ASYMPTOTIC EXPANSION OF THE VOLTAGE POTENTIAL UNDER SMALL ROBIN PERTURBATIONS OF ITS BOUNDARY CONDITIONS

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Abstract. Inspired by questions related to inverse problems and shape optimization, we derive an asymptotic expansion of the voltage potential, solution to a model elliptic second-order partial differential equation, under small perturbations of its boundary conditions. More precisely, the homogeneous Dirichlet or homogeneous Neumann boundary condition in a fixed, reference configuration of the problem is replaced by a Robin boundary condition with admittance $k_\varepsilon > 0$ on a "small" subset ω_ε of the boundary of the ambient domain, vanishing at the limit $\varepsilon \to 0$. In each of these two situations, a general asymptotic formula is established for the voltage potential, which rests on minimal assumptions about the shape of the vanishing subset ω_ε and the parameter k_ε. The scalings of these expansions, capturing the intensity of the perturbation, are measured by new quantities called "Robin-Dirichlet" capacity or a "Robin-Neumann" capacity, which depend on the geometry of ω_ε and on the value of the parameter k_ε. We analyze how these quantities compare to more classical measures of the "smallness" of ω_ε, such as the capacity or the Neumann capacity of ω_ε, according to the behavior of k_ε as $\varepsilon \to 0$.

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Contents

1. Introduction

The analysis of the asymptotic effect of "small" perturbations of the features of a physical model, accounted for by a partial differential equation, is handful for a variety of purposes. On the one hand, it can be used to make the system robust with respect to uncertain fluctuations of its parameters [15]. On the other hand, this sensitivity is the basic ingredient of multiple inverse problems techniques for the gradient-based reconstruction of such features from observational data [5, 37], or for the optimization of the shape of the system with respect to a performance criterion [36, 46].

The present article aims to analyze one particular class of such perturbations. To set ideas, we consider throughout the following reference elliptic boundary value problem posed in a smooth bounded domain Ω in the Euclidean space $\mathbb{R}^d$, where $d = 2$ or 3:

$$\left\{ \begin{array}{ccll} \text{div}(\gamma \nabla u_0) & = & f & \text{in } \Omega, \\ u_0 & = & 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial u_0}{\partial n} & = & 0 & \text{on } \Gamma_N. \end{array} \right. \tag{1.1}$$

Here, Γ_D and Γ_N are disjoint and complementary parts of $\partial\Omega$, respectively bearing homogeneous Dirichlet and Neumann boundary conditions and f is a smooth source term. The conductivity γ of the medium is a smooth function satisfying:

$$0 \ < \ \gamma^- \ \leq \ \gamma(x) \ \leq \ \gamma^+ < \infty, \quad \text{ for a.e. } \ x \in \Omega, \tag{1.2}$$

for some fixed positive constants γ^-, γ^+.

Volumic perturbations. A large body of work in the literature is devoted to the analysis of the asymptotic behavior of the solution to an elliptic equation of the form (1.1) when its material coefficient γ is perturbed in a small subset ω_ε of the domain Ω. In the seminal paper [21], a perturbed version of (1.1) is considered where γ is replaced with another coefficient inside a diametrically small subset ω_ε. Precisely, the conductivity γ in (1.1) is replaced with that γ_ε defined by:

$$\gamma_\varepsilon(x) \ = \ \gamma(x) + (\widetilde{\gamma}(x) - \gamma(x))\, \mathbb{1}_{\omega_\varepsilon}(x), \quad x \in \Omega,$$

where $\widetilde{\gamma}$ is another smooth function also satisfying (1.2) and $\mathbb{1}_A$ stands for the characteristic function of a set A. The set $\omega_\varepsilon := x_0 + \varepsilon\omega$ is a translated and rescaled version of a fixed inclusion pattern $\omega \subset \mathbb{R}^d$; ω_ε lies uniformly far from the boundary $\partial\Omega$ in the sense that;

$$\forall \varepsilon > 0, \quad \text{dist}(\omega_\varepsilon, \partial\Omega) \ \geq \ d_0 > 0,$$

for some fixed distance $d_0 > 0$. In this context, it is shown in [21] that at a point x lying "far" from all the subsets ω_ε, the perturbed solution u_ε satisfies

$$u_\varepsilon(x) \ = \ u_0(x) \ + \ |\omega_\varepsilon| \mathcal{M} \nabla u_0(x_0) \cdot \nabla_y G(x, x_0) \ + \ \text{o}(|\omega_\varepsilon|), \tag{1.3}$$

where $G(x, y)$ is the Green's function of the background problem (1.1), see Section 2. The polarization matrix $\mathcal{M} \in \mathbb{R}^{d \times d}$ in this formula is defined by:

$$\mathcal{M}_{ij} \ = \ \int_\omega \frac{\partial \varphi_j}{\partial y_i}(y)\, \text{d}y, \quad i, j = 1, \ldots, d,$$

involving the solutions φ_j to d auxiliary problems set in the infinite space $\mathbb{R}^d$:

$$-\text{div}\Big(\gamma_1(y)\nabla(\varphi_j(y)+y_j)\Big) \;=\; 0 \quad \text{in } \mathbb{R}^d, \text{ where } \;\gamma_1(y) = \gamma(x_0) + (\widetilde{\gamma}(x_0) - \gamma(x_0))\,\mathbb{1}_\omega(y).$$

Asymptotic expansions of the form (1.3) were derived in more challenging physical contexts, described by e.g. the linear elasticity system [7], or the Helmholtz and Maxwell equations [9], and in the case of eigenvalue problems [8]. These analyses were also extended to the situations of cracks or strip-like perturbations [13, 14, 30]. On a different note, assuming both conductivities γ and $\widetilde{\gamma}$ to be constant, layer potential techniques allow to derive higher-order expansions of the perturbed field u_ε, see [6] and the references therein.

Intuitively, formula (1.3) expresses that, up to first order terms in $|\omega_\varepsilon|$, the set ω_ε asymptotically acts like a dipole placed at x_0, with moment $\mathcal{M}\nabla u_0(x_0)$. In the context of inverse problems, this formula contains the information that one may expect to recover from measurements of $u_\varepsilon - u_0$: the singularity of $\nabla_y G(x,x_0)$ allows to determine the position x_0 of the center of ω_ε, while some partial information about the geometry of ω and on the contrast between material coefficients is encoded in $\mathcal{M}$. Asymptotic expansions of the form (1.3) have indeed been used to design efficient detection algorithms akin to linear sampling methods [4, 19], and to propose approximate cloaking mechanisms [38].

The general structure of the first-order correction to the background potential induced by a perturbation of the conductivity coefficient within a "small" set ω_ε of arbitrary shape was elucidated in [20]. This work proceeds under the following minimal assumptions about ω_ε:

- For each $\varepsilon > 0$, ω_ε may be the reunion of $n_\varepsilon < \infty$ connected components,
- The ω_ε are all contained in a fixed compact set $K_0 \Subset \Omega$, and thus stay uniformly away from $\partial\Omega$,
- The volume $|\omega_\varepsilon|$ tends to 0 as $\varepsilon \to 0$.

In this setting, there exists a subsequence of the indices ε, still denoted by ε for simplicity, a non-trivial Radon measure μ with support in K_0, and a polarization tensor field $\mathcal{M} \in L^2(\Omega; \mathrm{d}\mu)$ such that the following asymptotic expansion holds true at each point $x \in \Omega \setminus K_0$:

$$u_\varepsilon(x) \;=\; u_0(x) + |\omega_\varepsilon|\langle \mu(y), \mathcal{M}(y)\nabla u_0(y)\cdot\nabla G(x,y)\rangle \;+\; \mathrm{o}(|\omega_\varepsilon|). \tag{1.4}$$

This compensated compactness result echoes to typical results from two-phase homogenization theory [3, 11, 43]; it somehow shows that "small" perturbations are "dilute" limits of the homogenization phenomenon, i.e. when the fraction of one of the phases at play tends to 0.

Counterpart structure theorems were later proved in the more challenging physical contexts of elasticity and electromagnetism, see [12] and [34], respectively. From the applications viewpoint, these developments have led to the notion of topological derivative, appraising how a physical quantity, involving the voltage potential, depends on the introduction of a small heterogeneity (or a hole) within the ambient medium, see [23, 31, 46, 47].

Perturbations of boundary conditions. Our recent work [17] deals with another type of perturbations of the model background equation (1.1), which consists in altering its boundary conditions on a "small" region ω_ε of $\partial\Omega$. This work considers two different situations:

(i) The sets ω_ε are contained in the region Γ_N and the Neumann boundary condition imposed in there is replaced by a homogeneous Dirichlet condition, leading to a perturbed potential $u_\varepsilon^{N,\infty}$.

(ii) The sets ω_ε are contained in Γ_D and the Dirichlet condition imposed in there is replaced by a homogeneous Neumann condition, leading to a perturbed potential $u_\varepsilon^{D,0}$.

