f(\zeta)-\lambda c(\xi,\zeta)}{\lambda\varepsilon}},$$

and the desired result follows.

A convex duality result (III)

- The entropy-regularized distributionally robust optimization problem rewrites:

$$\min_{\substack{h \in \mathcal{U}_{\text{ad}},\\ \lambda \geq \mathbf{0}}} \mathcal{D}(h, \lambda), \text{ where}$$

$$\mathcal{D}(h, \lambda) := \lambda m + \lambda \varepsilon \int_{\Xi} \log \left(\int_{\Xi} e^{\frac{\mathcal{C}(h,\zeta) - \lambda c(\xi,\zeta)}{\lambda \varepsilon}} \mathrm{d}\nu_{\xi}(\zeta) \right) \mathrm{d}\mathbb{P}(\xi).$$

- This problem can be solved by a standard optimization algorithm based on the derivatives of the functional $\mathcal{D}(h, \lambda)$ with respect to h and λ.

- Constraints can be added to the problem without additional conceptual difficulty.

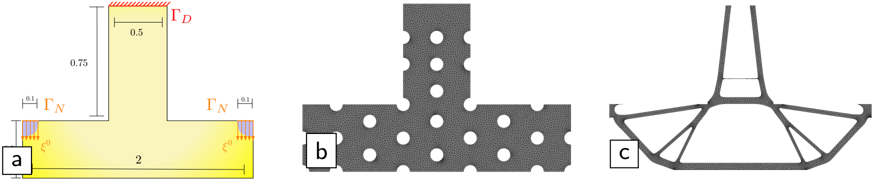
Distributionally robust design of a T-shaped beam (I)

- We optimize the geometry of a 2d T-shaped beam Ω.
- The applied loads $\xi \in \mathbb{R}^2$ are uncertain, with ideal value $\xi^0 = (0, -1)$.
- The law of ξ is itself uncertain, with nominal approximation $\mathbb{P} = \delta_{\xi^0}$.
- We minimize the total stress within Ω under a volume constraint:

$$\min_{\Omega} \Sigma(\Omega, \xi^0) \text{ s.t. } \mathrm{Vol}(\Omega) = V_T, \text{ where } V_T = 0.3,$$

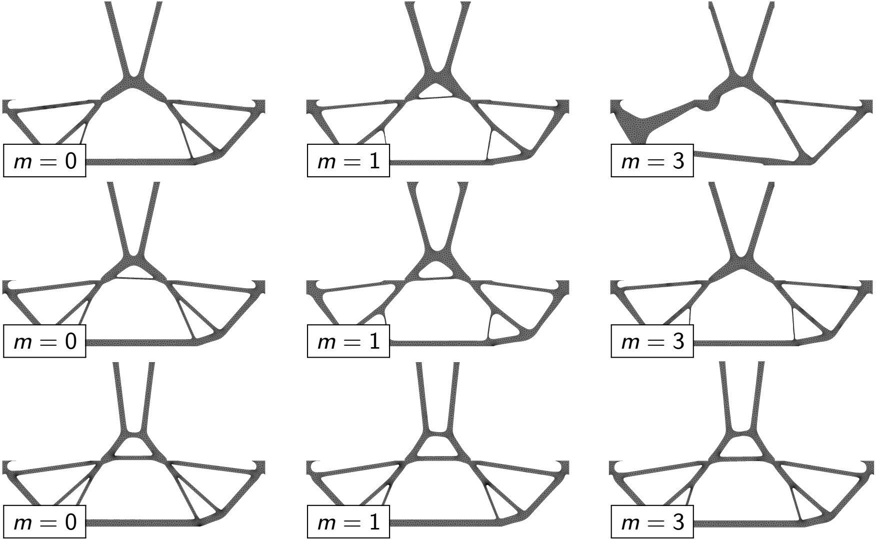
where the stress function is defined by

$$\Sigma(\Omega, \xi) = \int_{\Omega} \chi(x) ||\sigma(u_{\Omega,\xi})||^2 \, \mathrm{d}x.$$



(a) Setting of the minimization of stress within a T-shaped beam; (b) Mesh of the initial shape; (c) Optimized shape Ω^*_{det} assuming a perfect knowledge of the loads.

