Homogenization and the eigenvalues of the Neumann-Poincaré operator

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Localized plasmonic resonances (I)

A localized plasmon resonance is a phenomenon caused by the interaction between an electromagnetic wave and a nanoparticle in a dielectric medium.

The Lycurgus cup is encrusted with gold nanoparticles. It looks (left) green when seen in reflection, and (right) red when seen in transmission.

Localized plasmonic resonances (II)

- When the nanoparticle is excited by an electromagnetic wave whose frequency is close to a plasmonic resonance,
  - the absorbing and scattering properties of the particle are strongly enhanced,
  - the electric field blows up in the vicinity of the particle.
- Localized plasmonic resonances occur only in specific situations:
  - The size of the nanoparticle has to be much smaller than the wavelength,
  - The electric permittivity of the particle must have negative real part, as is the case, e.g. of metallic particles (gold, silver) at optical frequencies.
- The great sensitivity of plasmonic resonances to the local environment of the particle has been used as an ingredient in accurate imaging processes [Ma]:
  - biosensors, gold nanoparticles being harmless for health;
  - spectroscopy devices in biochemistry, to image molecular adsorption on DNA, polymers, etc.

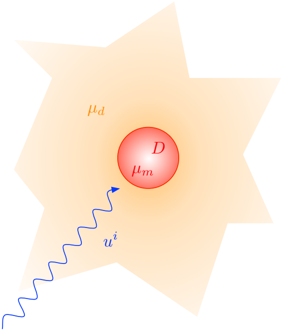
Mathematical model for plasmonic resonances (I)

The field u scattered by an incident wave u^i is solution to the TM Helmholtz equation:

$$\begin{cases} \mathrm{div}(\frac{1}{\mu_D}\nabla u) + \omega^2 \varepsilon_D u = 0 & \text{on } D \cup (\mathbb{R}^d \setminus \overline{D}), \\ u^+ = u^- & \text{on } \partial D, \\ \frac{1}{\mu_d}\frac{\partial u^+}{\partial n} = \frac{1}{\mu_m}\frac{\partial u^-}{\partial n} & \text{on } \partial D, \\ u - u^i & \text{satisfies the Sommerfeld condition at infinity.} \end{cases}$$

When the frequency ω is fixed and $|D| \to 0$, a rescaling shows that plasmonic resonances are governed by the existence of non trivial solutions to the quasi-static equation [AmMiRuiZha, AmRuiYuZha]:

$$\begin{cases} \Delta u = 0 & \text{on } D \cup (\mathbb{R}^d \setminus \overline{D}), \\ u^+ = u^- & \text{on } \partial D, \\ \frac{1}{\mu_d}\frac{\partial u^+}{\partial n} = \frac{1}{\mu_m}\frac{\partial u^-}{\partial n} & \text{on } \partial D, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$



Mathematical model for plasmonic resonances (II)

This quasi-static problem is often investigated using potential theory; u is represented as a single layer potential $u = \mathcal{S}_D\phi$, $\phi \in H^{-1/2}(\partial D)$:

$$u(x) = \mathcal{S}_D\phi(x) := \int_{\partial D} G(x,y)\phi(y)\, ds, \text{ where}$$

$$G(x,y) = \begin{cases} \frac{1}{2\pi}\log|x-y| & \text{if } d = 2, \\ \frac{|x-y|^{2-d}}{\omega_d(2-d)} & \text{if } d \geq 3, \end{cases} \quad \text{is the Newtonian potential.}$$

Using the Plemelj jump relations on ∂D:

$$\frac{\partial u^{\pm}}{\partial n} = \pm\frac{1}{2}\phi + \mathcal{K}_D^*\phi,$$

where $\mathcal{K}_D^* : H^{-1/2}(\partial D) \to H^{-1/2}(\partial D)$ is the Neumann-Poincaré operator of D:

$$\mathcal{K}_D^*\phi(x) = \int_{\partial D} \frac{\partial G}{\partial n_x}(x,y)\, \phi(y)\, ds(y),$$

the search for plasmonic resonances boils down to the eigenvalue problem:

$$\text{Find } \phi \in H^{-1/2}(\partial D) \text{ s.t. } \lambda\phi - \mathcal{K}_D^*\phi = 0, \text{ where } \lambda = \frac{1}{2}\frac{\mu_d + \mu_m}{\mu_d - \mu_m}.$$

More about the Neumann-Poincaré operator

Proposition 1.

If D is of class $\mathcal{C}^{1,\alpha}$,

- *the operator $\mathcal{K}_D^*$ is compact.*
- *The spectrum $\sigma(\mathcal{K}_D^*)$ is contained in $(-\frac{1}{2}, \frac{1}{2}]$. It consists of a discrete sequence with 0 as unique accumulation point.*

The Neumann-Poincaré operator is a key tool in the study of many interface problems with various origins; see [Kan] and references therein:

- Detection and imaging of inhomogeneities embedded in an ambient medium,
- passive cloaking, and cloaking by anomalous localized resonances,
- Analysis of stress concentration between close-to-touching inclusions (metallic particles, elastic fibers, etc.).

Purposes of the present work

1. Investigate the plasmonic resonances associated to a large collection $D_1, \ldots, D_N$ of N small particles: do interactions between particles stir new resonance effects?