In both situations, the perturbed voltage potential has a similar asymptotic structure to that in (1.4). Precisely, up to extraction of a subsequence (still indexed by ε), there exists a Radon measure μ supported on $\partial\Omega$, such that for any smooth cut-off function η, with support strictly inside Γ_N, it holds:

$$u_\varepsilon^{N,\infty}(x) \;=\; u_0(x) + \text{cap}(\omega_\varepsilon)\,\langle \mu(y), \eta(y)u_0(y)\gamma(y)G(x,y)\rangle \;+\; \mathrm{o}\big(\text{cap}(\omega_e)\big), \tag{1.5}$$

and for any smooth cut-off function η, whose support is now strictly contained in Γ_D:

$$u_\varepsilon^{D,0}(x) \;=\; u_0(x) + e(\omega_\varepsilon)\left\langle \mu(y), \eta(y)\gamma(y)\frac{\partial u_0}{\partial n}(y)\frac{\partial G}{\partial n_y}(x,y)\right\rangle \;+\; \mathrm{o}\big(e(\omega_\varepsilon)\big). \tag{1.6}$$

The scalings featured in the asymptotic expressions (1.5) and (1.6) are different. The replacement of a homogeneous Neumann by a Dirichlet boundary condition impacts the voltage potential by a term of the order of the capacity of ω_ε, which is defined by [36]:

$$\text{cap}(\omega_\varepsilon) = \inf\left\{ ||v||^2_{H^1(\mathbb{R}^d)},\ \ v \in H^1(\mathbb{R}^d),\ \ v = 1 \text{ in a neighborhood of } \omega_\varepsilon \right\}. \tag{1.7}$$

In the second situation (1.6), the difference $u_\varepsilon^{D,0} - u_0$ scales like the "Neumann capacity" $e(\omega_\varepsilon)$ defined in our previous work [17] as:

$$e(\omega_\varepsilon) \quad = \quad \inf \int_{\mathbb{R}^d \setminus \overline{\omega_\varepsilon}} \left(|\nabla z|^2 + z^2 \right) \mathrm{d}x,$$

where the infimum is taken over all functions $z \in H^1(\mathbb{R}^d \setminus \overline{\omega_\varepsilon})$ that solve

$$\begin{cases} -\Delta z + z &= \ 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\omega_\varepsilon}, \\ \frac{\partial z}{\partial n} &= \ \pm 1 \quad \text{on each of the connected components of } \omega_\varepsilon. \end{cases}$$

For particular geometries of the sets ω_ε, the measure μ can be fully explicited. These results apply to shape optimization problems where one seeks to optimize subsets of the boundary $\partial\Omega$ where particular boundary conditions are imposed: in [16], the asymptotic formulas (1.5) and (1.6) are used to calculate "topological derivatives" for a quantity of interest to be minimized that depends on the voltage potential, and a gradient descent strategy is implemented from this datum.

Contents of this article. The present work elaborates on the analysis of [17] and considers the replacement of a homogeneous Neumann or Dirichlet boundary condition in the background problem (1.1) by a Robin boundary condition with admittance parameter k_ε. More precisely, we consider two different types of perturbations of the problem (1.1):

(i) Introducing a family of subsets $\omega_\varepsilon \Subset \Gamma_N$, we denote by $u_\varepsilon^{N,k_\varepsilon}$ the solution to the perturbation of (1.1) where the homogeneous Neumann boundary condition is replaced by a Robin condition with admittance parameter k_ε on the subset ω_ε, see (2.4) below.

(ii) We consider the perturbation of (1.1) where the homogeneous Dirichlet boundary condition is replaced by a Robin condition with parameter k_ε on a subset $\omega_\varepsilon \subset \Gamma_D$; the perturbed potential $u_\varepsilon^{D,k_\varepsilon}$ is then the solution to (2.6) below.

In both situations, the Robin parameter $k_\varepsilon > 0$ may depend on ε. Formally, the Robin boundary condition "interpolates" between Dirichlet and Neumann conditions: when ε is fixed, one expects that when k_ε is large, $u_\varepsilon^{N,k_\varepsilon}$ approaches a function that satisfies a homogeneous Dirichlet boundary condition on ω_ε, while $u_\varepsilon^{D,k_\varepsilon}$ should approach a function that satisfies a homogeneous Neumann condition on ω_ε as $k_\varepsilon \to 0$. We refer to e.g. [27] about the physical interpretation of the Robin boundary condition.

The purpose of this article is to derive asymptotic expansions of the differences $u_\varepsilon^N - u_0$ and $u_\varepsilon^D - u_0$ between the reference and perturbed voltage potentials by the replacement of Neumann or Dirichlet by Robin boundary conditions with coefficient k_ε in a vanishingly small subset $\omega_\varepsilon \subset \partial\Omega$. These expansions express the magnitude of these differences in terms of new quantities $\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$, $\text{cap}^D_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ depending on the subset ω_ε and on the admittances k_ε, that deserve the names "Robin capacities". We further investigate how these asymptotic measures of smallness transition between the more usual quantities $e(\omega_\varepsilon)$ and $\text{cap}(\omega_e)$, depending on the behavior of the parameter sequence k_ε.

This work is related to that in [40], where a matched asymptotic expansion methods is used to conduct the asymptotic analysis of a Robin perturbation within a vanishing subset ω_ε of a Neumann boundary with a fixed admittance parameter in 2d. This study is extended to 3d in [41], where layer potential techniques are used. Let us also mention work on homogenization of periodic networks of inhomogeneities with Robin boundary conditions with high contrast, see [33] and the references therein.

Organization of the article. In the next Section 2, we describe the setup and recall some necessary background material, before introducing the two notions of Robin-Neumann and Robin-Dirichlet capacities $\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ and $\text{cap}^D_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$. Section 3 is then devoted to the statement of our main results, firstly when the Neumann region Γ_N is perturbed, secondly when the Dirichlet one Γ_D is perturbed. In each case we describe the asymptotic structure of the voltage potential and its proper scaling, see Theorems 3.1 and 3.4. We next compare the Robin capacities $\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ and $\text{cap}^D_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ to those obtained in our previous work [17] when a homogeneous Dirichlet or homogeneous Neumann condition was imposed on ω_ε instead of a Robin condition. In particular, we show in Theorem 3.3 that the transition from $\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ to $\text{cap}(\omega_\varepsilon)$ is continous as k_ε tends to infinity sufficiently fast. A similar result, whereby $\text{cap}^D_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ tends to $e(\omega_\varepsilon)$ as $k_\varepsilon \to 0$ is the subject of Theorem 3.5. Sections 4 and 5 are devoted to the proofs of these results in the case where the perturbations bear on Γ_N, while Section 6 concerns the case of perturbations impinging on Γ_D. In Section 7, we introduce equivalent measures of the Robin capacities $\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ and $\text{cap}^D_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$, that are intrinsic to the perturbation sets ω_ε, and in particular that do not depend on the rest of $\partial\Omega$. The article ends with an appendix dedicated to a technical lemma involved in the proof of Theorem 3.2 and that is of interest in its own right.

2. Setting of the problem and preliminaries

This section introduces the physical setting of the article and sets the main notation used throughout. After the formal statement of the problems under scrutiny in Section 2.1, we recall basic definitions about fractional Sobolev spaces in Section 2.2. Then, in Section 2.3, we recall a few results about the notions of capacity related to perturbations by Dirichlet or Neumann boundary conditions, before finally introducing the quantities adapted to account for the smallness of perturbations by Robin boundary conditions in Section 2.4.

2.1. Statement of the problems

Let Ω be a smooth bounded domain in $\mathbb{R}^d$ ($d = 2$ or 3), whose boundary is made of two disjoint regions with non-empty interiors:

$$\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}.$$

Here, Γ_D and Γ_N are relatively open subsets of $\partial\Omega$ such that:

- Homogeneous Dirichlet boundary conditions are applied on Γ_D,
- Homogeneous Neumann boundary conditions are applied on Γ_N.

Assuming the presence of a smooth source $f \in \mathcal{C}^\infty(\overline{\Omega})$, the voltage potential u_0 within Ω in this reference, "background" situation is the unique solution in the space

$$H^1_{\Gamma_D}(\Omega) := \left\{ u \in H^1(\Omega),\ u = 0 \text{ on } \Gamma_D \right\},$$

to the conductivity equation (1.1), that is reproduced below for convenience:

$$\left\{ \begin{array}{ccll} \text{div}(\gamma \nabla u_0) & = & f & \text{in } \Omega, \\ u_0 & = & 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial u_0}{\partial n} & = & 0 & \text{on } \Gamma_N, \end{array} \right. \tag{2.1}$$

where the conductivity $\gamma \in \mathcal{C}^\infty(\overline{\Omega})$ is elliptic, in the sense that it satisfies (1.2). Let us recall that, as a classical consequence of the elliptic regularity theory, the function u_0 is smooth except at the points of $\overline{\Gamma_D} \cap \overline{\Gamma_N}$, where the boundary conditions change types in (2.1):

$$\text{For all } \ x \in \overline{\Omega} \ \setminus \left(\overline{\Gamma_D} \cap \overline{\Gamma_N} \right), \text{ there exists an open neighborhood } \ \mathcal{O} \text{ of } x \text{ s.t. } u_0 \in \mathcal{C}^\infty(\overline{\Omega} \cap \mathcal{O}),$$

see e.g. [18, 32].

