Optimization of the shape and topology of regions supporting boundary conditions

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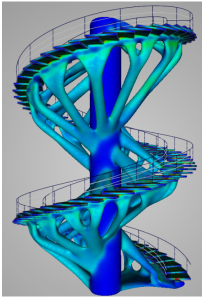
20th May, 2025

Foreword (I)

- Shape and topology optimization techniques are ubiquitous in industry and academics.

- Usually in practice,
  - A domain $\Omega \subset \mathbb{R}^d$ is optimized, representing e.g. a mechanical structure, a fluid device.
  - The performance of Ω is evaluated by an objective function $J(\Omega)$.
  - $J(\Omega)$ is expressed in terms of the solution u_Ω to a boundary value problem posed on Ω.
  - The regions of $\partial\Omega$ supporting specific boundary conditions are not subject to optimization.

- We investigate a variant of this setting, where not only the shape Ω, but also the subsets of $\partial\Omega$ bearing boundary conditions are optimized.



Optimization of a staircase (courtesy of Ansys).

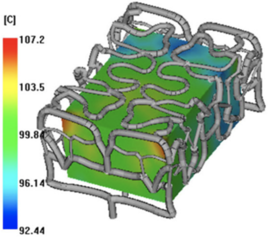
"Optimized" front-end of the Qatar National Convention Center.

Foreword (II)

Examples:

- In thermal conduction,
  - The temperature $u_\Omega : \Omega \to \mathbb{R}$ inside Ω is the solution to the conductivity equation;
  - Dirichlet b.c. account for a known profile,
  - Neumann b.c. represent an imposed heat flux.
- When Ω is a mechanical structure,
  - The displacement $u_\Omega : \Omega \to \mathbb{R}^d$ of Ω is solution to the linear elasticity system;
  - Ω is attached at the regions equipped with homogeneous Dirichlet b.c.
  - Neumann b.c. represent applied surface loads.
- Other applications arise in acoustics, in fluid mechanics, etc.

Optimization of the screws of a mandibular prosthesis [LaBa].



Optimized cooling process for a structure produced by molding [WeWuShi].

Contents

A model shape optimization problem (I)

- The considered shapes Ω are smooth, bounded domains in $\mathbb{R}^d$, with boundaries:

$$\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma}.$$

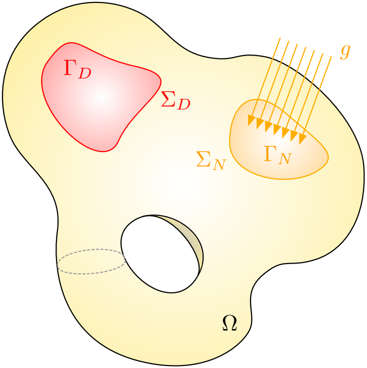
- We assume that $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$ and denote

$$\Sigma_D = \partial\Gamma_D, \text{ and } \Sigma_N = \partial\Gamma_N.$$

- The behavior of Ω is encoded in the solution $u \in H^1(\Omega)$ to the conductivity equation:

$$\begin{cases} -\text{div}(\gamma \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma, \\ \gamma \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N, \end{cases}$$

where
- γ is the conductivity of the medium
- $f \in L^2(\Omega)$ is a source (or a sink),
- $g \in L^2(\Gamma_N)$ is a heat flux.



A model shape optimization problem (II)

- A "classical" shape optimization problem then reads:

$$\boxed{\min_{\Omega} J(\Omega) \text{ s.t. } \; C(\Omega) \leq 0.}$$

Here, $J(\Omega)$ and $C(\Omega)$ are objective and constraint functions of the domain, e.g.

$$J(\Omega) := \int_{\Omega} j(u_{\Omega}) \, \mathrm{d}x,$$

where u_{Ω} is the solution to the conductivity equation and $j : \mathbb{R} \to \mathbb{R}$ is smooth.

- In the present application, Ω is fixed, and we consider problems of the form:

$$\boxed{\min_{G \subset \partial\Omega} J(G) \text{ s.t. } \; C(G) \leq 0.}$$

For instance, $G = \Gamma_D$ or Γ_N in the conductivity equation, and:

$$J(G) = \int_{\Omega} j(u_G) \, \mathrm{d}x.$$

Aims of this work

- We introduce notions of shape and topological derivatives for functions $J(G)$ depending on a region G of the boundary $\partial\Omega$.