2. Investigate the quasi-static limit of the Helmholtz equation,

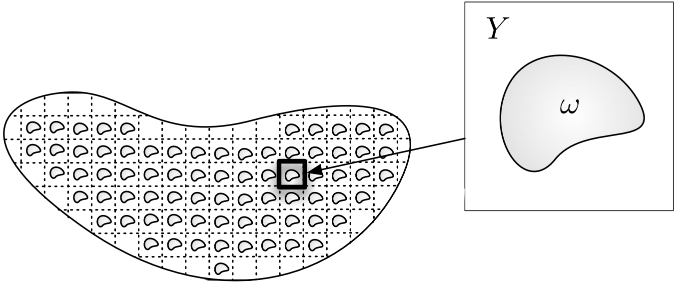
$$\left\{ \begin{array}{ll} -\text{div}(A_D \nabla u) = f & \text{in } \mathbb{R}^d, \\ + \text{ conditions at infinity} & \end{array} \right. ,$$

$$\text{where } A_D(x) = \left\{ \begin{array}{ll} a & \text{if } x \in D_1 \cup \ldots \cup D_N, \\ 1 & \text{otherwise,} \end{array} \right.$$

when the conductivity a is negative, and the number N of particles grows to ∞.

The homogenization setting

Microscopic inclusions with rescaled pattern $\omega \subset Y := (0,1)^d$ are periodically distributed in a macroscopic domain $\Omega \subset \mathbb{R}^d$.



Homogenized setting for a periodic distribution of inclusions.

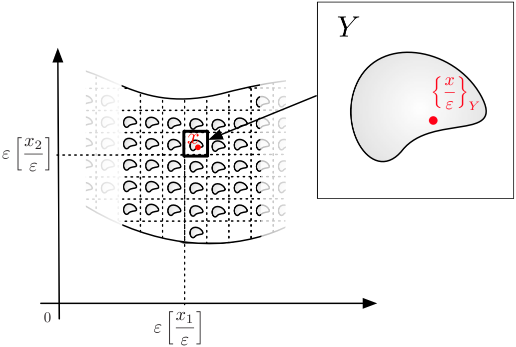
Working assumptions:

- ω is smooth and strongly included in Y: $\omega \Subset Y$;
- ω and $Y \setminus \overline{\omega}$ are connected.

Notations

- Macroscopic and microscopic 'coordinates' of $x \in \mathbb{R}^d$:
$$x = \varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon \left\{\frac{x}{\varepsilon}\right\}_Y,$$
for $\left[\frac{x}{\varepsilon}\right]_Y \in \mathbb{Z}^d$ and $\left\{\frac{x}{\varepsilon}\right\}_Y \in Y$.

- Indices of the cells that are strictly contained in Ω:
$$\Xi_\varepsilon = \left\{\xi \in \mathbb{Z}^d,\ \varepsilon(\xi + Y) \Subset \Omega\right\},$$
and corresponding region in Ω:
$$\mathcal{O}_\varepsilon = \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + Y).$$



- The considered set of inclusions is:
$$\omega_\varepsilon = \bigcup_{\xi \in \Xi_\varepsilon} \omega_\varepsilon^\xi, \text{ where } \omega_\varepsilon^\xi := \varepsilon(\xi + \omega).$$

The Poincaré variational principle

We consider the conductivity equation:

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{where } A_\varepsilon(x) = \begin{cases} a & \text{if } x \in \omega_\varepsilon, \\ 1 & \text{if } x \in \Omega \setminus \overline{\omega_\varepsilon}, \end{cases} \qquad (\mathcal{P}_\varepsilon)$$

and $f \in H^{-1}(\Omega)$ is a source.

A key tool in its study is the Poincaré variational operator $T_\varepsilon : H^1_0(\Omega) \to H^1_0(\Omega)$: for $u \in H^1_0(\Omega)$, $T_\varepsilon u$ is the unique element in $H^1_0(\Omega)$ such that:

$$\forall v \in H^1_0(\Omega), \int_\Omega \nabla(T_\varepsilon u) \cdot \nabla v \, dx = \int_{\omega_\varepsilon} \nabla u \cdot \nabla v \, dx.$$

Indeed, the conductivity equation $(\mathcal{P}_\varepsilon)$ is equivalent to:

$$(\lambda \mathrm{Id} - T_\varepsilon)u = \lambda g, \text{ where } \lambda = \frac{1}{1-a}, \text{ and } f = \nabla g.$$

In particular, $(\mathcal{P}_\varepsilon)$ is well-posed iff $\frac{1}{1-a} \notin \sigma(T_\varepsilon)$.