The background potential u_0 can be represented in terms of the Green's function $G(x, y)$ associated to the reference configuration (2.1). The latter is defined as follows: for each fixed point $x \in \Omega$, the function $y \mapsto G(x, y)$ is the solution to:

$$\left\{ \begin{array}{ccll} -\text{div}_y(\gamma(y) \nabla_y G(x, y)) & = & \delta_x(y) & \text{in } \Omega, \\ G(x, y) & = & 0 & \text{on } \Gamma_D, \\ \gamma(y) \frac{\partial G}{\partial n_y}(x, y) & = & 0 & \text{on } \Gamma_N; \end{array} \right. \tag{2.2}$$

we refer e.g. to [6, 28, 29] for the construction of such functions. It then holds:

$$u_0(x) = \int_\Omega G(x, y) f(y) \, \mathrm{d}y, \quad x \in \Omega.$$

As mentioned in the introduction, we analyze two different types of perturbations of the background problem (1.1):

(i) Let ω_ε be a family of subsets of the boundary $\partial\Omega$, indexed by $\varepsilon > 0$, which are "well-inside" the Neumann region Γ_N, that is:

$$\text{There exists a fixed compact subset } K_N \subset \Gamma_N \text{ s.t. } \forall\varepsilon > 0, \quad \omega_\varepsilon \subset K_N. \tag{2.3}$$

The homogeneous Neumann boundary condition in (1.1) is replaced by a Robin boundary condition on ω_ε, and the perturbed potential $u_\varepsilon^{N,k_\varepsilon}$ is the $H^1(\Omega)$ solution to:

$$\begin{cases} \operatorname{div}(\gamma\nabla u_\varepsilon^{N,k_\varepsilon}) & = & f & \text{in } \Omega, \\ u_\varepsilon^{N,k_\varepsilon} & = & 0 & \text{on } \Gamma_D, \\ \gamma(x)\dfrac{\partial u_\varepsilon^{N,k_\varepsilon}}{\partial n} & = & 0 & \text{on } \Gamma^N \setminus \overline{\omega_\varepsilon}, \\ \gamma\dfrac{\partial u_\varepsilon^{N,k_\varepsilon}}{\partial n} + k_\varepsilon u_\varepsilon^{N,k_\varepsilon} & = & 0 & \text{on } \omega_\varepsilon. \end{cases} \tag{2.4}$$

(ii) Let ω_ε be a family of subsets of $\partial\Omega$ which are "well-inside" the Dirichlet region Γ_D:

$$\text{There exists a fixed compact subset } K_D \subset \Gamma_D \text{ s.t. } \forall\varepsilon > 0, \quad \omega_\varepsilon \subset K_D. \tag{2.5}$$

The homogeneous Dirichlet boundary condition in (1.1) is replaced by a Robin boundary condition on ω_ε, and the perturbed potential $u_\varepsilon^{D,k_\varepsilon}$ is the $H^1(\Omega)$ solution to:

$$\begin{cases} \operatorname{div}(\gamma\nabla u_\varepsilon^{D,k_\varepsilon}) & = & f, & \text{in } \Omega, \\ u_\varepsilon^{D,k_\varepsilon} & = & 0 & \text{on } \Gamma_D \setminus \overline{\omega_\varepsilon}, \\ \gamma\dfrac{\partial u_\varepsilon^{D,k_\varepsilon}}{\partial n} & = & 0 & \text{on } \Gamma_N, \\ \gamma\dfrac{\partial u_\varepsilon^{D,k_\varepsilon}}{\partial n} + k_\varepsilon u_\varepsilon^{D,k_\varepsilon} & = & 0 & \text{on } \omega_\varepsilon. \end{cases} \tag{2.6}$$

Both problems (2.4) and (2.6) are well-posed, under the assumption that for each $\varepsilon > 0$, the admittance k_ε is positive and finite. However, we make no assumption about the behavior of k_ε as $\varepsilon \to 0$. In particular, it may very well tend to 0 or to ∞ in this limit.

2.2. Fractional Sobolev spaces on boundary regions

This section gathers classical definitions about fractional Sobolev spaces defined on (a subset of) the boundary of a smooth domain. This material is mainly excerpted from [25, 35, 42].

Let us first consider Sobolev spaces defined on the total boundary $\partial\Omega$ of a smooth bounded domain Ω in $\mathbb{R}^d$. For $0 < s < 1$, the space $H^s(\partial\Omega)$ is defined by:

$$H^s(\partial\Omega) = \left\{ u \in L^2(\partial\Omega),\ ||u||^2_{L^2(\partial\Omega)} + |u|^2_{H^s(\partial\Omega)} < \infty \right\}, \text{ where} \tag{2.7}$$
$$|u|^2_{H^s(\partial\Omega)} := \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{d-1+2s}}\, \mathrm{d}s(x)\mathrm{d}s(y).$$

This space is a Hilbert space, with norm:

$$||u||_{H^s(\partial\Omega)} = \left(||u||^2_{L^2(\partial\Omega)} + |u|^2_{H^s(\partial\Omega)} \right)^{1/2}.$$

Still for $0 < s < 1$, the negative index Sobolev space $H^{-s}(\partial\Omega)$ is the topological dual of $H^s(\partial\Omega)$.

Let now ω be an open Lipschitz subset of $\partial\Omega$. For $0 < s < 1$, the space $H^s(\omega)$ is defined in a similar manner to (2.7):

$$H^s(\omega) = \left\{ u \in L^2(\omega),\ ||u||^2_{L^2(\omega)} + |u|^2_{H^s(\omega)} < \infty \right\}, \text{ where } |u|^2_{H^s(\omega)} := \int_\omega\int_\omega \frac{|u(x)-u(y)|^2}{|x-y|^{d-1+2s}}\, \mathrm{d}s(x)\mathrm{d}s(y).$$

Again, $H^s(\omega)$ is a Hilbert space when equipped with the norm

$$||u||_{H^s(\omega)} = \left(||u||^2_{L^2(\omega)} + |u|^2_{H^s(\omega)} \right)^{1/2}. \tag{2.8}$$

It is well-known that any function in $H^s(\omega)$ can be extended into a function in $H^s(\partial\Omega)$, and that $||\cdot||_{H^s(\omega)}$ is equivalent to the quotient norm:

$$u \longmapsto \inf\left\{||v||_{H^s(\partial\Omega)},\ v \in H^s(\partial\Omega) \text{ s.t. } v = u \text{ on } \omega\right\}, \tag{2.9}$$

the equivalence constants between both norms being dependent on ω and $\partial\Omega$.

The situation of a strict subset $\omega \subset \partial\Omega$ features another type of Sobolev space of interest for our purpose, namely that $\widetilde{H}^s(\omega)$ gathering the elements in $H^s(\omega)$ whose extension to $\partial\Omega$ by 0 belongs to $H^s(\partial\Omega)$. This space is equipped with the norm:

$$||u||_{\widetilde{H}^s(\omega)} = ||\widetilde{u}||_{H^s(\partial\Omega)}, \text{ where } \widetilde{u} = \begin{cases} u & \text{on } \omega, \\ 0 & \text{on } \partial\Omega \setminus \overline{\omega}. \end{cases}$$

Let us eventually discuss Sobolev spaces with negative index on the subset ω of $\partial\Omega$. For $0 < s < 1$, $H^{-s}(\omega)$ is still defined as the space of the restrictions to ω of distributions in $H^{-s}(\partial\Omega)$, and it is again equipped with the quotient norm, see (2.9). Equivalently, $H^{-s}(\omega)$ is an isometric realization of the dual of $\widetilde{H}^s(\omega)$, through the following pairing:

$$\text{For all } u \in H^{-s}(\omega),\ v \in \widetilde{H}^s(\omega), \quad \langle u, v\rangle_{H^{-s}(\omega),\widetilde{H}^s(\omega)} = \langle U, \widetilde{v}\rangle_{H^{-s}(\omega),\widetilde{H}^s(\omega)},$$

where U is an arbitrary element in $H^{-s}(\partial\Omega)$ such that $U|_\omega = u$ and $\widetilde{v} \in H^s(\partial\Omega)$ is the extension of v by 0 to $\partial\Omega$. Likewise, $\widetilde{H}^{-s}(\omega)$ is defined as the space of distributions in $H^{-s}(\partial\Omega)$ with compact support inside $\overline{\omega}$. It is the dual space of $H^s(\omega)$ through the pairing:

$$\text{For all } u \in \widetilde{H}^{-s}(\omega),\ v \in H^s(\omega), \quad \langle u, v\rangle_{\widetilde{H}^{-s}(\omega),H^s(\omega)} = \langle \widetilde{u}, V\rangle_{\widetilde{H}^{-s}(\omega),H^s(\omega)},$$

where $\widetilde{u} \in H^{-s}(\partial\Omega)$ is the extension of u by 0 to $\partial\Omega$ and V is an arbitrary element in $H^s(\partial\Omega)$ such that $V|_\omega = v$.

2.3. Capacity and equilibrium distributions attached to Dirichlet and Neumann boundary conditions

In this section, we summarize some classical material about the notion of capacity, as well as some findings from our previous work [17] that are handful for our purpose; we refer the reader to the books [1, 26, 39] about these concepts.

At first, let ω be a Lipschitz subset of the region Γ_N; the Dirichlet equilibrium potential $\chi^\infty = \chi^\infty(\omega)$ is the unique $H^1(\Omega)$ solution to the following boundary-value problem:

$$\begin{cases} -\text{div}(\gamma\nabla\chi^\infty) & = & 0 & \text{in } \Omega, \\ \chi^\infty & = & 0 & \text{on } \Gamma_D, \\ \gamma\frac{\partial\chi^\infty}{\partial n} & = & 0 & \text{on } \Gamma_N \setminus \overline{\omega}, \\ \chi^\infty & = & 1 & \text{on } \omega. \end{cases} \tag{2.10}$$

Intuitively, χ^∞ is the function in the admissible space $H^1_{\Gamma_D}(\Omega)$ for (2.1) that nearly realizes the minimum value in the definition (1.7) of capacity.

In the same spirit, let now ω be a Lipschitz open subset of the region Γ_D. We define the Neumann equilibrium potential $\xi^0 = \xi^0(\omega)$ as the $H^1(\Omega)$ solution to:

$$\begin{cases} -\text{div}(\gamma\nabla\xi^0) & = & 0 & \text{in } \Omega, \\ \xi^0 & = & 0 & \text{on } \Gamma_D \setminus \overline{\omega}, \\ \gamma\frac{\partial\xi^0}{\partial n} & = & 0 & \text{on } \Gamma_N, \\ \gamma\frac{\partial\xi^0}{\partial n} & = & 1 & \text{on } \omega. \end{cases} \tag{2.11}$$

When ω_ε is a collection of subsets of Γ_N or Γ_D indexed by a real number ε, we shall denote $\chi^\infty(\omega_\varepsilon) = \chi^\infty_\varepsilon$ and $\xi^0(\omega_\varepsilon) = \xi^0_\varepsilon$.