- We propose "simple" mathematical methods to achieve their calculation.

- We implement them in combination with:
  - A constrained optimization algorithm to infer a descent direction [FeAlDa];
  - A mesh evolution method to track the evolving region G [BriDa, AlDaFre].

- We apply these ideas in different physical situations:
  - Thermal or electric conduction (conductivity equation);
  - Structure mechanics (linear elasticity system);
  - Acoustics (Helmholtz equation).

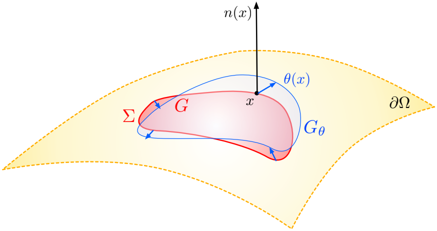
Shape derivatives (I): definition

The method of Hadamard relies on variations of a region $G \subset \partial\Omega$ of the form

$$G_\theta := (\mathrm{Id} + \theta)(G),$$

where θ is a "small" tangential vector field:

$$\theta \cdot n = 0 \text{ on } \partial\Omega.$$



Tangential deformations modify G within $\partial\Omega$.

Definition 1.

The shape derivative of a function $J(G)$ is the Fréchet derivative at $\theta = 0$ of the underlying mapping

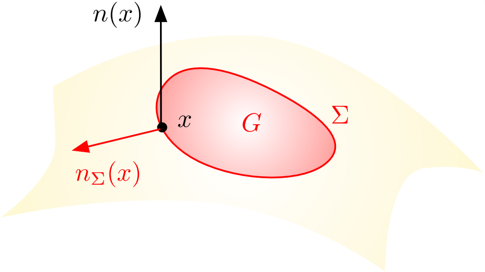
$$\theta \longmapsto J(G_\theta) \in \mathbb{R}.$$

The following expansion holds:

$$J(G_\theta) = J(G) + J'(G)(\theta) + \mathrm{o}(\theta), \text{ where } \mathrm{o}(\theta) \text{ is a "small" remainder.}$$

Shape derivatives (II): Examples

- "Simple" functions of the region G are:
  - Its surface area $\text{Area}(G) = \int_G \mathrm{d}s$;
  - The length $\text{Cont}(G) = \int_\Sigma \mathrm{d}\ell$ of its contour Σ.



- Their shape derivatives read, for any tangential deformation θ:

$$\text{Area}'(G)(\theta) = \int_\Sigma \theta \cdot n_\Sigma \, \mathrm{d}\ell, \text{ and } \text{Cont}'(G)(\theta) = \int_\Sigma \kappa(\theta \cdot n_\Sigma) \, \mathrm{d}\ell,$$

where $\kappa := \text{div}_{\partial\Omega}(n_\Sigma)$ is the mean curvature of Σ.

- The derivative of a general function, involving a physical state u_G is of the form:

$$J'(G) = \int_\Sigma v(u_G, p_G) \, (\theta \cdot n_\Sigma) \, \mathrm{d}\ell,$$

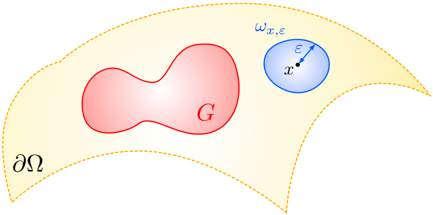
where $v(u_G, p_G)$ is a scalar quantity, depending on a suitable adjoint state p_G.

Topological derivatives

The notion of topological derivative features variations of $G \subset \partial\Omega$ of the form

$$G_{x,\varepsilon} := G \cup \omega_{x,\varepsilon},$$

where $\omega_{x,\varepsilon}$ is the surface disk with center $x \in \partial\Omega \setminus \overline{G}$ and radius $\varepsilon \ll 1$.



Definition 2.

The function $J(G)$ has a topological derivative at $x \in \partial\Omega \setminus \overline{G}$ if there exist a function $\rho(\varepsilon) \xrightarrow{\varepsilon \to} 0$ and a number $\mathrm{d}J_T(\Omega)(x) \in \mathbb{R}$ such that:

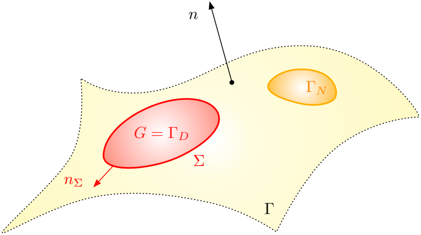
$$J(G_{x,\varepsilon}) = J(G) + \rho(\varepsilon)\mathrm{d}J_T(G)(x) + \mathrm{o}(\rho(\varepsilon)).$$

Remark The rate $\rho(\varepsilon)$ of the expansion depends on the application.