The spectrum of T_ε (I)

- T_ε is a positive, self-adjoint operator with norm $||T_\varepsilon|| \leq 1$.
- The following orthogonal decomposition holds:

$$H_0^1(\Omega) = \mathrm{Ker}(T_\varepsilon) \oplus \mathfrak{h}_\varepsilon \oplus \mathrm{Ker}(\mathrm{Id} - T_\varepsilon),$$

  where

  - $\mathrm{Ker}(T_\varepsilon) = \left\{ u \in H_0^1(\Omega), \ u = \text{ a cste in each connected component of } \omega_\varepsilon, \right\}$
  - $\mathrm{Ker}(\mathrm{Id} - T_\varepsilon) = \left\{ u \in H_0^1(\Omega), \ u = 0 \text{ on } \Omega \setminus \overline{\omega_\varepsilon}, \right\}$
  - $\mathfrak{h}_\varepsilon$ ($\approx$ the space of single layer potentials) contains the $u \in H_0^1(\Omega)$ such that:

$$\left\{ \begin{array}{cl} \Delta u = 0 & \text{on } \omega_\varepsilon \cup (\Omega \setminus \overline{\omega_\varepsilon}), \\ \int_{\partial \omega_\varepsilon^\xi} \frac{\partial u^+}{\partial n} \, ds = 0 & \text{for each connected component } \omega_\varepsilon^\xi \text{ of } \omega_\varepsilon. \end{array} \right.$$

The spectrum of T_ε (II)

Proposition 2 ([BonTri, KhaPuSha]).

The spectrum of $T_\varepsilon : \mathfrak{h}_\varepsilon \to \mathfrak{h}_\varepsilon$ is a translate of that $\sigma(\mathcal{K}_\varepsilon^*)$ of the Neumann-Poincaré operator; it is a discrete sequence of eigenvalues with $\frac{1}{2}$ as unique accumulation point.

$$0 < \lambda_1^- \leq \lambda_2^- \leq \ldots \leq \frac{1}{2}, \text{ and } \frac{1}{2} \leq \ldots \leq \lambda_2^+ \leq \lambda_1^+ < 1.$$

If $\{w_i^\pm\}_{i\geq 1}$ are the associated eigenfunctions, the min-max formulae hold:

$$\lambda_i^- = \min_{\substack{u\in\mathfrak{h}_\varepsilon\setminus\{0\}\\ u\perp w_1^-,\ldots,w_{i-1}^-}} \frac{\displaystyle\int_{\omega_\varepsilon} |\nabla u|^2\,dx}{\displaystyle\int_\Omega |\nabla u|^2\,dx} = \max_{\substack{F_i\subset\mathfrak{h}_\varepsilon\\ \dim(F_i)=i-1}} \min_{u\in F_i^\perp\setminus\{0\}} \frac{\displaystyle\int_{\omega_\varepsilon} |\nabla u|^2\,dx}{\displaystyle\int_\Omega |\nabla u|^2\,dx},$$

and

$$\lambda_i^+ = \max_{\substack{u\in\mathfrak{h}_\varepsilon\setminus\{0\}\\ u\perp w_1^+,\ldots,w_{i-1}^+}} \frac{\displaystyle\int_{\omega_\varepsilon} |\nabla u|^2\,dx}{\displaystyle\int_\Omega |\nabla u|^2\,dx} = \min_{\substack{F_i\subset\mathfrak{h}_\varepsilon\\ \dim(F_i)=i-1}} \max_{u\in F_i^\perp\setminus\{0\}} \frac{\displaystyle\int_{\omega_\varepsilon} |\nabla u|^2\,dx}{\displaystyle\int_\Omega |\nabla u|^2\,dx}.$$

Hint of the proof: The operator $R_\varepsilon := T_\varepsilon - \frac{1}{2}\mathrm{Id}$ is related to $\mathcal{K}_\varepsilon^*$ as:

$$2R_\varepsilon u = (\mathcal{S}_\varepsilon \circ \mathcal{K}_\varepsilon^* \circ S_\varepsilon^{-1})(u|_{\partial\omega_\varepsilon}), \;\; u \in \mathfrak{h}_\varepsilon.$$

More remarks about T_ε

- For $u \in H_0^1(\Omega)$, $T_\varepsilon u$ only depends on $u|_{\omega_\varepsilon} \in H^1(\omega_\varepsilon)$, modulo a function in

$$C(\omega_\varepsilon) := \left\{ u \in H^1(\omega_\varepsilon),\ \exists c_\xi \in \mathbb{R},\ u = c_\xi \text{ in } \omega_\varepsilon^\xi,\ \xi \in \Xi_\varepsilon \right\}.$$

- The values of $T_\varepsilon u$ on $\Omega \setminus \overline{\omega_\varepsilon}$ may be 'easily recovered' from its values inside ω_ε (since $T_\varepsilon u$ is harmonic on $\Omega \setminus \overline{\omega_\varepsilon}$).

- The spectrum of the Neumann-Poincaré operator can be studied from two complementary points of view:
  - by using integral equations, posed on $\partial\omega_\varepsilon$,
  - by variational methods, involving the operator T_ε (posed on a fixed domain).

Goals of this work

The two concurrent goals pursued in this work rewrite, in the homogenization setting:

1. Analyze the asymptotic behavior of the spectrum $\sigma(T_\varepsilon)$ in terms of the limit spectrum:
$$\lim_{\varepsilon \to 0} \sigma(T_\varepsilon) = \left\{ \lambda \in [0,1], \text{ s.t. } \exists \varepsilon_j \downarrow 0, \ \lambda_{\varepsilon_j} \in \sigma(T_{\varepsilon_j}), \ \lambda_{\varepsilon_j} \to \lambda \right\}.$$

2. Explore the well-posedness of the conductivity equation for the voltage potential,
$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases},$$
when the conductivity a inside the inclusions is negative, in the limit $\varepsilon \to 0$.