These capacity functions are related to the scalings that appear in the asymptotic expansions (1.5) and (1.6). We indeed recall the following estimates from our previous work [17].

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain, whose boundary is made of two disjoint open regions Γ_N and Γ_D as in Section 2.1, and let K_N, K_D be two compact subsets of Γ_N and Γ_D, respectively. Then,*

(i) Let ω be a Lipschitz subset of Γ_N lying far from Γ_D in the sense that (2.3) holds. There exists a constant $C > 0$, depending on Ω, Γ_N, Γ_D and the compact K_N but not on ω, such that:

$$\frac{1}{C}\text{cap}(\omega)^{1/2} \leq ||\chi^\infty||_{H^1(\Omega)} \leq C\text{cap}(\omega)^{1/2} \quad \text{and} \quad ||\chi^\infty||_{L^2(\Omega)} \leq C\,\text{cap}(\omega)^{3/4}.$$

(ii) Let ω be a Lipschitz subset of Γ_D satisfying (2.5). There exists a constant $C > 0$, depending on Ω, Γ_N, Γ_D and the compact K_D but not on ω, such that:

$$\frac{1}{C}e(\omega)^{1/2} \leq ||\xi^0||_{H^1(\Omega)} \leq Ce(\omega)^{1/2} \quad \text{and} \quad ||\xi^0||_{L^2(\Omega)} \leq C\,e(\omega)^{3/4}.$$

2.4. **Capacity and equilibrium distributions attached to Robin boundary conditions**

This section introduces the equilibrium distributions and the associated notions of capacity involved in our analysis of Robin boundary conditions.

Let us first consider a subset $\omega \Subset \Gamma_N$ and let $k > 0$. The Robin-Neumann potential $\chi = \chi(\omega, k)$ is defined as the unique $H^1(\Omega)$ solution to the following boundary-value problem:

$$\left\{\begin{array}{rcll} -\text{div}(\gamma\nabla\chi) &=& 0 & \text{in } \Omega, \\ \chi &=& 0 & \text{on } \Gamma_D, \\ \gamma\frac{\partial\chi}{\partial n} &=& 0 & \text{on } \Gamma_N \setminus \overline{\omega}, \\ \gamma\frac{\partial\chi}{\partial n} + k\chi &=& k & \text{on } \omega. \end{array}\right. \tag{2.12}$$

Equivalently, this function satisfies the following variational problem:

$$\text{Find } \chi \in H^1_{\Gamma_D}(\Omega) \text{ s.t. } \forall v \in H^1_{\Gamma_D}(\Omega), \quad \int_\Omega \gamma\nabla\chi\cdot\nabla v \,\mathrm{d}x + k\int_\omega \chi v \,\mathrm{d}s = k\int_\omega v\,\mathrm{d}s. \tag{2.13}$$

Thence, the Robin-Neumann capacity $\text{cap}^N_{\text{Rob}}(\omega, k)$ of the set ω is defined by

$$\text{cap}^N_{\text{Rob}}(\omega, k) = \int_\omega \gamma\frac{\partial\chi}{\partial n}\,\mathrm{d}s = k\int_\omega (1-\chi)\,\mathrm{d}s. \tag{2.14}$$

Let us now consider a subset $\omega \Subset \Gamma_D$ and let $k > 0$. The Robin-Dirichlet potential $\xi = \xi(\omega, k)$ is the H^1 solution to the following boundary-value problem:

$$\left\{\begin{array}{rcll} -\text{div}(\gamma\nabla\xi) &=& 0 & \text{in } \Omega, \\ \xi &=& 0 & \text{on } \Gamma_D \setminus \omega, \\ \gamma\frac{\partial\xi}{\partial n} &=& 0 & \text{on } \Gamma_N, \\ \gamma\frac{\partial\xi}{\partial n} + k\xi &=& 1 & \text{on } \omega, \end{array}\right. \tag{2.15}$$

or equivalently, in variational form

$$\forall v \in H^1_{\Gamma_D\setminus\overline{\omega}}(\Omega), \quad \int_\Omega \gamma\nabla\xi\cdot\nabla v\,\mathrm{d}x + k\int_\omega \xi v\,\mathrm{d}s = \int_\omega v\,\mathrm{d}s.$$

The Robin-Dirichlet capacity $\text{cap}^D_{\text{Rob}}(\omega, k)$ of ω and $k > 0$ is then defined by

$$\text{cap}^D_{\text{Rob}}(\omega, k) = \int_\Omega \gamma|\nabla\xi|^2\,\mathrm{d}x + k\int_\omega \xi^2\mathrm{d}s = \int_\omega \xi\mathrm{d}s. \tag{2.16}$$

Again, when ω is a collection ω_ε of Lipschitz subsets of Γ_N or Γ_D and k accounts for a sequence k_ε of real numbers, we simply write $\chi(\omega_\varepsilon, k_\varepsilon) = \chi_\varepsilon$ and $\xi_\varepsilon(\omega_\varepsilon, k_\varepsilon) = \xi_\varepsilon$.

Remark 2.1. *The above definitions (2.14) and (2.16) of the Robin capacities* $\text{cap}^N_{\text{Rob}}(\omega, k)$ *and* $\text{cap}^D_{\text{Rob}}(\omega, k)$ *may look a little awkward at first glance, since they depend on the boundary-value problems (2.12) and (2.15), while, ideally, one would expect a measure of smallness that depends only on the Robin condition, i.e. on* ω *and* k*. As a matter of fact, Section 7 below remedies this conceptual drawback, as it proposes equivalent quantities (up to constants depending on* Ω*) that are only defined in terms of* ω *and* k*. For technical convenience however, we stick to the definitions (2.14) and (2.16) of the Robin capacities in our analysis.*

3. Statement of the main results

This section gathers and comments the main findings of this article. At first, Section 3.1 deals with the situation where the homogeneous Neumann boundary condition featured in the background problem (2.1) is replaced by a Robin boundary condition, and Section 3.2 is devoted to the alternative situation where the homogeneous Dirichlet boundary condition is perturbed. The claims of this section are proved in the subsequent Sections 4 to 6.

3.1. Robin perturbation of the Neumann boundary condition on Γ_N

In this section, we consider the situation where the "small" perturbation set ω_ε satisfies $\omega_\varepsilon \Subset \Gamma_N$. The perturbed version of the background problem (2.1) under scrutiny is (2.4).

Our first result shows that the Robin-Neumann capacity $\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ is the adequate quantity to appraise the "smallness" of a Robin perturbation with parameter k_ε on a subset ω_ε of Γ_N.

Theorem 3.1. *Let* Ω *be a smooth bounded domain of* $\mathbb{R}^d$ *whose boundary is divided into two disjoint open subsets* Γ_N *and* Γ_D*, and let* $K_N \subset \Gamma_N$ *be a fixed compact set, see Section 2.1. Let* ω_ε *be a sequence of subsets of* Γ_N *which is well-separated from* Γ_D *in the sense that (2.3) holds true, and let* k_ε *be any sequence of positive real numbers. Assume that the Robin-Neumann capacity* $\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ *in (2.14) tends to* 0 *as* $\varepsilon \to 0$*. Then, there exists a subsequence, still denoted by* ε*, as well as a positive Radon measure* μ *with unit mass, supported on* Γ_N*, such that the following asymptotic expansion of the perturbed potential* $u^{N,k_\varepsilon}_\varepsilon$ *in (2.4) holds true:*

$$\text{(3.1)} \qquad \text{For } x \in \Omega, \quad u^{N,k_\varepsilon}_\varepsilon(x) = u_0(x) - \text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)\, \langle \mu, u_0(\cdot) G(x,\cdot)\rangle + \text{o}\left(\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)\right).$$

Our next result aims to better appraise the relation between the Robin capacity $\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ and the classical capacity $\text{cap}(\omega_\varepsilon)$ as k_ε varies. Such a relation is to be expected, as $\text{cap}(\omega_\varepsilon)$ is the scaling obtained in [17] when a mere Dirichlet condition is imposed on ω_ε. In view of the asymptotic formula (1.3) for "volumic" perturbations of the conductivity γ, we are also interested in how $\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ compares to the Lebesgue measure of ω_ε.

Theorem 3.2. *Let* Ω *be a smooth bounded domain of* $\mathbb{R}^d$ *whose boundary is divided into two disjoint open subsets* Γ_N *and* Γ_D*, and let* K_N *be a fixed compact subset of* Γ_N*. Then, the following facts hold true:*

(i) *There exists a constant* $C > 0$*, depending on* Ω*,* Γ_N*,* Γ_D *and the compact set* K_N *such that*

$$\text{(3.2)} \qquad \forall\, k > 0, \ \ \forall\, \omega \subset K_N, \quad \text{cap}^N_{\text{Rob}}(\omega, k) \ \leq \ C\, \text{cap}(\omega).$$

(ii) *Let* $M > 0$ *be a fixed constant. Then, there exists* $C > 0$*, depending on* Ω *and* M*, such that*

$$\text{(3.3)} \qquad \forall 0 \leq k \leq M, \ \ \forall\, \omega \Subset \Gamma_N, \quad Ck\,|\omega| \ \leq \ \text{cap}^N_{\text{Rob}}(\omega, k) \ \leq \ k\,|\omega|.$$

(iii) *Let* $\omega \Subset \Gamma_N$ *be a fixed subset and let* k_ε *be a sequence of positive numbers such that* $k_\varepsilon \to \infty$*. Then*

$$\text{(3.4)} \qquad \frac{1}{C} \ \leq \ \frac{\text{cap}^N_{\text{Rob}}(\omega, k_\varepsilon)}{\text{cap}(\omega)} \ \leq \ C,$$

for a certain constant C *which depends on* ω*.*

The next result, which is proved in Section 5 under additional assumptions on the sets ω_ε, improves point (iii) in Theorem 3.2: it shows that the asymptotic expression (3.1) is formally consistent with that (1.5) obtained in the case of a Dirichlet perturbation of Γ_N when k_ε tends to ∞ sufficiently fast.