Setting

- We address the "most difficult" case, where $G = \Gamma_D$ in the conductivity equation.

- We consider the model functional



$$J(G) = \int_\Omega j(u_G) \, \mathrm{d}x,$$

where $u_G \in H^1(\Omega)$ is the solution to the conductivity equation.

- We aim to calculate the shape derivative $J'(G)(\theta)$.

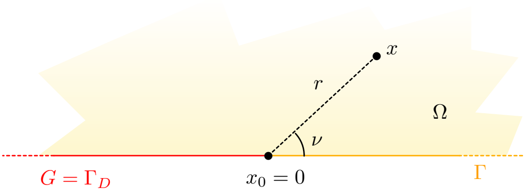
- For notational simplicity, $\theta \equiv 0$ on Γ_N.

- The analysis is difficult because of the weakly singular character of u_G near Σ...

A taste of the regularity of u_G

- The function u_G is smooth except near Σ, where the b.c. change types.
- If $d = 2$, let V be a neighborhood of $x_0 = 0 \in \Sigma$ such that:

$$x_0 = 0, \;\; \Omega \cap V = \{x \in V, \text{ s.t. } x_2 > 0\}, \text{ and } \quad G \cap V = \{x \in V, \text{ s.t. } x_2 = 0, \; x_1 < 0\}.$$



Then u_G is no more regular than $H^{3/2-\eta}(V)$ (for any $\eta > 0$), and

$$u_G = u_r + c_u S \text{ on } \Omega \cap V, \text{ where } u_r \text{ is "regular", } c_u \in \mathbb{R} \text{ and } S(r,\nu) = r^{\frac{1}{2}} \cos\left(\frac{\nu}{2}\right).$$

- The shape derivative $J'(G)(\theta)$ depends on the coefficients c_u, c_p of the state and adjoint functions!

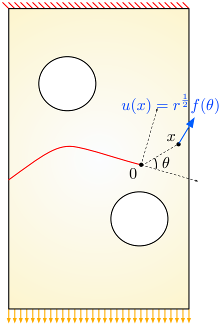
P. Grisvard, *Elliptic problems in nonsmooth domains*, SIAM, (2011).

Remarks

- A related phenomenon occurs in fracture mechanics:

  - The weak singularity of the elastic displacement near the tip of a crack defines the stress intensity factor;

  - This determines the energy release rate, i.e. the derivative of the energy w.r.t. the position of the crack.



- This dependence of $J'(G)(\theta)$ on the singularities of u_G and p_G makes its numerical evaluation awkward.

  $\Rightarrow$ Need to construct smooth approximations $u_{G,\varepsilon}$ and $J_\varepsilon(G)$ of u_G and $J(G)$.

The geodesic signed distance function

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain.

- The geodesic distance $d^{\partial\Omega}(x, y)$ on $\partial\Omega$ between two points $x, y \in \partial\Omega$ is:

$$d^{\partial\Omega}(x,y) = \inf_{\substack{\gamma:[0,1]\to\partial\Omega, \\ \gamma(0)=x,\ \gamma(1)=y}} \ell(\gamma), \text{ where } \ell(\gamma) = \int_0^1 |\gamma'(t)|\,\mathrm{d}t.$$

- The geodesic distance $d^{\partial\Omega}(x, K)$ of $x \in \partial\Omega$ to a compact subset $K \subset \partial\Omega$ is:

$$d^{\partial\Omega}(x,K) = \inf_{y\in K} d^{\partial\Omega}(x,y).$$

- When the minimizer is unique in the above definition, it is denoted by $p_K(x)$ and called the projection of x onto K.