One part of the following result was observed in [BuRam]:

Theorem 3.

There exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$,

$$(\lambda \in \sigma(T_\varepsilon),\ \ \lambda \notin \{0,1\}) \Rightarrow m \leq \lambda \leq M,$$

where $0 < m < M < 1$ are explicit constants:

$$m = \min_{\substack{u \in \widehat{\mathfrak{h}}_0 \\ u \neq 0}} \frac{\displaystyle\int_\omega |\nabla_y u|^2\, dy}{\displaystyle\int_Y |\nabla_y u|^2\, dy}, \text{ and } M = \max_{\substack{u \in \widehat{\mathfrak{h}}_0 \\ u \neq 0}} \frac{\displaystyle\int_\omega |\nabla_y u|^2\, dy}{\displaystyle\int_Y |\nabla_y u|^2\, dy},$$

and $\widehat{\mathfrak{h}}_0 \subset H^1(Y)/\mathbb{R}$ is the Hilbert space defined by:

$$\widehat{\mathfrak{h}}_0 = \left\{ u \in H^1(Y)/\mathbb{R},\ \ \Delta_y u = 0 \text{ in } \omega \cup (Y \setminus \overline{\omega}),\ \text{and } \int_{\partial\omega} \frac{\partial u^+}{\partial n_y}\, ds = 0 \right\}.$$

Hint of the proof: Use the min-max formulae for the eigenvalues of $T_\varepsilon : \mathfrak{h}_\varepsilon \to \mathfrak{h}_\varepsilon$.

How to study the limiting behavior of sequences $\lambda_\varepsilon \in \sigma(T_\varepsilon)$?

- T_ε converges weakly to the trivial operator $|\omega|\mathrm{Id}$:

$$\forall u \in H_0^1(\Omega), \quad T_\varepsilon u \xrightarrow{\varepsilon \to 0} |\omega| u, \text{ weakly in } H_0^1(\Omega).$$

- This poor convergence allows to infer nothing about the spectrum $\sigma(T_\varepsilon)$.
- As is well-known in homogenization theory, one remedy to get a stronger convergence uses correctors, describing the oscillations of the functions $T_\varepsilon u$ at the ε-scale.
- These correctors can be used in the study of eigenvalues - see [SanVo, MosVo] - but this approach seems difficult in our context.
- Our work is largely inspired by that of [AlCon] about Bloch wave homogenization. T_ε is rescaled into an operator

$$\mathbb{T}_\varepsilon : L^2(\Omega, H^1(\omega)/\mathbb{R}) \to L^2(\Omega, H^1(\omega)/\mathbb{R}),$$

which 'does the same' as T_ε, but acts on functions $\phi(x, y)$ depending on both macroscopic and microscopic variables x and y.

Definition 1 ([AlCon, CioDamGri]).

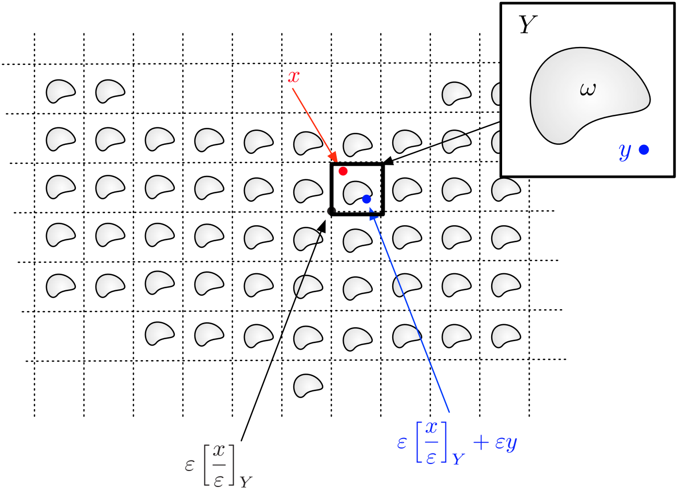
- *The extension operator* $E_\varepsilon : L^2(\Omega) \to L^2(\Omega \times Y)$ *is defined by:*

$$E_\varepsilon u(x,y) = \begin{cases} u(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y) & \text{if } x \in \mathcal{O}_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

- *The projection operator* $P_\varepsilon : L^2(\Omega \times Y) \to L^2(\Omega)$ *is defined by:*

$$P_\varepsilon \phi(x) = \begin{cases} \displaystyle\int_Y \phi(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y)\, dz & \text{if } x \in \mathcal{O}_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

The extension and projection operators E_ε and P_ε (II)



The operator E_ε rescales the content of each cell to size 1.

The extension and projection operators E_ε and P_ε (III)

- E_ε and P_ε are bounded with norm 1.

- $E_\varepsilon : L^2(\Omega) \to L^2(\Omega \times Y)$ and $P_\varepsilon : L^2(\Omega \times Y) \to L^2(\Omega)$ are adjoint operators.