Theorem 3.3. *Under the assumptions of Theorem 3.2, let ω_ε be a sequence of subsets of Γ_N satisfying (2.3), and such that, additionally:*

- *If $d = 2$, the set ω_ε is an arc of length ε;*
- *If $d = 3$, the set ω_ε is the image $\Phi(B_\varepsilon)$ of the planar disk*
$$\mathbb{D}_\varepsilon := \left\{ x = (x_1, x_2, 0) \in \mathbb{R}^3, \quad x_1^2 + x_2^2 < \varepsilon^2 \right\},$$
by a fixed, smooth diffeomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$.

Assume that $\text{cap}(\omega_\varepsilon) \to 0$ and that k_ε tends to infinity faster than $|\omega_\varepsilon|^3$ goes to 0, i.e. $k_\varepsilon |\omega_\varepsilon|^3 \to \infty$ as $\varepsilon \to 0$. Then,
$$C \leq \frac{\text{cap}_{\text{Rob}}^N(\omega_\varepsilon, k_\varepsilon)}{\text{cap}(\omega_\varepsilon)} \leq 1,$$
for a certain constant $C > 0$ which is independent of ε.

This result expresses that the expansion (3.1) of $u_\varepsilon - u_0$ has the formal limit (1.5) when k_ε tends to ∞ sufficiently fast. This aligns with the intuition that the Robin boundary condition $\gamma \frac{\partial u_\varepsilon}{\partial n} + k_\varepsilon u_\varepsilon = 0$ on ω_ε "degenerates" to a Dirichlet boundary condition $u_\varepsilon = 0$ when $k_\varepsilon \to \infty$ sufficiently fast. Note that the particular form of the sets ω_ε in this section implies that $|\omega_\varepsilon| = \text{o}\big(\text{cap}(\omega_\varepsilon)\big)$.

Remark 3.1.

(i) Under the assumptions on the shapes of the ω_ε in Theorem 3.3, the measure μ that appears in the expansion (1.5) corresponding to a "pure" Dirichlet perturbation of the Neumann region can be fully identified, see [17]. It would be interesting to conduct an analogous study in the present context of a Robin perturbation: an explicit formula for the first-order expansion of $u_\varepsilon^{N,k_\varepsilon}$ which holds true in a uniform fashion with respect to the parameter k_ε would explain the smooth transition between the rates $k_\varepsilon|\omega_\varepsilon|$ and $\text{cap}(\omega_\varepsilon)$ in (3.3) and (3.2) when k_ε passes from 0 to ∞. We refer to e.g. [24, 44] about such uniform expansions in the context of volumic perturbations of the coefficients of an elliptic equation. In particular, we conjecture that the rate at which k_ε should tend to ∞ exhibited by Theorem 3.3 is not optimal. This would be elucidated by such an explicit expansion.

(ii) In [16], it is shown that $u_\varepsilon - u_0$ scales like $|\omega_\varepsilon|$, when one replaces a homogeneous Neumann boundary condition by an inhomogeneous one on $\omega \Subset \Gamma_N$, which is consistent with (3.3) for the Robin condition $\gamma \frac{\partial u}{\partial n} + ku$ with "small" k.

We conclude this section with a result about the monotonicity of the Robin-Neumann capacity $\text{cap}_{\text{Rob}}^N(\omega, k)$ with respect to the support ω and the parameter k of the Robin boundary condition.

Proposition 3.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^d$, whose boundary is divided into two disjoint open subsets Γ_N and Γ_D.*

(i) Let ω be a Lipschitz open subset of the Neumann region Γ_N and let $0 < k_1 \leq k_2$ be positive real numbers. It holds:
$$\text{cap}_{\text{Rob}}^N(\omega, k_1) \leq \text{cap}_{\text{Rob}}^N(\omega, k_2).$$

(ii) Let k be a fixed real number and let $\omega_1 \subset \omega_2$ be two Lipschitz open subsets of Γ_N. It holds:
$$\text{cap}_{\text{Rob}}^N(\omega_1, k) \leq \text{cap}_{\text{Rob}}^N(\omega_2, k).$$

3.2. Robin perturbation of the Dirichlet boundary condition on Γ_D

In this section, we consider the situation where $\omega_\varepsilon \Subset \Gamma_D$, and where the perturbed potential $u_\varepsilon^{D,k_\varepsilon}$ results from the perturbation of the homogeneous Dirichlet condition on ω_ε by a Robin condition with parameter k_ε, see (2.6). In this situation, the asymptotic behavior of $u_\varepsilon^{D,k_\varepsilon}$ is described by the following result:

Theorem 3.4. *Let Ω be a smooth bounded domain of $\mathbb{R}^d$ whose boundary is divided into two disjoint open subsets Γ_N and Γ_D, and let $K_D \subset \Gamma_D$ be a fixed compact set, see Section 2.1. Let ω_ε be a sequence of subsets of Γ_D that satisfies the separation assumption (2.5) and let k_ε be any sequence of positive real numbers. Assume that the Robin-Dirichlet capacity $\text{cap}_{\text{Rob}}^D(\omega_\varepsilon, k_\varepsilon)$ tends to 0 as $\varepsilon \to 0$. Then, there exists* *a subsequence, still denoted by ε, as well as a positive Radon measure μ with unit mass, supported in Γ_D, such that the following asymptotic expansion holds true:*

$$\text{(3.5)} \qquad \text{For } x \in \Omega, \quad u_\varepsilon^{D,k_\varepsilon}(x) = u_0(x) + \text{cap}_{\text{Rob}}^D(\omega_\varepsilon, k_\varepsilon) \left\langle \mu, \gamma \frac{\partial u_0}{\partial n} \frac{\partial G}{\partial n_y}(x, \cdot) \right\rangle + \text{o}\left(\text{cap}_{\text{Rob}}^D(\omega_\varepsilon, k_\varepsilon)\right).$$

Alongside Theorem 3.4, we show the following result.

Theorem 3.5. *Let Ω be a smooth bounded domain of $\mathbb{R}^d$ whose boundary is divided into two disjoint open subsets Γ_N and Γ_D, and let K_D be a fixed compact subset of Γ_D. Then, the following facts hold true:*

(i) *For any $k > 0$, and any $\omega \subset K_D$, there exists $C > 0$ such that:*

$$\text{cap}_{\text{Rob}}^D(\omega, k) \ \leq \ C\, e(\omega). \tag{3.6}$$

The constant C depends on Ω, Γ_N, Γ_D and K_D, but it is independent of ω and k.

(ii) *For any $\omega \Subset \Gamma_D$, and for any $k > 0$, it holds:*

$$\text{cap}_{\text{Rob}}^D(\omega, k) \ \leq \ \frac{1}{k}|\omega|. \tag{3.7}$$

(iii) *Let ω_ε denote a sequence of subsets of a fixed compact set $K_D \subset \Gamma_D$ and let $k_\varepsilon > 0$ be a sequence of positive numbers. Assume that there exists $M > 0$ such that $k_\varepsilon \leq M$. Then*

$$\frac{1}{C} \leq \frac{\text{cap}_{\text{Rob}}^D(\omega_\varepsilon, k_\varepsilon)}{e(\omega_\varepsilon)} \ \leq \ C, \tag{3.8}$$

for a constant $C > 0$ depending on Ω, Γ_N, Γ_D, K_D and the constant M but not on ε.

Remark 3.2. *The estimate (3.8) shows that when $k_\varepsilon \to 0$, $\text{cap}_{\text{Rob}}^D(\omega_\varepsilon, k_\varepsilon)$ scales like the Neumann capacity $e(\omega_\varepsilon)$, consistently with (1.6). In other words, the Robin boundary condition on ω_ε has an effect similar here to that of a Neumann condition in this regime.*

Eventually, the Robin-Dirichlet capacity $\text{cap}_{\text{Rob}}^D(\omega, k)$ satisfies monotonicity properties with respect to the set ω and the parameter k of the Robin boundary condition, akin to those stated in Proposition 3.1 in the case of the Robin-Neumann capacity.

Proposition 3.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^d$, whose boundary is divided into two disjoint open subsets Γ_N and Γ_D.*

(i) Let ω be a Lipschitz open subset of the Dirichlet region Γ_D and let $0 < k_1 \leq k_2$ be positive real numbers. It holds:

$$\text{cap}_{\text{Rob}}^D(\omega, k_2) \leq \text{cap}_{\text{Rob}}^D(\omega, k_1).$$

(ii) Let k be a fixed real number and let $\omega_1 \subset \omega_2$ be two Lipschitz open subsets of Γ_D. It holds:

$$\text{cap}_{\text{Rob}}^D(\omega_1, k) \leq \text{cap}_{\text{Rob}}^D(\omega_2, k).$$

4. Proofs of the results about Robin perturbation of the Neumann boundary Γ_N

This section takes place in the situation of Section 3.1: ω_ε is a sequence of subsets of Γ_N which is well-separated from Γ_D in the sense of (2.3), and the perturbed version of the background problem (2.1) under scrutiny is that (2.4) where the Neumann boundary condition on ω_ε is replaced by a Robin boundary condition with parameter k_ε. We aim to prove Theorems 3.1 and 3.2 and Proposition 3.1.