Distributionally robust design of a T-shaped beam (II)



Distributionally robust optimized designs obtained in the T-shaped beam example, associated to various values of σ^2*; (1st row)* $\sigma^2 = 1e{-}0.5$*; (2nd row)* $\sigma^2 = 1e{-}1$*; (3rd row)* $\sigma^2 = 1e{-}3$*.*

Moment-based ambiguity sets (I)

- We assume that only the first and second-order moments $\mu^0 \in \mathbb{R}^k$ and $\Sigma^0 \in \mathbb{S}_+^k$ of the true law $\mathbb{P}_{\text{true}}$ have been reconstructed [DeYe]:

$$\mu^0 = \int_\Xi \xi \, \mathrm{d}\mathbb{P}_{\text{true}}(\xi), \text{ and } \Sigma^0 = \int_\Xi (\xi - \mu^0) \otimes (\xi - \mu^0) \, \mathrm{d}\mathbb{P}_{\text{true}}(\xi).$$

- The ambiguity set of interest is then:

$$\mathcal{A}_M = \Big\{ \mathbb{Q} \in \mathcal{P}(\Xi), \ \Big| \int_\Xi \xi \, \mathrm{d}\mathbb{Q}(\xi) - \mu^0 \Big| \leq m_1, \\ \text{and} \quad \int_\Xi (\xi - \mu^0) \otimes (\xi - \mu^0) \, \mathrm{d}\mathbb{Q}(\xi) \leq m_2 \Sigma^0 \Big\},$$

where $m_1 > 0$ and $m_2 > 0$ are given bounds.

Moment-based ambiguity sets (II)

- We define the entropy $H(\mathbb{Q})$ of a probability measure $\mathbb{Q} \in \mathcal{P}(\Xi)$ by:

$$H(\mathbb{Q}) = \begin{cases} \displaystyle\int_\Xi \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}_0}\,\mathrm{d}\mathbb{Q} & \text{if } \mathbb{Q} \text{ is absolutely continuous w.r.t. } \mathbb{Q}_0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathbb{Q}_0 \in \mathcal{P}(\Xi)$ is a given reference law.

Ex: $\mathbb{Q}_0$ is the k-variate Gaussian law with mean value μ^0 and covariance matrix Σ^0.

- This leads to the moment-based distributionally robust optimal design problem:

$$\min_{h \in \mathcal{U}_{\mathrm{ad}}} J_{\mathrm{M}}(h), \text{ where } J_{\mathrm{M}}(h) := \sup_{\mathbb{Q} \in \mathcal{A}_{\mathrm{M}}} \left(\int_\Xi \mathcal{C}(h,\xi)\,\mathrm{d}\mathbb{Q}(\xi) - \varepsilon H(\mathbb{Q}) \right).$$

Moment-based ambiguity sets (III)

Proposition 2.

Let $\Xi \subset \mathbb{R}^k$ be compact, and let $f : \Xi \to \mathbb{R}$ be a continuous function. It holds:

$$\sup_{\mathbb{Q}\in\mathcal{A}_{\mathrm{M}}} \left(\int_\Xi f(\xi)\, \mathrm{d}\mathbb{Q}(\xi) - \varepsilon H(\mathbb{Q}) \right) = \inf_{\substack{\lambda\geq \mathbf{0},\ |\tau|\leq \mathbf{1},\\ S\in\mathbb{S}^k_+}} \left\{ \lambda m_1 - \lambda\tau\cdot\mu^0 + m_2 S : \Sigma^0 + \varepsilon \log\left(\int_\Xi \left(e^{\frac{f(\xi)+\lambda\tau\cdot\xi - S:(\xi-\mu^{\mathbf{0}})\otimes(\xi-\mu^{\mathbf{0}})}{\varepsilon}} \right) \mathrm{d}\mathbb{Q}_0(\xi) \right) \right\}.$$

The moment-based distributionally robust problem thus rewrites:

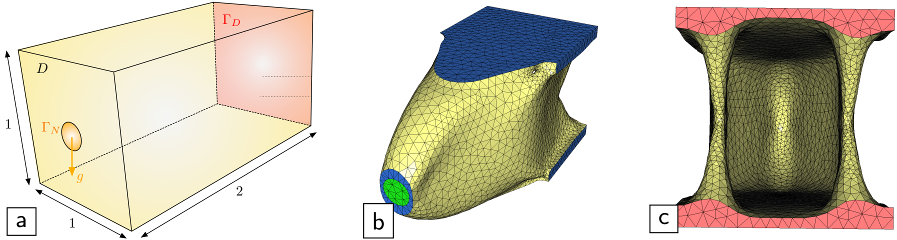
$$\min_{\substack{h\in\mathcal{U}_{\mathrm{ad}},\ |\tau|\leq \mathbf{1},\\ \lambda\geq \mathbf{0},\ S\in\mathbb{S}^k_+}} \mathcal{D}_{\mathrm{M}}(h,\lambda,\tau,S), \text{ where } \mathcal{D}_{\mathrm{M}}(h,\lambda,\tau,S) := \lambda m_1 - \lambda\tau\cdot\mu^0 + m_2 S : \Sigma^0 + \varepsilon \log\left(\int_\Xi \left(e^{\frac{C(h,\xi)+\lambda\tau\cdot\xi - S:(\xi-\mu^{\mathbf{0}})\otimes(\xi-\mu^{\mathbf{0}})}{\varepsilon}} \right) \mathrm{d}\mathbb{Q}_0(\xi) \right).$$

This is a standard optimal design problem, posed over an augmented set of variables.

Distributionally robust design of a 3d beam under moment ambiguity (I)

- We optimize the shape Ω of a 3d cantilever beam under uncertain loads $\xi \in \mathbb{R}^3$.
- In an ideal situation, the applied load $\xi = \xi^0 := (0, 0, -1)$ is known perfectly.
- We then consider the following compliance minimization problem:

$$\min_{\Omega} \; C(\Omega, \xi^0) \quad \text{s.t.} \quad \mathrm{Vol}(\Omega) = V_T, \; \text{where } V_T = 0.45.$$



(a) Setting of the shape optimization example of a 3d cantilever beam; (b,c) Optimized structure Ω^*_{det} when the applied load $\xi \equiv \xi^0$ is perfectly known.

Distributionally robust design of a 3d beam under moment ambiguity (II)

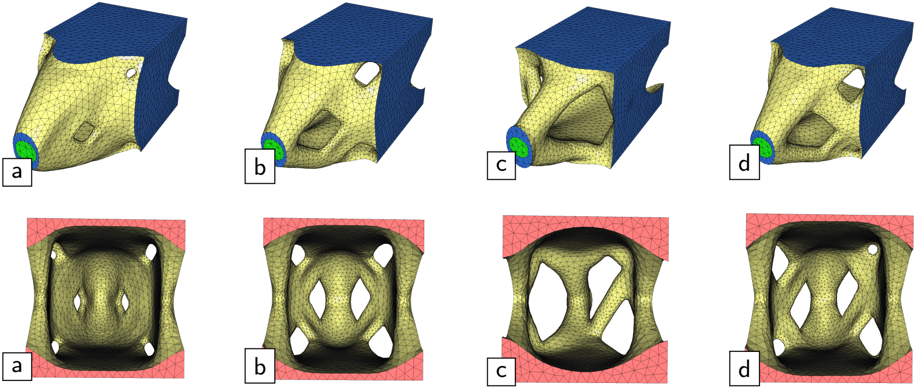
- We now assume that the applied load ξ is uncertain, with unknown probability law.

- The only pieces of information about $\mathbb{P}_{\textbf{true}}$ are:
  - The mean value $\mu^0 = \xi^0$;
  - The covariance matrix $\Sigma^0 = \sigma^2 \mathrm{I}$, where $\sigma^2 = 0.01$.