- E_ε and P_ε are 'almost inverse' from one another:
  - For $u \in L^2(\Omega)$, $P_\varepsilon E_\varepsilon u(x) = \begin{cases} u(x) & \text{if } x \in \mathcal{O}_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$
  - For $\phi \in L^2(\Omega \times Y)$, $E_\varepsilon P_\varepsilon \phi \to \phi$ strongly in $L^2(\Omega \times Y)$, as $\varepsilon \to 0$.

- *(Two-scale convergence):* (see also [Al, Ngue]) Let u_ε be a bounded sequence in $H^1(\Omega)$; then, up to a subsequence, there exist $u_0 \in H^1(\Omega)$ and $\widehat{u} \in L^2(\Omega, H^1_\#(Y))$ such that:

$$u_\varepsilon \to u_0 \text{ weakly in } H^1(\Omega), \text{ and } E_\varepsilon(\nabla u_\varepsilon) \to \nabla u_0 + \nabla_y \widehat{u} \text{ weakly in } L^2(\Omega \times Y).$$

The rescaled operator $\mathbb{T}_\varepsilon$

We define the rescaled operator

$$\mathbb{T}_\varepsilon = E_\varepsilon T_\varepsilon P_\varepsilon : L^2(\Omega, H^1(\omega)/\mathbb{R}) \to L^2(\Omega, H^1(\omega)/\mathbb{R}),$$

which is possible since T_ε is defined modulo functions in $\mathrm{Ker}(T_\varepsilon)$ (i.e. functions that are constant inside each inclusion ω_ε^ξ).

Proposition 4.

The rescaled operator $\mathbb{T}_\varepsilon$ has the following properties.

- *$\mathbb{T}_\varepsilon$ is self-adjoint.*
- *The spectrum $\sigma(\mathbb{T}_\varepsilon)$ of $\mathbb{T}_\varepsilon$ coincides with that $\sigma(T_\varepsilon)$ of T_ε except that it does not contain the eigenvalue* 0

Properties of the rescaled operator $\mathbb{T}_\varepsilon$.

Proposition 5.

The operator $\mathbb{T}_\varepsilon$ *converges pointwise to a limit* $\mathbb{T}_0$*:*

$$\forall \phi \in L^2(\Omega, H^1(\omega)/\mathbb{R}), \ \mathbb{T}_\varepsilon \phi \xrightarrow{\varepsilon \to 0} \mathbb{T}_0 \phi, \ \text{strongly in } L^2(\Omega, H^1(\omega)/\mathbb{R}).$$

The operator $\mathbb{T}_0$ *is defined by*

$$\mathbb{T}_0 \phi(x,y) = Q\left(\nabla v_0(x) \cdot y + \widehat{v}(x,y)\right), \ \text{where}$$

- $Q : L^2(\Omega, H^1(Y)) \to L^2(\Omega, H^1(\omega)/\mathbb{R})$ *is the natural restriction,*
- $\widehat{v}$ *is the unique solution in* $L^2(\Omega, H^1_\#(Y)/\mathbb{R})$ *to the equation:*

$$-\Delta_y \widehat{v}(x,y) = -\mathrm{div}_y(\mathbb{1}_\omega(y) \nabla_y \phi)(x,y) \ \text{in } H^1_\#(Y), \ \text{a.e. in } x \in \Omega,$$

- v_0 *is the unique solution in* $H^1_0(\Omega)$ *to:*

$$-\Delta v_0 = -\mathrm{div}\left(\int_\omega \nabla_y \phi(x,y)\, dy\right).$$

Hint of proof: This is implied by two-scale convergence: the strong convergence of sequences $\mathbb{T}_\varepsilon u$ shows that $\mathbb{T}_0 u$ 'keeps track' of the ε-oscillations of the $\mathbb{T}_\varepsilon u$.

Single cell resonances.

This convergence result allows to identify one part of $\lim_{\varepsilon \to 0} \sigma(T_\varepsilon)$, corresponding to the resonance modes of a single inclusion $\omega \subset Y$.

Theorem 6.

The limit spectrum $\lim_{\varepsilon \to 0} \sigma(T_\varepsilon)$ *contains the cell spectrum, i.e. the spectrum of the operator* $T_0 : H^1_\#(Y)/\mathbb{R} \to H^1_\#(Y)/\mathbb{R}$ *defined by: for* $u \in H^1_\#(Y)/\mathbb{R}$,

$$\forall v \in H^1_\#(Y)/\mathbb{R}, \ \int_Y \nabla_y (T_0 u) \cdot \nabla_y v \, dy = \int_\omega \nabla_y u \cdot \nabla_y v \, dy.$$

Hint of the proof: The pointwise convergence $\mathbb{T}_\varepsilon \to \mathbb{T}_0$ allows to infer:

$$\lim_{\varepsilon \to 0} (\sigma(T_\varepsilon) \setminus \{0\}) = \lim_{\varepsilon \to 0} \sigma(\mathbb{T}_\varepsilon) \supset \sigma(\mathbb{T}_0).$$

In addition, it readily follows from the definition of $\mathbb{T}_0$ that $\sigma(T_0) \subset \sigma(\mathbb{T}_0)$.