For notational brevity, throughout this section, we use the shorthands $\chi_\varepsilon \equiv \chi(\omega_\varepsilon, k_\varepsilon)$ and $u_\varepsilon = u_\varepsilon^{N,k_\varepsilon}$ for the equilibrium distribution and the perturbed potential, solutions to (2.4) and (2.12), respectively.

4.1. **Preliminary estimates**

Let us first recall the following basic version of the Poincaré inequality for functions in $H^1_{\Gamma_D}(\Omega)$: there exists a constant $C>0$, which depends only on Ω, Γ_D and the bounds γ_-, γ_+ on the conductivity γ in (1.2), such that:

$$\forall v \in H^1_{\Gamma_D}(\Omega), \quad \frac{1}{C}||v||^2_{H^1(\Omega)} \leq \int_\Omega \gamma |\nabla v|^2 \,\mathrm{d}x \leq ||v||^2_{H^1(\Omega)}. \tag{4.1}$$

Given $g \in \mathcal{C}^\infty(\mathbb{R}^d)$, we now define φ_ε as the $H^1(\Omega)$ solution to the following boundary-value problem:

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\gamma \nabla \varphi_\varepsilon) &=& 0 & \text{in } \Omega, \\ \varphi_\varepsilon &=& 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial \varphi_\varepsilon}{\partial n} &=& 0 & \text{on } \Gamma_N \setminus \overline{\omega_\varepsilon}, \\ \gamma \frac{\partial \varphi_\varepsilon}{\partial n} + k_\varepsilon \varphi_\varepsilon &=& k_\varepsilon g & \text{on } \omega_\varepsilon. \end{array} \right. \tag{4.2}$$

Equivalently, the variational form of this problem is

$$\text{Find } \varphi_\varepsilon \in H^1_{\Gamma_D}(\Omega) \text{ s.t. } \forall v \in H^1_{\Gamma_D}(\Omega), \quad \int_\Omega \gamma \nabla \varphi_\varepsilon \cdot \nabla v \,\mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} \varphi_\varepsilon v \,\mathrm{d}s = k_\varepsilon \int_{\omega_\varepsilon} g v \,\mathrm{d}s. \tag{4.3}$$

Note that when $g=1$, φ_ε is the solution χ_ε to (2.12) when $\omega = \omega_\varepsilon$ and $k = k_\varepsilon$.

We next estimate the function φ_ε.

Proposition 4.1. *Let ω_ε be a sequence of subsets of Γ_N satisfying the separation assumption (2.3) and let k_ε be any sequence of positive real numbers; let $g \in \mathcal{C}^\infty(\mathbb{R}^d)$, and let φ_ε be the unique solution to (4.2). Then,*

(i) The following pointwise bounds hold true:

$$\min\left(0, \inf_{x \in \overline{\Omega}} g(x)\right) \leq \varphi_\varepsilon(x) \leq \max\left(0, \sup_{x\in\overline{\Omega}} g(x)\right) \text{ for a.e. } x \in \overline{\Omega}. \tag{4.4}$$

(ii) The following H^1 estimate holds true:

$$\int_\Omega \gamma |\nabla \varphi_\varepsilon|^2 \,\mathrm{d}x \leq C ||g||^2_{\mathcal{C}^1(\mathbb{R}^d)} \mathrm{cap}^N_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon), \tag{4.5}$$

where the constant C depends on Ω, Γ_N, Γ_D, the compact set K_N, but not on ε.

(iii) The following improved L^2 estimate holds true:

$$\int_\Omega \varphi_\varepsilon^2 \,\mathrm{d}x \leq C ||g||^2_{\mathcal{C}^1(\mathbb{R}^d)} \mathrm{cap}^N_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)^{\frac{3}{2}}. \tag{4.6}$$

Again, C depends on Ω, Γ_N, Γ_D, K_N, but not on ε.

Proof. Proof of (i). This result is a variant of the "classical" maximum principle applied to the present boundary-value problem (4.2). Its proof is inspired by the arguments in e.g §9.7 in [18]. For notational brevity, let M denote the quantity in the right-hand side of (4.4), and let us set $\psi_\varepsilon(x) := \max(\varphi_\varepsilon(x) - M, 0)$. Since the mapping $\mathbb{R} \ni t \mapsto \max(t,0)$ is Lipschitz continuous, the function ψ_ε belongs to $H^1_{\Gamma_D}(\Omega)$ and its derivative reads:

$$\nabla \psi_\varepsilon = \mathbb{1}_{\varphi_\varepsilon \geq M} \nabla \varphi_\varepsilon,$$

see e.g. §3.1.2 in [36]. Insertion of ψ_ε as test function into (4.3) leads to:

$$\int_\Omega \gamma |\nabla \varphi_\varepsilon|^2 \mathbb{1}_{\{\varphi_\varepsilon \geq M\}} \,\mathrm{d}x + \int_{\omega_\varepsilon} k_\varepsilon \varphi_\varepsilon (\varphi_\varepsilon - M) \mathbb{1}_{\{\varphi_\varepsilon \geq M\}} \,\mathrm{d}s = \int_{\omega_\varepsilon} k_\varepsilon g (\varphi_\varepsilon - M) \mathbb{1}_{\{\varphi_\varepsilon \geq M\}} \,\mathrm{d}s.$$

By re-arranging this equality, we obtain:

$$\int_\Omega \gamma |\nabla \varphi_\varepsilon|^2 \mathbb{1}_{\{\varphi_\varepsilon \geq M\}} \,\mathrm{d}x = \int_{\omega_\varepsilon} k_\varepsilon (g - \varphi_\varepsilon)(\varphi_\varepsilon - M) \mathbb{1}_{\{\varphi_\varepsilon \geq M\}} \,\mathrm{d}s \leq 0,$$

where the inequality is a consequence of the definition of M. It follows that $\psi_\varepsilon(x) = 0$ a.e. and so $\varphi_\varepsilon(x) \leq M$ for a.e. $x \in \overline{\Omega}$, as desired. The left-hand inequality in (4.4) is proved by similar means.

Proof of (ii). Taking $v = \varphi_\varepsilon$ as test function in (4.3), we obtain the following a priori estimate for φ_ε:

$$\int_\Omega \gamma |\nabla \varphi_\varepsilon|^2 \,\mathrm{d}x = k_\varepsilon \int_{\omega_\varepsilon} (g - \varphi_\varepsilon)\varphi_\varepsilon \,\mathrm{d}s.$$

Introducing the function χ_ε in (2.12) into the above right-hand side, it follows:

$$\begin{aligned}
\int_\Omega \gamma |\nabla \varphi_\varepsilon|^2 \,\mathrm{d}x &= \int_{\omega_\varepsilon} k_\varepsilon (g - \varphi_\varepsilon)\varphi_\varepsilon \,\mathrm{d}s \\
&= \int_{\omega_\varepsilon} \left(\gamma \frac{\partial \chi_\varepsilon}{\partial n} + k_\varepsilon \chi_\varepsilon \right)(g - \varphi_\varepsilon)\varphi_\varepsilon \,\mathrm{d}s \\
&= \int_{\omega_\varepsilon} \gamma \frac{\partial \chi_\varepsilon}{\partial n} g\varphi_\varepsilon \,\mathrm{d}s \;-\; \int_{\omega_\varepsilon} \gamma \frac{\partial \chi_\varepsilon}{\partial n} \varphi_\varepsilon^2 \,\mathrm{d}s \;-\; \int_{\omega_\varepsilon} k_\varepsilon \chi_\varepsilon (g - \varphi_\varepsilon)^2 \,\mathrm{d}s \;+\; \int_{\omega_\varepsilon} k_\varepsilon \chi_\varepsilon (g - \varphi_\varepsilon) g \,\mathrm{d}s \\
&\leq \int_{\partial\Omega} \gamma \frac{\partial \chi_\varepsilon}{\partial n} g\varphi_\varepsilon \,\mathrm{d}s + \int_{\partial\Omega} \gamma \frac{\partial \varphi_\varepsilon}{\partial n} g\chi_\varepsilon \,\mathrm{d}s,
\end{aligned}$$

where we have used the Robin boundary condition for χ_ε in (2.12) to pass from the first line to the second one and the Robin boundary condition for φ_ε in (4.2) to get the third one. We have also used the fact that $0 \leq \chi_\varepsilon \leq 1$ as a consequence of the pointwise bounds (4.4), and the fact that the normal derivative $\gamma \frac{\partial \chi_\varepsilon}{\partial n}$ is therefore nonnegative on ω_ε. Now using the Green's formula, the Cauchy-Schwarz inequality and the Poincaré's inequality (4.1), we obtain:

$$\int_\Omega \gamma |\nabla \varphi_\varepsilon|^2 \,\mathrm{d}x \;\leq\; C ||g||_{\mathcal{C}^1(\overline{\Omega})} ||\nabla \varphi_\varepsilon||_{L^2(\Omega)^d} ||\nabla \chi_\varepsilon||_{L^2(\Omega)^d},$$

and so, after cancellation of $||\nabla \varphi_\varepsilon||_{L^2(\Omega)^d}$ on both sides and squaring:

$$\int_\Omega \gamma |\nabla \varphi_\varepsilon|^2 \,\mathrm{d}x \;\leq\; C ||g||^2_{\mathcal{C}^1(\overline{\Omega})} \int_\Omega \gamma |\nabla \chi_\varepsilon|^2 \,\mathrm{d}x.$$

Now, a simple calculation yields:

$$\begin{aligned}
\int_\Omega \gamma |\nabla \chi_\varepsilon|^2 \,\mathrm{d}x &= k_\varepsilon \int_{\omega_\varepsilon} (1 - \chi_\varepsilon)\chi_\varepsilon \,\mathrm{d}s \\
&\leq k_\varepsilon \int_{\omega_\varepsilon} (1 - \chi_\varepsilon) \,\mathrm{d}s \\
&= \mathrm{cap}^N_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon),
\end{aligned}$$

as follows from another application of the pointwise bounds (4.4) and the definition (2.14) of $\mathrm{cap}^N_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)$. This terminates the proof of (4.5).