- We thus consider the distributionally robust version of the previous problem:

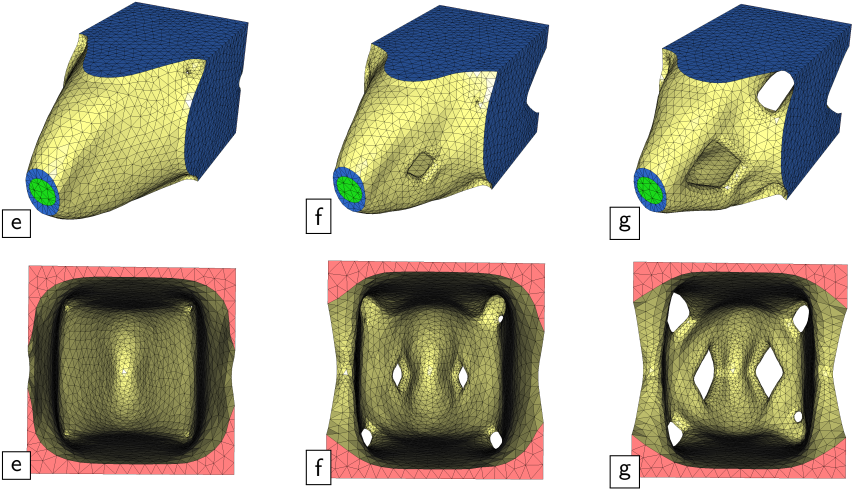
$$\min_{\Omega} \left(\sup_{\mathbb{Q} \in \mathcal{A}_{\mathbf{M}}} \int_{\Xi} C(\Omega, \xi) \, \mathrm{d}\mathbb{Q}(\xi) - \varepsilon H(\mathbb{Q}) \right) \text{ s.t. } \mathrm{Vol}(\Omega) = V_T$$

- ... or rather its tractable reformulation.



Optimized 3d cantilever beams under distributional uncertainties using a moment ambiguity set; front and back views for the parameters (a) $m_1 = 0,\ m_2 = 1$*; (b)* $m_1 = 1,\ m_2 = 1$*; (c)* $m_1 = 2,\ m_2 = 1$*, and (d)* $m_1 = 5,\ m_2 = 1$*.*

Distributionally robust design of a 3d beam under moment ambiguity (IV)



Optimized 3d cantilever beams under distributional uncertainties using a moment ambiguity set; front and back views for the parameters (e) $m_1 = 0,\ m_2 = 2$; (f) $m_1 = 2,\ m_2 = 1$, and (g) $m_1 = 5,\ m_2 = 5$.

Robustification of another statistical quantity of the cost

- The previous distributionally robust problems are of the form:

$$\min_{h \in \mathcal{U}_{\rm ad}} \sup_{\mathbb{Q} \in \mathcal{A}} \int_{\Xi} \mathcal{C}(h, \xi) \, \mathrm{d}\mathbb{Q}(\xi),$$

  i.e. the worst value of the expectation of $\mathcal{C}(h, \cdot)$ is minimized.

- We "robustify" another statistical quantity of $\mathcal{C}(h, \cdot)$: its conditional value at risk.

- This indicator better appraises "how much" $\mathcal{C}(h, \cdot)$ deviates from its mean value.

- In particular, it allows for a surrogate expression of failure probabilities.

A glimpse of the notion of Conditional Value at Risk (I)

Let $h \in \mathcal{U}_{\mathrm{ad}}$ and $\mathbb{P} \in \mathcal{P}(\Xi)$ be the (known) law of the uncertain parameter $\xi \in \Xi$.

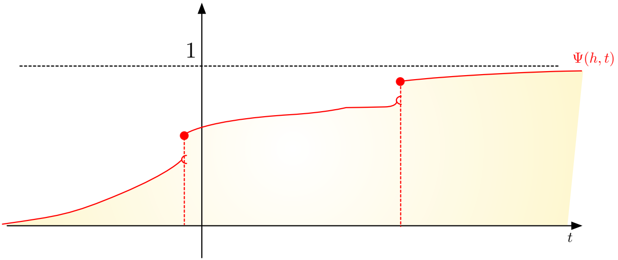
- The cumulative distribution function $t \mapsto \Psi(h,t)$ of the cost $\mathcal{C}(h,\cdot)$ is:

$$\forall t \in \mathbb{R}, \quad \Psi(h,t) = \mathbb{P}\Big\{\xi \in \Xi,\ \mathcal{C}(h,\xi) \leq t\Big\}.$$

- This function is non decreasing, with limits:

$$\lim_{t\to-\infty} \Psi(h,t) = 0, \text{ and } \lim_{t\to+\infty} \Psi(h,t) = 1.$$

- It is continuous from the right, but it may be discontinuous from the left at $t \in \mathbb{R}$ if $\mathcal{C}(h,\cdot) = t$ on a subset of Ξ with positive measure.
- (For simplicity) We suppose that $\Psi(h,\cdot)$ is continuous.