□

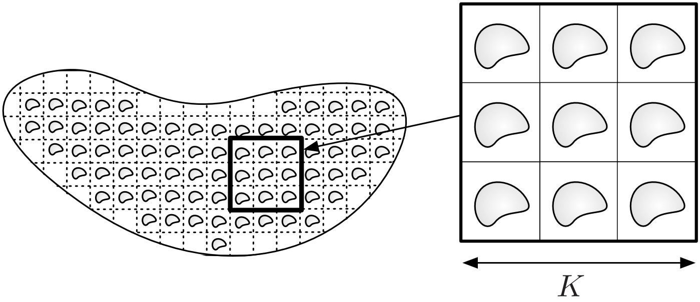
Rescaling T_ε over packs of cells (I)

Following [AlCon, Plan], the previous rescaling procedure can be performed over packs of K^d cells, $K > 1$.

We define new extension and projection operators over K^d cells:

$$E_\varepsilon^K : L^2(\Omega) \to L^2(\Omega \times KY), \text{ and } P_\varepsilon^K : L^2(\Omega \times KY) \to L^2(\Omega),$$

which satisfy analogous properties to those of their single-cell counterparts.



Rescaling over a pack of K^d cells.

Rescaling T_ε over packs of cells (II)

- We introduce the collection of K^d copies of ω:

$$\omega^K := \bigcup_{0\le j\le K-1} (j+\omega) \subset KY,$$

and the quotient Hilbert space $H^K := H^1(\omega^K)/C(\omega^K)$, where

$$C(\omega^K) := \left\{ u \in H^1(\omega^K),\ \exists c_j \in \mathbb{R},\ u \equiv c_j \text{ on } (j+\omega),\ 0 \le j \le K-1 \right\}.$$

- The rescaled operator $\mathbb{T}_\varepsilon^K : L^2(\Omega, H^K) \to L^2(\Omega, H^K)$ is now $\mathbb{T}_\varepsilon^K = E_\varepsilon^K T_\varepsilon P_\varepsilon^K$.

- Again, we prove the the pointwise convergence of $\mathbb{T}_\varepsilon^K$ to a limit operator $\mathbb{T}_0^K$:

$$\text{For all } \phi \in H^K,\ \ \mathbb{T}_\varepsilon^K \phi \xrightarrow{\varepsilon\to 0} \mathbb{T}_0^K \phi, \text{ strongly in } H^K.$$

- The spectrum $\sigma(\mathbb{T}_0^K)$ contains that of the operator $T_0^K : H^K \to H^K$ defined by:

$$\forall v \in H^K,\ \int_{KY} \nabla_y (T_0^K u) \cdot \nabla_y v \, dy = \int_{\omega^K} \nabla_y u \cdot \nabla_y v \, dy.$$

The Bloch spectrum.

The spectrum $\sigma(T_0^K)$ is analyzed using a discrete Bloch decomposition [AguiCon]:

Theorem 7.

Let u in $L^2_\#(KY)$. Then, there exist K^d complex-valued functions $u_j(y) \in L^2_\#(Y)$, $j = (j_1, ..., j_d)$, $j_1, ..., j_d = 0, ..., K-1$, such that:

$$u(z) = \sum_{0 \le j \le K-1} u_j(z)\, e^{\frac{2i\pi j}{K} \cdot z}, \text{ a.e. } z \in KY;$$

The u_j are unique and are given by:

$$u_j(y) = \sum_{0 \le j' \le K-1} u(y + j') e^{-2i\pi \frac{j}{K} \cdot (y + j')}, \text{ a.e. } y \in Y.$$

Furthermore, the Parseval identity holds:

$$\forall u, v \in L^2_\#(KY),\ \frac{1}{K^d} \int_{KY} u(z) \overline{v(z)}\, dx = \sum_{0 \le j \le K-1} \int_Y u_j(y) \overline{v_j(y)}\, dy.$$

The Bloch spectrum.

Bloch decomposition behaves well with functions $u \in H^1(\omega^K)$, and diagonalizes operators with Y-periodic coefficients. Hence,

$$\sigma(T_0^K) = \bigcup_{0 \le j \le K-1} \sigma(T_{\eta_j}), \text{ for } \eta_j = \frac{j}{K},$$

and where the operators T_η are defined by:

- For $\eta \neq 0$, $T_\eta : H^1_\#(Y) \to H^1_\#(Y)$ is given by:

$$\forall v \in H^1_\#(Y), \int_Y (\nabla_y(T_\eta u) + 2i\pi\eta(T_\eta u)) \cdot \overline{(\nabla_y v + 2i\pi\eta v)}\, dy = \int_\omega (\nabla_y(T_\eta u) + 2i\pi\eta u) \cdot \overline{(\nabla_y v + 2i\pi\eta v)}\, dy.$$

- $T_0 : H^1_\#(Y)/\mathbb{R} \to H^1_\#(Y)/\mathbb{R}$ is the the same as in the case of a single cell:

$$\forall v \in H^1_\#(Y)/\mathbb{R}, \int_Y \nabla_y(T_0 u) \cdot \overline{\nabla_y v}\, dy = \int_\omega \nabla_y u \cdot \overline{\nabla_y v}\, dy.$$

The Bloch spectrum.

Theorem 8.