Proof of (iii). We rely on the so-called Aubin-Nitsche duality trick, see [10, 45] or [22] in the context of the Finite Element Method. Let $p_\varepsilon \in H^1_{\Gamma_D}(\Omega)$ be the unique solution to the following boundary value problem:

$$\left\{ \begin{array}{rll} -\mathrm{div}(\gamma \nabla p_\varepsilon) = -\varphi_\varepsilon & \text{in } \Omega, \\ p_\varepsilon = 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial p_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N. \end{array} \right. \tag{4.7}$$

We introduce two open sets $\mathcal{O}_1 \Subset \mathcal{O}_2$ such that $K_N \subset \mathcal{O}_1$ and $\overline{\mathcal{O}_2} \cap \overline{\Gamma_D} = \emptyset$, as well as a smooth cut-off function η satisfying $\eta \equiv 1$ on $\mathcal{O}_1$ and $\eta \equiv 0$ on $\mathbb{R}^d \setminus \mathcal{O}_2$. We now use classical elliptic regularity theory [18, 32] and the Sobolev embedding theorem [2] to see that that $p_\varepsilon \in H^3(\mathcal{O}_2 \cap \Omega)$ with

$$||p_\varepsilon||_{\mathcal{C}^1(\mathcal{O}_2 \cap \Omega)} \leq C ||p_\varepsilon||_{H^3(\mathcal{O}_2 \cap \Omega)} \leq C ||\varphi_\varepsilon||_{H^1(\Omega)}. \tag{4.8}$$

It follows that:

$$\begin{aligned}
\int_\Omega \varphi_\varepsilon^2 \,\mathrm{d}x &= -\int_\Omega \gamma \nabla p_\varepsilon \cdot \nabla \varphi_\varepsilon \,\mathrm{d}x \\
&= k_\varepsilon \int_{\omega_\varepsilon} (g - \varphi_\varepsilon) p_\varepsilon \,\mathrm{d}s,
\end{aligned}$$

where we have used (4.7) to obtain the second line. As in the proof of (ii), introducing the equilibrium distribution χ_ε via its Robin boundary condition on ω_ε, we obtain:

$$\begin{aligned}\int_\Omega \varphi_\varepsilon^2 \,\mathrm{d}x &= \int_{\omega_\varepsilon} \gamma \frac{\partial \chi_\varepsilon}{\partial n}(g-\varphi_\varepsilon)p_\varepsilon \,\mathrm{d}s + \int_{\omega_\varepsilon} k_\varepsilon \chi_\varepsilon (g-\varphi_\varepsilon)p_\varepsilon \,\mathrm{d}s \\ &= \int_{\omega_\varepsilon} \gamma \frac{\partial \chi_\varepsilon}{\partial n}(g-\varphi_\varepsilon)p_\varepsilon \,\mathrm{d}s + \int_{\omega_\varepsilon} \gamma \frac{\partial \varphi_\varepsilon}{\partial n}\chi_\varepsilon p_\varepsilon \,\mathrm{d}s \\ &= \int_{\omega_\varepsilon} \gamma \frac{\partial \chi_\varepsilon}{\partial n}(g-\varphi_\varepsilon)p_\varepsilon \,\mathrm{d}s + \int_\Omega \gamma \nabla\varphi_\varepsilon \cdot \nabla(\chi_\varepsilon p_\varepsilon \eta)\,\mathrm{d}x.\end{aligned}$$

Now, the non negativity of $\frac{\partial \chi_\varepsilon}{\partial n}$ and the definition (2.14) of the Robin-Neumann capacity together imply that:

$$\begin{aligned}\int_\Omega \varphi_\varepsilon^2 \,\mathrm{d}x &\leq ||p_\varepsilon||_{\mathcal{C}(\omega_\varepsilon)} \int_{\omega_\varepsilon} \gamma \frac{\partial \chi_\varepsilon}{\partial n}|g-\varphi_\varepsilon| \,\mathrm{d}s + \int_\Omega \gamma \nabla\varphi_\varepsilon \cdot \nabla(\chi_\varepsilon p_\varepsilon \eta)\,\mathrm{d}x \\ &\leq ||p_\varepsilon||_{\mathcal{C}(\omega_\varepsilon)}||g||_{\mathcal{C}^1(\partial\Omega)} \int_{\omega_\varepsilon} \gamma \frac{\partial \chi_\varepsilon}{\partial n} \,\mathrm{d}s + ||p_\varepsilon||_{\mathcal{C}^1(\mathcal{O}_2\cap\overline{\Omega})} ||\nabla\varphi_\varepsilon||_{L^2(\Omega)^d}||\nabla\chi_\varepsilon||_{L^2(\Omega)^d} \\ &\leq C||p_\varepsilon||_{\mathcal{C}^1(\mathcal{O}_2\cap\overline{\Omega})}||g||_{\mathcal{C}^1(\partial\Omega)} \mathrm{cap}^N_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon),\end{aligned}$$

where we have used the H^1 estimate (4.5). The desired result follows from (4.8). □

4.2. **The representation formula**

The aim of this section is to prove the representation formula (3.1) for the perturbed potential u_ε, solution to (2.4). The argument is based on the idea of compensated compactness [43]. It is quite reminiscent of that used on the proof of the representation formula (1.4) for volumic perturbations of the conductivity considered in [20], see also [17].

Proof of Theorem 3.1. Let $r_\varepsilon := u_\varepsilon - u_0$ denote the error between the perturbed and background potentials. This function is the unique $H^1(\Omega)$ solution to the problem:

$$\left\{\begin{aligned} -\mathrm{div}(\gamma\nabla r_\varepsilon) = 0 & \quad \text{in } \Omega, \\ r_\varepsilon = 0 & \quad \text{on } \Gamma_D, \\ \gamma\tfrac{\partial r_\varepsilon}{\partial n} = 0 & \quad \text{on } \Gamma_N \setminus \overline{\omega_\varepsilon}, \\ \gamma\tfrac{\partial r_\varepsilon}{\partial n} + k_\varepsilon r_\varepsilon = -k_\varepsilon u_0 & \quad \text{on } \omega_\varepsilon. \end{aligned}\right. \tag{4.9}$$

Let $x \in \Omega$ be a fixed point. Using the definition (2.2) of the Green's function $G(x,y)$ to represent the value $r_\varepsilon(x)$ of the error at x, we obtain, after integration by parts:

$$\begin{aligned} r_\varepsilon(x) &= -\int_\Omega \mathrm{div}_y(\gamma(y)\nabla_y G(x,y))\, r_\varepsilon(y)\,\mathrm{d}y \\ &= -\int_{\partial\Omega} \gamma(y)\frac{\partial G}{\partial n_y}(x,y) r_\varepsilon(y)\,\mathrm{d}s(y) + \int_\Omega \gamma(y)\nabla_y G(x,y)\cdot\nabla r_\varepsilon(y)\,\mathrm{d}y \\ &= \int_\Omega \gamma(y)\nabla_y G(x,y)\cdot\nabla r_\varepsilon(y)\,\mathrm{d}y,\end{aligned}$$

where we have used the fact that $\gamma(y)\frac{\partial G}{\partial n_y}(x,y)$ vanishes for $y \in \Gamma_N$ and that r_ε vanishes on Γ_D. Another integration by parts yields:

$$\begin{aligned} r_\varepsilon(x) &= -\int_\Omega \mathrm{div}(\gamma\nabla r_\varepsilon)(y)\, G(x,y)\,\mathrm{d}y + \int_{\partial\Omega}\gamma(y)\frac{\partial r_\varepsilon}{\partial n}(y) G(x,y)\,\mathrm{d}s(y) \\ &= -k_\varepsilon \int_{\omega_\varepsilon} r_\varepsilon(y) G(x,y)\,\mathrm{d}s(y) - k_\varepsilon\int_{\omega_\varepsilon} u_0(y)G(x,y)\,\mathrm{d}s(y).\end{aligned}$$

Now introducing the Robin equilibrium potential χ_ε defined in (2.12), we obtain:

$$r_\varepsilon(x) = -\int_{\omega_\varepsilon} \gamma \frac{\partial \chi_\varepsilon}{\partial n}(y) r_\varepsilon(y) G(x,y)\, \mathrm{d}s(y) - k_\varepsilon \int_{\omega_\varepsilon} \chi_\varepsilon(y) r_\varepsilon(y) G(x,y)\, \mathrm{d}s(y) - k_\varepsilon \int_{\omega_\varepsilon} u_0(y) G(x,y)\, \mathrm{d}s(y).$$