- The geodesic signed distance function $d_G^{\partial\Omega}$ to an open region $G \subset \partial\Omega$ is:

$$\forall x \in \partial\Omega, \quad d^{\partial\Omega}(x) = \begin{cases} -d^{\partial\Omega}(x,\partial G) & \text{if } x \in G, \\ 0 & \text{if } x \in \partial G, \\ d^{\partial\Omega}(x,\partial G) & \text{if } x \in \partial\Omega \setminus \overline{G}. \end{cases}$$

Remark "Many" basic properties of $d_G^{\partial\Omega}$ are mere adaptations of those of the "usual" signed distance function to a domain of $\mathbb{R}^d$.

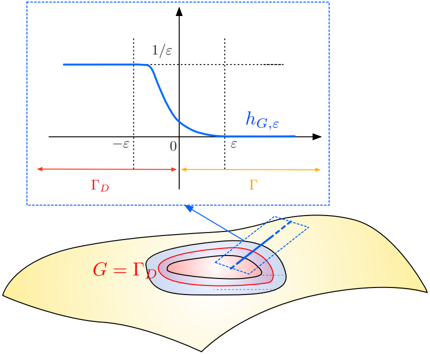
An approximate optimization problem (I)

- Let the approximate conductivity equation:

$$\begin{cases} -\text{div}(\gamma \nabla u_{G,\varepsilon}) = f & \text{in } \Omega, \\ \gamma \frac{\partial u_{G,\varepsilon}}{\partial n} + h_{G,\varepsilon} u_{G,\varepsilon} = 0 & \text{on } \Gamma \cup \Gamma_D, \\ \gamma \frac{\partial u_{G,\varepsilon}}{\partial n} = g & \text{on } \Gamma_N. \end{cases}$$

- $h_{G,\varepsilon}(x) := \frac{1}{\varepsilon} h\left(\frac{d_G^{\partial\Omega}(x)}{\varepsilon}\right)$ is made from a smooth profile $h : \mathbb{R} \to \mathbb{R}$ such that:

$$0 \leq h \leq 1, \quad \begin{cases} h \equiv 1 & \text{on } (-\infty, -1], \\ h(0) > 0, & \\ h \equiv 0 & \text{on } [1, \infty). \end{cases}$$



- Intuitively,
  - $h_{G,\varepsilon} = 0$ well inside Γ ($\approx$ homogeneous Neumann b.c.),
  - $h_{G,\varepsilon} = \frac{1}{\varepsilon} \approx \infty$ well inside Γ_D ($\approx$ homogeneous Dirichlet b.c.).

- For a fixed $\varepsilon > 0$, standard elliptic regularity implies that $u_{G,\varepsilon}$ is smooth on $\overline{\Omega}$.

An approximate optimization problem (II)

An approximation of the original shape optimization problem features the function:

$$J_\varepsilon(G) = \int_\Omega j(u_{G,\varepsilon})\,\mathrm{d}x,$$

whose shape derivative can be computed by classical techniques.

Proposition 1.

The shape derivative of the functional $J_\varepsilon(G)$ equals:

For all tangential θ s.t. $\theta = 0$ on Γ_N,

$$\begin{aligned} J'_\varepsilon(G)(\theta) &= -\frac{1}{\varepsilon^2}\int_{\Gamma\cup\Gamma_D} h'\left(\frac{d_G^{\partial\Omega}(x)}{\varepsilon}\right)\theta(p_\Sigma(x))\cdot n_\Sigma(p_\Sigma(x)) u_{G,\varepsilon}(x) p_{G,\varepsilon}(x)\,\mathrm{d}s(x) \\ &\approx -\frac{1}{\varepsilon}\int_\Sigma \theta(x)\cdot n_\Sigma(x)\, u_{G,\varepsilon}(x)\, p_{G,\varepsilon}(x)\,\mathrm{d}\ell(x), \end{aligned}$$

where the adjoint state $p_{G,\varepsilon}$ is the unique solution in $H^1(\Omega)$ to the problem:

$$\left\{ \begin{array}{rl} -\mathrm{div}(\gamma\nabla p_{G,\varepsilon}) = -j(u_{G,\varepsilon}) & \text{in } \Omega, \\ \gamma\frac{\partial p_{G,\varepsilon}}{\partial n} + h_{G,\varepsilon} p_{G,\varepsilon} = 0 & \text{on } \Gamma_D \cup \Gamma, \\ \gamma\frac{\partial p_{G,\varepsilon}}{\partial n} = 0 & \text{on } \Gamma_N. \end{array}\right.$$

An approximate optimization problem (III)

The consistence of this approximation process holds true under "mild" assumptions:

- The function $u_{G,\varepsilon}$ converges to u_G strongly in $H^1(\Omega)$: for any $0 < s < \frac{1}{4}$,

$$||u_{G,\varepsilon} - u_G||_{H^1(\Omega)} \leq C_s \varepsilon^s ||f||_{L^2(\Omega)}.$$

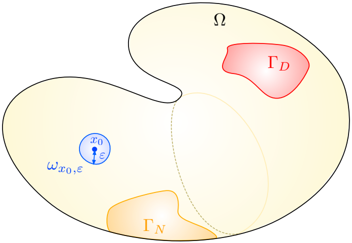
- As a result, for any given region G, the approximate shape functional $J_\varepsilon(G)$ converges to its exact counterpart $J(G)$.

- Going further, the approximate shape derivative $J'_\varepsilon(G)$ converges to its exact counterpart $J'(G)$, i.e.:

$$\sup_{||\theta|| \leq 1} |J'_\varepsilon(G)(\theta) - J'(G)(\theta)| \xrightarrow{\varepsilon \to 0} 0.$$

The model setting

We aim to calculate the topological derivative of $J(G) = \int_\Omega j(u_G)\,\mathrm{d}x$, when $G = \Gamma_D$.



- Ω is a smooth bounded domain in $\mathbb{R}^d$, $d = 2, 3$;
- Its boundary is the reunion of 3 disjoint parts:

$$\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma}.$$

- The subset $\omega_{x_0,\varepsilon}$ is a surface disk, centered at $x_0 \in \Gamma$, with radius ε.

The background and perturbed potentials u_G and $u_{G_{x_0,\varepsilon}} \in H^1(\Omega)$ are solution to:

$$\begin{cases} -\mathrm{div}(\gamma \nabla u_G) = f & \text{in } \Omega, \\ u_G = 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial u_G}{\partial n} = g & \text{on } \Gamma_N, \\ \gamma \frac{\partial u_G}{\partial n} = 0 & \text{on } \Gamma, \end{cases} \quad \text{and} \quad \begin{cases} -\mathrm{div}(\gamma \nabla u_{G_{x_0,\varepsilon}}) = f & \text{in } \Omega, \\ u_{G_{x_0,\varepsilon}} = 0 & \text{on } \Gamma_D \cup \omega_{x_0,\varepsilon}, \\ \gamma \frac{\partial u_{G_{x_0,\varepsilon}}}{\partial n} = g & \text{on } \Gamma_N, \\ \gamma \frac{\partial u_{G_{x_0,\varepsilon}}}{\partial n} = 0 & \text{on } \Gamma \setminus \overline{\omega_{x_0,\varepsilon}}. \end{cases}$$

Asymptotic formulas for the potential

The calculation of the expansions of $u_{G_{x_0,\varepsilon}}$ and $J(G_{x_0,\varepsilon})$ relies on asymptotic analysis.

Theorem 2.

The following asymptotic expansion holds, at any point $x \in \overline{\Omega}$, $x \notin \Sigma \cup \{x_0\}$:

$$u_{G_{x_0,\varepsilon}}(x) = u_G(x) - \frac{\pi}{|\log \varepsilon|}\gamma(x_0)u_G(x_0)N(x,x_0) + \mathrm{o}\left(\frac{1}{|\log \varepsilon|}\right) \quad \text{if } \; d = 2,$$

and

$$u_{G_{x_0,\varepsilon}}(x) = u_G(x) - 4\varepsilon\gamma(x_0)u_G(x_0)N(x,x_0) \quad \text{if } d = 3,$$

where $N(x,y)$ is the Green's function of the background problem.

E. Bonnetier, C. Brito-Pacheco, C. Dapogny and R. Estevez, *Numerical shape and topology optimization of regions supporting the boundary conditions of a physical problem*, to appear in ESAIM: Control, Optimization, Calculus of Variations, (2025).

Asymptotic formulas for a quantity of interest

The shape functional evaluated at the perturbed shape reads:

$$J(G_{x_0,\varepsilon}) = \int_\Omega j(u_{G_{x_0,\varepsilon}})\,\mathrm{d}x.$$

Corollary 3.