The spectrum $\sigma(T_\eta)$ is composed of a discrete sequence of real eigenvalues:

$$0 < \lambda_1^-(\eta) \leq \lambda_2^-(\eta) \leq ... \leq \frac{1}{2} \leq ... \leq \lambda_2^+(\eta) \leq \lambda_1^+(\eta) \leq 1.$$

Moreover, for any $i = 1, ...,$ the mapping $\overline{Y} \ni \eta \mapsto \lambda_i^\pm(\eta)$ is Lipschitz continuous.

Hint of the proof:

- This rests on an adapted, quasi-periodic version of the Poincaré variational principle, which is tightly connected to the (compact) quasi-periodic version of the Neumann-Poincaré operator.
- The Lipschitz continuity of the mappings $\overline{Y} \ni \eta \mapsto \lambda_i^\pm(\eta)$ relies on the (adapted) min-max formulae, and on an argument of [Ge].

□

The Bloch spectrum.

Theorem 9.

The limit spectrum $\lim_{\varepsilon\to 0}\sigma(T_\varepsilon)$ *contains the Bloch spectrum* σ_{Bloch} *defined by*

$$\sigma_{\mathrm{Bloch}} = \bigcup_{i=1}^{\infty}\left[\min_{\eta\in[0,1]^d}\lambda_i^-(\eta),\ \max_{\eta\in[0,1]^d}\lambda_i^-(\eta)\right]\cup\bigcup_{i=1}^{\infty}\left[\min_{\eta\in[0,1]^d}\lambda_i^+(\eta),\ \max_{\eta\in[0,1]^d}\lambda_i^+(\eta)\right].$$

Hint of the proof:

- For every $K\geq 1$, the pointwise convergence $\mathbb{T}_\varepsilon^K \to \mathbb{T}_0^K$ shows that:

$$\lim_{\varepsilon\to 0}\sigma(T_\varepsilon) \quad\supset\quad \sigma(\mathbb{T}_0^k) \quad\supset\quad \sigma(T_0^K) \quad=\quad \bigcup_{0\leq j\leq K-1}\sigma(T_{\eta_j}).$$

- Hence, the limit spectrum contains

$$\bigcup_{i=1}^{\infty}\left\{\lambda_i^{\pm}(\eta_j)\right\}_{\substack{K\geq 1\\ 0\leq j\leq K-1}},$$

which gives the desired band structure since the mappings $\eta\mapsto\lambda_i^{\pm}(\eta)$ are continuous and $\lim_{\varepsilon\to 0}\sigma(T_\varepsilon)$ is a closed set.

The completeness result

The remainder of $\lim_{\varepsilon\to 0}\sigma(T_\varepsilon)$ gathers the limit behaviors of the eigenvectors of T_ε which spend a 'not too small' part of their energy near the macroscopic boundary $\partial\Omega$.

Theorem 10.

The limit spectrum is decomposed as:

$$\lim_{\varepsilon\to 0}\sigma(T_\varepsilon) = \{0,1\} \cup \sigma_{\partial\Omega} \cup \sigma_{\mathrm{Bloch}},$$

where the boundary layer spectrum $\sigma_{\partial\Omega}$ is the set of the $\lambda \in (0,1)$ such that, for any sequence $\lambda_\varepsilon \in \sigma(T_\varepsilon)$ with $\lambda_\varepsilon \to \lambda$, and any corresponding (normalized) eigenvector sequence $u_\varepsilon \in H^1_0(\Omega)$:

$$\forall s > 0, \;\; \lim_{\varepsilon\to 0} \varepsilon^{-(1-1/d+s)} ||\nabla u_\varepsilon||_{L^2(\mathcal{U}_\varepsilon)} = \infty,$$

where $\mathcal{U}_\varepsilon := \{x \in \Omega, \;\; d(x,\partial\Omega) < \varepsilon\}$ is the tubular neighborhood of $\partial\Omega$ with width ε.

The difficulty to characterize more precisely $\sigma_{\partial\Omega}$ reveals very strong interactions between the macroscopic boundary Ω and the inclusions; see [CasZua, MosVo].

General setting

- We now turn to the conductivity equation:

$$\left\{ \begin{array}{ll} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{array} \right. \quad \text{where } A_\varepsilon(x) = \left\{ \begin{array}{ll} a & \text{if } x \in \omega_\varepsilon, \\ 1 & \text{otherwise}, \end{array} \right. \qquad (\mathcal{P}_\varepsilon)$$

and the source f is in $H^{-1}(\Omega)$.

- When $a > 0$, the classical homogenization theory states that u_ε converges weakly in $H_0^1(\Omega)$ to the unique solution u_* of

$$\left\{ \begin{array}{ll} -\mathrm{div}(A^* \nabla u_*) = f & \text{in } \Omega, \\ u_* = 0 & \text{on } \partial\Omega, \end{array} \right. \qquad (\mathcal{P}^*)$$

where the positive definite homogenized tensor is defined by:

$$A^*_{ij} = \int_Y A(y)(\nabla_y w_i + e_i) \cdot (\nabla_y w_j + e_j)\, dy, \ \text{where } A(y) = \left\{ \begin{array}{ll} a & \text{if } y \in \omega, \\ 1 & \text{if } y \in Y \setminus \overline{\omega}. \end{array} \right.$$

and the cell functions $w_i \in H^1_\#(Y)/\mathbb{R}$ solve

$$-\mathrm{div}(A(y)(\nabla w_i + e_i)) = 0 \text{ in } Y, \quad i = 1, ..., d.$$

- What happens when $a < 0$?