A glimpse of the notion of Conditional Value at Risk (II)

Definition 2.

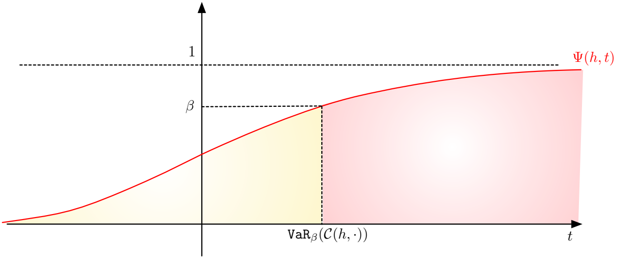
For a given threshold $\beta \in (0,1)$,

- *The β-value at risk, or β-quantile* $\mathtt{VaR}_\beta(\mathcal{C}(h,\cdot))$ *of* $\mathcal{C}(h,\cdot)$ *is the smallest value* $t \in \mathbb{R}$ *majorizing* $\mathcal{C}(h,\cdot)$ *with probability* β*:*

$$\mathtt{VaR}_\beta(\mathcal{C}(h,\cdot)) = \inf\Big\{ t \in \mathbb{R},\ \ \Psi(h,t) \geq \beta \Big\}.$$

- *The β-conditional value at risk* $\mathtt{CVaR}_\beta(\mathcal{C}(h,\cdot))$ is the average of $\mathcal{C}(h,\cdot)$ over the events where it exceeds $\mathtt{VaR}_\beta(\mathcal{C}(h,\cdot))$:

$$\mathtt{CVaR}_\beta(\mathcal{C}(h,\cdot)) = \frac{1}{1-\beta} \int_{\left\{\xi\in\Xi,\ \mathcal{C}(h,\xi)\geq \mathtt{VaR}_\beta(\mathcal{C}(h,\cdot))\right\}} \mathcal{C}(h,\xi)\,\mathrm{d}\mathbb{P}(\xi).$$



The conditional value at risk can be expressed as a minimum value.

Theorem 3.

The following representation formula holds true:

$$\mathtt{CVaR}_\beta(\mathcal{C}(h,\cdot)) = \inf_{\alpha\in\mathbb{R}} \left\{ \alpha + \frac{1}{1-\beta} \int_\Xi [\mathcal{C}(h,\xi) - \alpha]_+ \, \mathrm{d}\mathbb{P}(\xi) \right\},$$

where $[t]_+ := \max(t, 0)$.

The minimum in the above program is uniquely attained at $\alpha = \mathtt{VaR}_\beta(\mathcal{C}(h,\cdot))$.

R. T. Rockafellar and S. Uryasev, *Conditional value-at-risk for general loss distributions*, Journal of banking & finance, 26 (2002), pp. 1443–1471.

Conditional Value at Risk and probability of failure (I)

- The conditional value at risk allows for a convenient reformulation of failure probability constraints, of the form

$$\mathbb{P}\Big\{\xi \in \Xi,\ \mathcal{C}(h,\xi) \geq C_T\Big\} \leq 1-\beta,$$

  for a parameter $\beta \in (0,1)$ and a safety threshold $C_T \in \mathbb{R}$.

- Indeed, the definition of $\Psi(h,t)$ implies that:

$$\mathbb{P}\Big\{\xi \in \Xi,\ \mathcal{C}(h,\xi) \geq C_T\Big\} \leq 1-\beta \iff \Psi(h, C_T) \geq \beta.$$

- Since $\texttt{VaR}_\beta(\mathcal{C}(h,\cdot))$ is the smallest value $t \in \mathbb{R}$ such that $\mathcal{C}(h,\xi) \leq t$ with probability β, this is in turn equivalent to:

$$\texttt{VaR}_\beta(\mathcal{C}(h,\cdot)) \leq C_T.$$

- A conservative surrogate for this requirement is:

$$\texttt{CVaR}_\beta(\mathcal{C}(h,\cdot)) \leq C_T.$$

Conditional Value at Risk and probability of failure (II)