The quantity $J(G_{x_0,\varepsilon})$ has the following asymptotic expansion at 0:

$$\textit{If } d = 2, \quad J(G_{x_0,\varepsilon}) = J(G) + \frac{\pi}{|\log \varepsilon|}\gamma(x_0)u_G(x_0)p_G(x_0) + \mathrm{o}\left(\frac{1}{|\log \varepsilon|}\right),$$

and

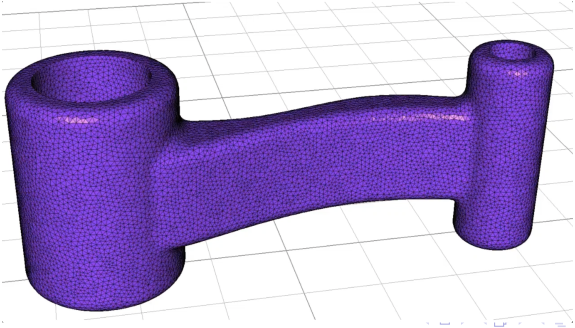
$$\textit{If } d = 3, \quad J(G_{x_0,\varepsilon}) = J(G) + 4\varepsilon\gamma(x_0)u_G(x_0)p_G(x_0) + \mathrm{o}(\varepsilon),$$

where p_G is the unique solution in $H^1(\Omega)$ to the adjoint problem:

$$\begin{cases} -\mathrm{div}(\gamma\nabla p_G) = -j'(u_G) & \text{in } \Omega, \\ p_G = 0 & \text{on } \Gamma_D, \\ \gamma\frac{\partial p_G}{\partial n} = 0 & \text{on } \Gamma_N \cup \Gamma. \end{cases}$$

Numerical algorithm

- At each iteration $n = 0, \ldots$, the fixed shape Ω is equipped with a mesh $\mathcal{T}^n$.
- Its surface part $\mathcal{S}^n$ contains a sub-triangulations $\mathcal{S}^n_G$ for the optimized region G^n.
- The finite element computations for u_{G^n} and p_{G^n} are conducted on $\mathcal{T}^n$.
- A descent direction θ^n is obtained from $J'_\varepsilon(G^n)$, $C'_\varepsilon(G^n)$.
- The updates $\mathcal{T}^n \to \mathcal{T}^{n+1}$ leverage a mesh evolution algorithm [BriDa, AlDaFre].
- Topological derivatives are periodically used to add small disks to G.

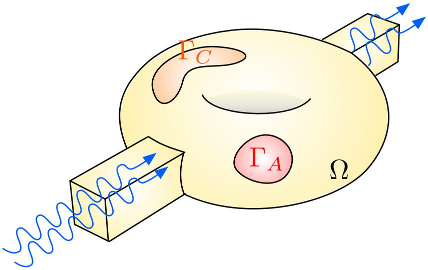


Optimization of a micro-osmotic mixer (I)

- Electro-osmotic mixers achieve the mixture of two fluids inside a device Ω by maximizing the electric field induced by electrodes on $\partial\Omega$.
- The boundary of Ω is decomposed as:

$$\partial\Omega = \overline{\Gamma_C} \cup \overline{\Gamma_A} \cup \overline{\Gamma},$$

where
  - Γ_C is the cathode,
  - Γ_A is the anode,
  - Ω is insulated on Γ.



- The potential inside Ω is the solution to:

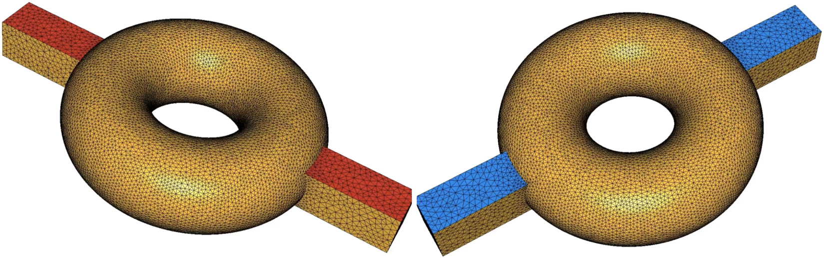
$$\begin{cases} -\mathrm{div}(\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_C, \\ u = u_{\text{in}} & \text{on } \Gamma_A, \\ \gamma \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma. \end{cases}$$

- We aim to maximize the electric power inside Ω with respect to Γ_A and Γ_C:

$$J(\Gamma_A, \Gamma_C) = -\int_\Omega |\gamma \nabla u_{\Gamma_A,\Gamma_C}|^2 \, \mathrm{d}x,$$

under constraints on the surface measures of Γ_A and Γ_C.