The formal, homogenized tensor

The cell problems

$$-\text{div}(A(y)(\nabla_y w_i + e_i)) = 0 \text{ in } Y, \quad i = 1, ..., d.$$

are well-posed provided $\lambda := \frac{1}{1-a}$ does not belong to the spectrum $\sigma(T_0)$ of the cell operator $T_0 : H^1_\#(Y)/\mathbb{R} \to H^1_\#(Y)/\mathbb{R}$:

$$\forall v \in H^1_\#(Y)/\mathbb{R}, \ \int_Y \nabla_y(T_0 u) \cdot \nabla_y v \, dy = \int_\omega \nabla_y u \cdot \nabla_y v \, dy.$$

It then makes sense to define the (formal) homogenized tensor

$$A^*_{ij} = \int_Y A(y)(\nabla_y w_i + e_i) \cdot (\nabla_y w_j + e_j) \, dy$$

as soon as $a \notin \Sigma_\omega := \left\{ a \in \mathbb{C}, \ \frac{1}{1-a} \in \sigma(T_0) \right\}$.

Theorem 11.

Let $a \in \mathbb{C} \setminus \Sigma_\omega$*; then,*

- *If* $u_\varepsilon \in H_0^1(\Omega)$ *is a sequence of solutions to* $(\mathcal{P}_\varepsilon)$ *such that*

$$||\nabla u_\varepsilon||_{L^2(\Omega)} \leq C,$$

  then up to a subsequence, u_ε *converges weakly in* $H_0^1(\Omega)$ *to a solution of* $(\mathcal{P}^*)$.

- *Conversely, if* $u \in H_0^1(\Omega)$ *is one solution to* $(\mathcal{P}^*)$ *(if any), then for any sequence* $a_\varepsilon \to a$, $a_\varepsilon \notin \Sigma_\omega$, *there exists a sequence* $f_\varepsilon \in H^{-1}(\Omega)$ *converging pointwise to* f *such that the voltage potentials* u_ε, *solution to:*

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}, \text{ where } A_\varepsilon(x) = \begin{cases} a_\varepsilon & \text{if } x \in \omega_\varepsilon, \\ 1 & \text{otherwise.} \end{cases}$$

  converge to u *weakly in* $H_0^1(\Omega)$

This indicates that no 'good' solution to $(\mathcal{P}^*)$ can be singled out via such a limiting process.

Partial identification of the limit spectrum

Proposition 12.

Let $a \in \mathbb{C} \setminus \{0\}$*. Then* $\frac{1}{1-a}$ *belongs to the limit spectrum* $\lim_{\varepsilon \to 0} \sigma(T_\varepsilon)$ *if and only if there exists* $f \in H^{-1}(\Omega)$ *and a sequence* $f_\varepsilon \in H^{-1}(\Omega)$ *with* $f_\varepsilon \to f$ *pointwise, such that the solution* $u_\varepsilon \in H_0^1(\Omega)$ *of* $(\mathcal{P}_\varepsilon)$ *with* f_ε *as a source term satisfies* $||\nabla u_\varepsilon||_{L^2(\Omega)^d} \to +\infty$.

Corollary 13.

Let $a \in \mathbb{C} \setminus \Sigma_\omega$*, and let* A^* *be the corresponding homogenized tensor. If*

- *either there exists* $u \in H_0^1(\Omega)$*,* $u \neq 0$ *such that* $-\mathrm{div}(A^* \nabla u) = 0$*,*
- *or there exists* $f \in H^{-1}(\Omega)$ *such that* $(\mathcal{P}^*)$ *does not have a solution,*

then

$$\frac{1}{1-a} \in \lim_{\varepsilon \to 0} \sigma(T_\varepsilon).$$

The particular case of high-contrast

The previous material reveals that the conductivity equation $(\mathcal{P}_\varepsilon)$ is uniformly well-posed as $\varepsilon \to 0$ when a is either 'very small' or 'very large'.

Theorem 14.

There exists a constant $0 < \alpha$ *such that, if the conductivity* a *belongs to* $(-\infty, -1/\alpha) \cup (-\alpha, 0)$*, then*

(i) *For* $0 < \varepsilon$*, the system* $(\mathcal{P}_\varepsilon)$ *for* u_ε *is well-posed, i.e. it has a unique solution for any source* $f \in H^{-1}(\Omega)$*, and* u_ε *depends continuously on* f*.*

(ii) *The homogenized tensor* A^* *is elliptic; in particular,* $(\mathcal{P}^*)$ *is well-posed.*

(iii) *For any source* $f \in H^{-1}(\Omega)$*, the unique solution* $u_\varepsilon \in H^1_0(\Omega)$ *to* $(\mathcal{P}_\varepsilon)$ *converges, weakly in* $H^1_0(\Omega)$*, to the unique solution* u_* *of* $(\mathcal{P}^*)$*.*

Thank you for your attention!

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