- Let us consider an optimal design problem featuring an objective function $J(h)$ and a constraint on the probability that the cost $\mathcal{C}(h,\xi)$ exceed C_T:

$$\min_{h\in\mathcal{U}_{\mathrm{ad}}} J(h) \quad \text{s.t.} \quad \mathbb{P}\Big\{\xi\in\Xi \text{ s.t. } \mathcal{C}(h,\xi)\geq C_T\Big\}\leq 1-\beta.$$

- A conservative surrogate for this problem is:

$$\min_{h\in\mathcal{U}_{\mathrm{ad}}} J(h) \ \text{ s.t. } \mathtt{CVaR}_\beta(\mathcal{C}(h,\cdot))\leq C_T.$$

- Using Theorem 3, this rewrites as a "classical" optimization over the pair (h,α):

$$\min_{\substack{h\in\mathcal{U}_{\mathrm{ad}},\\ \alpha\in\mathbb{R}}} J(h) \text{ s.t. } \alpha+\frac{1}{1-\beta}\int_{\xi\in\Xi}[\mathcal{C}(h,\xi)-\alpha]_+ \,\mathrm{d}\mathbb{P}(\xi)\leq C_T.$$

Conditional Value at Risk and probability of failure (III)

- The law of ξ is now uncertain: only a nominal law $\mathbb{P} \in \mathcal{P}(\Xi)$ is available.
- (For instance) Assuming an ambiguity set $\mathcal{A}_{\mathrm{W}}$ of Wasserstein type, the distributionally robust counterpart of the previous problem reads:

$$\min_{h \in \mathcal{U}_{\mathrm{ad}}} J(h) \text{ s.t. } \sup_{\mathbb{Q} \in \mathcal{A}_{\mathrm{W}}} \min_{\alpha \in \mathbb{R}} \left(\alpha + \frac{1}{1-\beta} \int_{\xi \in \Xi} [\mathcal{C}(h,\xi) - \alpha]_+ \, \mathrm{d}\mathbb{Q}(\xi) \right) \leq C_T.$$

- Using similar duality techniques as previously, this rewrites:

$$\min_{\substack{h \in \mathcal{U}_{\mathrm{ad}}, \\ \lambda \geq \mathbf{0}, \alpha \in \mathbb{R}}} J(h) \text{ s.t. } \mathcal{D}_{\mathrm{C}}(h,\lambda,\alpha) \leq C_T, \text{ where}$$

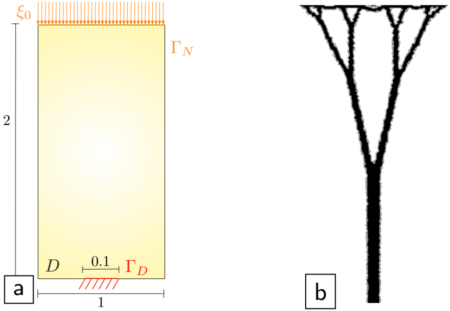
$$\mathcal{D}_{\mathrm{C}}(h,\lambda,\alpha) = \alpha + \frac{\lambda m}{1-\beta} + \frac{\lambda \varepsilon}{1-\beta} \int_{\Xi} \log \left(\int_{\Xi} e^{\frac{[\mathcal{C}(h,\zeta)-\alpha]_+ - \lambda c(\xi,\zeta)}{\lambda \varepsilon}} \, \mathrm{d}\nu_{\xi}(\zeta) \right) \mathrm{d}\mathbb{P}(\xi).$$

- This "classical" problem can be solved by standard numerical algorithms.

Numerical example (I)

- We optimize a 2d bridge, accounted for by a density function $h : D \to [0,1]$.
- It is clamped along the lower side Γ_D of ∂D.
- A load ξ is applied on its upper boundary Γ_N.
- At first, we optimize the compliance $C(h, \xi^0)$ of the structure under a volume constraint, in the ideal situation where the load is $\xi^0 := (0, -1)$:

$$\min_{h \in \mathcal{U}_{\text{ad}}} C(h, \xi^0) \text{ s.t. } \text{Vol}(h) = V_T, \text{ where } V_T = 0.245.$$



(a) Setting of the bridge problem; (b) Optimized design h^*_{det} in ideal conditions.