Optimization of a micro-osmotic mixer (II)

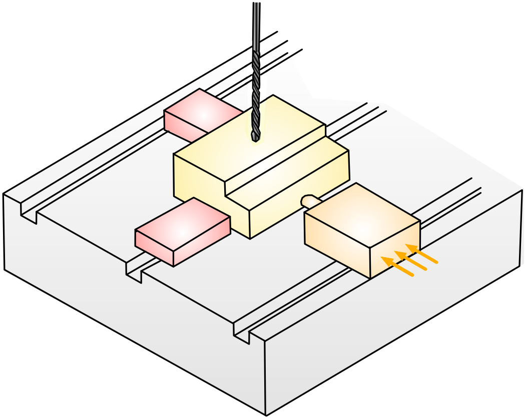


Optimization of (left) the anode, (right) the cathode of a micro-osmotic mixer.

Optimization of a fixture system (I)

During its construction, a mechanical structure $\Omega \subset \mathbb{R}^3$ is stilled by a clamp-locator system:

- Locators are regions of $\partial\Omega$ where the displacement is prevented;
- Clamps are regions where a surface load is applied to maintain the part.

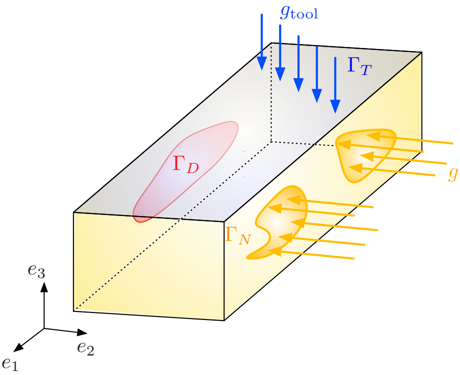


Optimization of a fixture system (II)

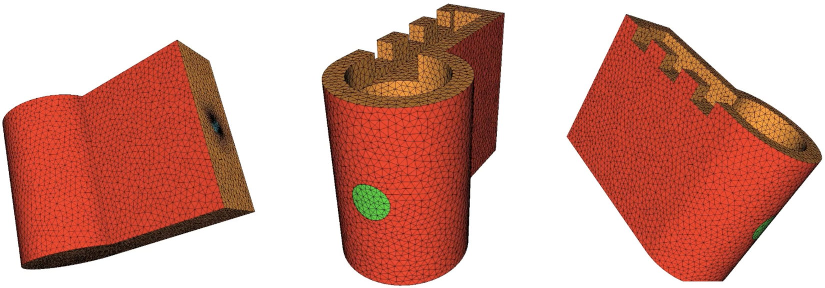
- Let $\Omega \subset \mathbb{R}^3$ be a fixed structure.
- A load g_{tool} is applied on $\Gamma_T \subset \partial\Omega$ by the machine tool.
- Ω is located on Γ_D.
- It is clamped on Γ_N: a load g is applied.
- The remaining boundary Γ is free.
- The displacement u of Ω is solution to the linear elasticity system.
- We aim to minimize the displacement of the structure,

$$J(\Gamma_D, \Gamma_N) = \int_\Omega |u_{\Gamma_D,\Gamma_N}|^2 \, \mathrm{d}x,$$

under constraints on the surfaces of Γ_D and Γ_N.



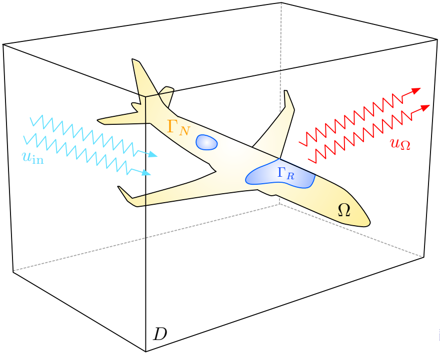
Optimization of a fixture system (III)



Optimal design of clamps and locators on the boundary of a manufactured mechanical part.