Numerical example (II)

- We now assume that the load $\xi \in \Xi \subset \mathbb{R}^2$ is uncertain.
- Its (known) law $\mathbb{P} \in \mathcal{P}(\Xi)$ is the Gaussian law centered at ξ^0 with variance σ^2.
- We solve the problem

$$\min_{h \in \mathcal{U}_{\mathrm{ad}}} \mathrm{Vol}(h) \text{ s.t. } \mathbb{P}\Big\{\xi \in \Xi \text{ s.t. } C(h,\xi) \geq C_T\Big\} \leq 1-\beta,$$

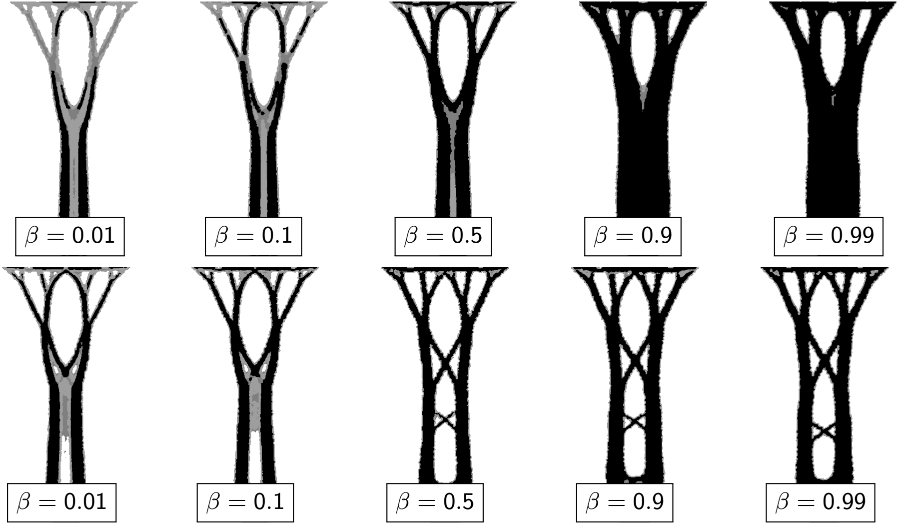
where the threshold $C_T := 40$ is close to $C(h^*_{\mathrm{det}}, \xi^0)$.

- We replace this problem by the following conservative version:

$$\min_{\substack{h \in \mathcal{U}_{\mathrm{ad}}, \\ \alpha \in \mathbb{R}}} \mathrm{Vol}(h) \quad \text{s.t.} \quad \alpha + \frac{1}{1-\beta} \int_{\xi \in \Xi} [C(h,\xi) - \alpha]_+ \, \mathrm{d}\mathbb{P}(\xi) \leq C_T.$$

- We solve this problem for several values of the variance σ^2 of the law $\mathbb{P}$ for ξ and the bound β.

Numerical example (III)



Optimized designs of the bridge under a safety constraint; (Upper row) $\sigma^2 = 1e{-}2.5$*; (Lower row)* $\sigma^2 = 1e{-}2$.

Numerical example (IV)

- We now turn to the situation where the law $\mathbb{P}$ of ξ is uncertain.
- A nominal law $\mathbb{P} := \delta_{\xi^{\mathbf{0}}}$ is reconstructed from the single ideal load.
- We consider the Wasserstein distributionally robust problem:

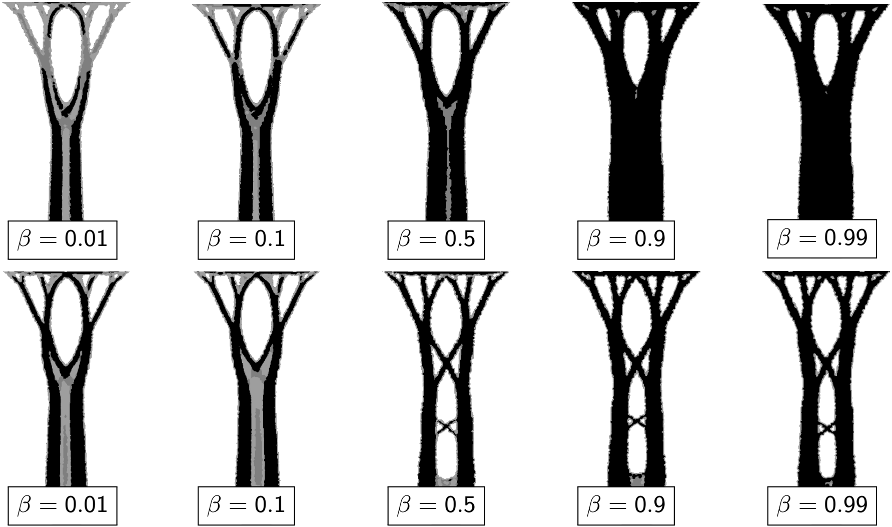
$$\min_{\substack{h\in\mathcal{U}_{\mathrm{ad}},\\ \alpha\in\mathbb{R}}} \mathrm{Vol}(h) \text{ s.t. } \sup_{\mathbb{Q}\in\mathcal{A}_{\mathbf{W}}} \min_{\alpha\in\mathbb{R}} \left(\alpha + \frac{1}{1-\beta}\int_{\xi\in\Xi} [\mathcal{C}(h,\xi)-\alpha]_+ \, \mathrm{d}\mathbb{Q}(\xi)\right) \le C_T.$$

- This problem has the following equivalent formulation:

$$\min_{\substack{h\in\mathcal{U}_{\mathrm{ad}},\\ \lambda\geq\mathbf{0},\alpha\in\mathbb{R}}} \mathrm{Vol}(h) \text{ s.t. } \mathcal{D}_{\mathsf{C}}(h,\lambda,\alpha) \le C_T, \text{ where}$$

$$\mathcal{D}_{\mathsf{C}}(h,\lambda,\alpha) = \alpha + \frac{\lambda m}{1-\beta} + \frac{\lambda\varepsilon}{1-\beta}\log\left(\int_{\Xi} e^{\frac{[C(h,\zeta)-\alpha]_+ - \lambda c(\xi^{\mathbf{0}},\zeta)}{\lambda\varepsilon}} \, \mathrm{d}\nu_{\xi^{\mathbf{0}}}(\zeta)\right).$$

Numerical example (V)



Distributionally robust designs of the bridge for a Wasserstein radius $m = 0.5$, *various values of the parameter* β *and (top row)* $\sigma^2 = 1e{-3}$, *(bottom row)* $\sigma^2 = 1e{-2.5}$.

Conclusion and perspectives

Conclusion:

- The distributionally robust designs systematically show:
  - Worse performance than their "simple" probabilistic counterparts in situations present in the nominal law;
  - Better performance in "out-of-sample" situations.
- Infinite-dimensional uncertain parameters can be considered after approximation (e.g. KL decomposition for a random field).
- Uncertainties over different data have been considered (material uncertainties, geometric uncertainties).

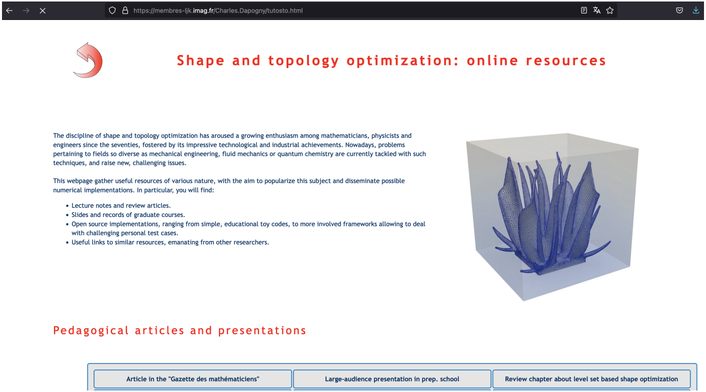
Perspectives:

- Treatment of geometric uncertainties in a distributionally robust way, with an optimal transport approach.

A word of advertisement

- All the numerical realizations are based on open-source libraries.
- A webpage gathering lecture notes, slides, demonstration codes, etc.

https://dapogny.org/tutosto.html



Thank you for your attention!

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