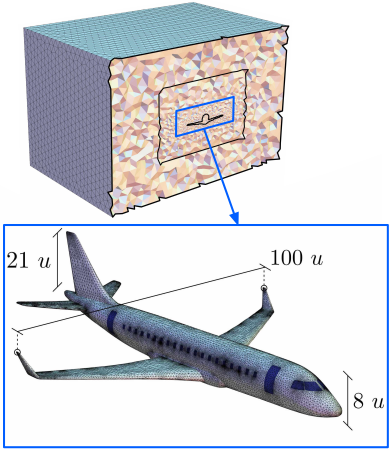
Optimal repartition of sound-soft and sound-hard materials (I)

- The interaction of an incident wave u_{in} (e.g. sound) with an obstacle Ω inside the medium D induces a scattered wave u.
- This scattering effect can be used to detect or reconstruct Ω, which is either
  - Desirable, as in medical imaging (tomography, etc.),
  - Undesirable, e.g. when Ω is a stealth military submarine or aircraft.
- We aim to optimize the repartition of a sound-soft and a sound-hard materials on $\partial\Omega$ to cloak Ω, i.e. make it invisible to measurements of the scattered wave.



- The obstacle Ω is an aircraft.
- The incoming wave reads $u_{\text{in}}(x) = e^{-i\omega\xi\cdot x}$.
- The boundary $\partial\Omega$ is decomposed as $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma_R}$, where:
  - Γ_N supports Neumann conditions;
  - Γ_R bears Robin conditions.
- The scattered wave $u(x)$ is solution to the Helmholtz equation:

$$\left\{ \begin{array}{rl} -\text{div}(\gamma\nabla u) - \omega^2 u = 0 & \text{in } \Omega \\ \gamma\frac{\partial u}{\partial n} = -\gamma\frac{\partial u_{\text{in}}}{\partial n} & \text{on } \Gamma_N, \\ \gamma\frac{\partial u}{\partial n} + \frac{i\omega}{z} u = -\gamma\frac{\partial u_{\text{in}}}{\partial n} - \frac{i\omega}{z} u_{\text{in}} & \text{on } \Gamma_R, \\ + \text{ radiation boundary conditions} & \text{on } \partial D. \end{array} \right.$$



- We optimize Γ_R to minimize the amplitude of the scattered wave:

$$J(\Gamma_R) = \int_{D\setminus\overline{\Omega}} |u_{\Gamma_R}|^2 \, dx,$$

under a constraint on its surface.

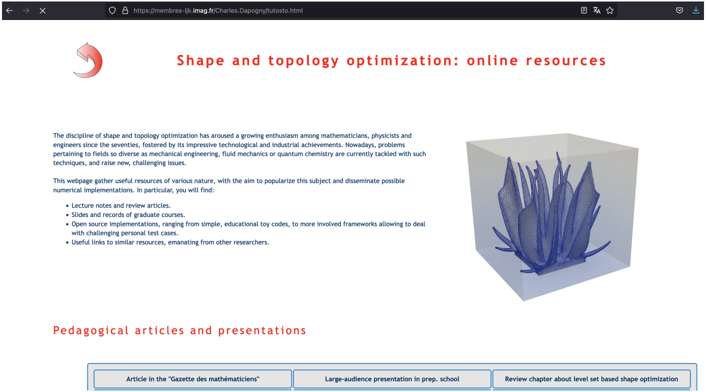
Optimal repartition of sound-soft and sound-hard materials (III)

Optimization of the repartition of a sound-soft and a sound-hard material on the body of an aircraft.

A word of advertisement

- All the numerical realizations are based on open-source libraries.
- A webpage gathering lecture notes, slides, demonstration codes, etc.

https://membres-ljk.imag.fr/Charles.Dapogny/tutosto.html


https://membres-ljk.imag.fr/Charles.Dapogny/tutosto.html

Shape and topology optimization: online resources

The discipline of shape and topology optimization has aroused a growing enthusiasm among mathematicians, physicists and engineers since the seventies, fostered by its impressive technological and industrial achievements. Nowadays, problems pertaining to fields so diverse as mechanical engineering, fluid mechanics or quantum chemistry are currently tackled with such techniques, and raise new, challenging issues.

This webpage gather useful resources of various nature, with the aim to popularize this subject and disseminate possible numerical implementations. In particular, you will find:

- Lecture notes and review articles.
- Slides and records of graduate courses.
- Open source implementations, ranging from simple, educational toy codes, to more involved frameworks allowing to deal with challenging personal test cases.
- Useful links to similar resources, emanating from other researchers.

Pedagogical articles and presentations

Article in the "Gazette des mathématiciens" | Large-audience presentation in prep. school | Review chapter about level set based shape optimization

Thank you!

Thank you for your attention!

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