Let now ϕ be a smooth function which coincides with $G(x,\cdot)$ on a fixed neighborhood of the compact set K_N containing the vanishing subsets ω_ε. Another use of Green's formula yields:

$$\begin{aligned} r_\varepsilon(x) &= -\int_\Omega \gamma \nabla \chi_\varepsilon \cdot \nabla(r_\varepsilon \phi)\, \mathrm{d}y - k_\varepsilon \int_{\omega_\varepsilon} \chi_\varepsilon r_\varepsilon \phi \, \mathrm{d}s - k_\varepsilon \int_{\omega_\varepsilon} u_0 \phi\, \mathrm{d}s \\ &= -\int_\Omega (\gamma \nabla \chi_\varepsilon \cdot \nabla r_\varepsilon)\phi\, \mathrm{d}y - k_\varepsilon \int_{\omega_\varepsilon} \chi_\varepsilon r_\varepsilon \phi \, \mathrm{d}s - k_\varepsilon \int_{\omega_\varepsilon} u_0 \phi\, \mathrm{d}s + R_{1,\varepsilon} \\ &= -\int_\Omega \gamma \nabla(\chi_\varepsilon \phi) \cdot \nabla r_\varepsilon\, \mathrm{d}y - k_\varepsilon \int_{\omega_\varepsilon} \chi_\varepsilon r_\varepsilon \phi \, \mathrm{d}s - k_\varepsilon \int_{\omega_\varepsilon} u_0 \phi\, \mathrm{d}s + R_{1,\varepsilon} + R_{2,\varepsilon}, \end{aligned}$$

where we have introduced the remainders

$$R_{1,\varepsilon} := -\int_\Omega (\gamma \nabla \chi_\varepsilon \cdot \nabla \phi) r_\varepsilon \, \mathrm{d}y, \text{ and } R_{2,\varepsilon} := \int_\Omega (\gamma \nabla r_\varepsilon \cdot \nabla \phi)\chi_\varepsilon \, \mathrm{d}y.$$

Yet another use of Green's formula reveals that

$$\begin{aligned} r_\varepsilon(x) &= -\int_{\partial\Omega} \gamma \frac{\partial r_\varepsilon}{\partial n} \chi_\varepsilon \phi\, \mathrm{d}s - k_\varepsilon \int_{\omega_\varepsilon} \chi_\varepsilon r_\varepsilon \phi\, \mathrm{d}s - k_\varepsilon \int_{\omega_\varepsilon} u_0 \phi \, \mathrm{d}s + R_{1,\varepsilon} + R_{2,\varepsilon} \\ &= -\int_{\omega_\varepsilon} \gamma \frac{\partial r_\varepsilon}{\partial n} \chi_\varepsilon \phi\, \mathrm{d}s - k_\varepsilon \int_{\omega_\varepsilon} \chi_\varepsilon r_\varepsilon \phi\, \mathrm{d}s - k_\varepsilon \int_{\omega_\varepsilon} u_0 \phi \, \mathrm{d}s + R_{1,\varepsilon} + R_{2,\varepsilon} \\ &= k_\varepsilon \int_{\omega_\varepsilon} (\chi_\varepsilon - 1) u_0 \phi\, \mathrm{d}s + R_{1,\varepsilon} + R_{2,\varepsilon}, \end{aligned}$$

Let us now write this equality under the form:

$$r_\varepsilon(x) = \mathrm{cap}^N_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon) \int_{\omega_\varepsilon} \frac{1}{\mathrm{cap}^N_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)} k_\varepsilon (\chi_\varepsilon - 1) u_0 \phi\, \mathrm{d}s + R_{1,\varepsilon} + R_{2,\varepsilon}. \tag{4.10}$$

Now, the sequence $\frac{1}{\mathrm{cap}^N_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)} k_\varepsilon(\chi_\varepsilon - 1)$ is bounded in $L^1(\partial\Omega)$, as an immediate consequence of the definition (2.14) of the Neumann-Robin capacity. Hence, there exists a subsequence of the ε (still denoted by ε) and a positive Radon measure μ with unit mass, supported on the fixed compact subset K_N (in particular, on Γ_N), such that

$$\forall \varphi \in \mathcal{C}(\partial\Omega), \quad \int_{\omega_\varepsilon} \frac{1}{\mathrm{cap}^N_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)} k_\varepsilon (\chi_\varepsilon - 1)\varphi\, \mathrm{d}s \xrightarrow{\varepsilon \to 0} -\int_{\partial\Omega} \varphi\, \mathrm{d}\mu.$$

Note that the sign of μ stems from the fact that $\chi_\varepsilon - 1 \leq 0$, as a consequence of the pointwise bounds (4.4). Moreover, using the H^1 and improved L^2 estimates in Proposition 4.1, we see that

$$R_{1,\varepsilon} + R_{2,\varepsilon} = \mathrm{o}\left(\mathrm{cap}^N_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)\right).$$

Combining this estimate with (4.10), we arrive at the desired formula (3.1). $\square$

4.3. Comparisons between scalings: proof of Theorem 3.2

Proof of (i): Let $\omega \Subset \Gamma_N$ and $k > 0$ be given; we use the shorthands $\chi = \chi(\omega, k)$ and $\chi^\infty = \chi^\infty(\omega)$ for the Robin and Dirichlet equilibrium distributions defined by (2.12) and (2.10). Let r be the difference $r = \chi - \chi^\infty$, which satisfies the following boundary-value problem:

$$\begin{cases} -\mathrm{div}(\gamma \nabla r) = 0 & \text{in } \Omega, \\ r = 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial r}{\partial n} = 0 & \text{on } \Gamma_N \setminus \overline{\omega}, \\ \gamma \frac{\partial r}{\partial n} + kr = -\gamma \frac{\partial \chi^\infty}{\partial n} & \text{on } \omega. \end{cases}$$

The corresponding variational problem reads:

$$\forall v \in H^1_{\Gamma_D}(\Omega), \quad \int_\Omega \gamma \nabla r \cdot \nabla v \,\mathrm{d}x \;+\; k \int_\omega r v \,\mathrm{d}s \;\;=\;\; -\int_\omega \gamma \frac{\partial \chi^\infty}{\partial n} v \,\mathrm{d}s.$$

On the one hand, by taking $v = r$ as a test function in this identity and using the fact that χ^∞ solves (2.10), we see that:

$$\begin{aligned}
\int_\Omega \gamma |\nabla r|^2 \,\mathrm{d}x \;+\; k \int_\omega r^2 \,\mathrm{d}s \;&=\; -\int_\omega \gamma \frac{\partial \chi^\infty}{\partial n} r \,\mathrm{d}s \\
&=\; -\int_{\partial\Omega} \gamma \frac{\partial \chi^\infty}{\partial n} r \,\mathrm{d}s \\
&=\; -\int_\Omega \gamma \nabla \chi^\infty \cdot \nabla r \,\mathrm{d}x \\
&\leq\; C \,||\nabla r||_{L^2(\Omega)^d} \,||\nabla \chi^\infty||_{L^2(\Omega)^d},
\end{aligned}$$

so that in particular:

$$||\nabla r||_{L^2(\Omega)^d} \;\leq\; C \,||\nabla \chi^\infty||_{L^2(\Omega)^d}. \tag{4.11}$$

On the other hand, elementary manipulations of the definitions (2.10) and (2.12) of χ^∞ and χ yield:

$$\begin{aligned}
\mathrm{cap}^N_{\mathrm{Rob}}(\omega, k) \;&=\; \int_\omega \gamma \frac{\partial \chi}{\partial n} \,\mathrm{d}s \\
&=\; \int_\omega \gamma \frac{\partial \chi}{\partial n} \chi^\infty \,\mathrm{d}s \\
&=\; \int_{\partial\Omega} \gamma \frac{\partial \chi}{\partial n} \chi^\infty \,\mathrm{d}s.
\end{aligned}$$

Now, by integration by parts, we see that:

$$\begin{aligned}
\mathrm{cap}^N_{\mathrm{Rob}}(\omega, k) \;&=\; \int_\Omega \gamma \nabla \chi \cdot \nabla \chi^\infty \,\mathrm{d}x. \\
&=\; \int_\Omega \gamma \nabla r \cdot \nabla \chi^\infty \,\mathrm{d}x \;+\; \int_\Omega \gamma \nabla \chi^\infty \cdot \nabla \chi^\infty \,\mathrm{d}x \\
&\leq\; C \left(||\nabla r||_{L^2(\Omega)^d} \;+\; ||\nabla \chi^\infty||_{L^2(\Omega)^d} \right) ||\nabla \chi^\infty||_{L^2(\Omega)^d} \\
&\leq\; C \,||\nabla \chi^\infty||^2_{L^2(\Omega)^d},
\end{aligned}$$

where the constant C depends only on Ω and the bounds γ^-, γ^+ in (1.2). The estimate (3.2) then follows from Proposition 2.1.

Proof of (ii): We again use the shorthand $\chi = \chi(\omega, k)$ for the solution to (2.12). From (4.4), we have $0 \leq \chi(x) \leq 1$ a.e. in Ω, so that:

$$\mathrm{cap}^N_{\mathrm{Rob}}(\omega, k) \;=\; k \int_\omega (1 - \chi) \,\mathrm{d}s \;\leq\; k\,|\omega|.$$

In order to obtain a lower bound for $\mathrm{cap}^N_{\mathrm{Rob}}(\omega, k)$, we use the Cauchy-Schwarz inequality and the Poincaré's inequality (4.1) to see that:

$$\int_\omega \chi \,\mathrm{d}s \;\;\leq\;\; |\omega|^{1/2} \Big(\int_\omega \chi^2 \,\mathrm{d}s \Big)^{1/2} \;\leq\; C |\omega|^{1/2} ||\nabla \chi||_{L^2(\Omega)^d},$$

where the constant C is independent of ω and k. In addition, using χ as test function in (2.13) shows that

$$\int_\Omega \gamma |\nabla \chi|^2 \,\mathrm{d}x = k \int_\omega \chi(1 - \chi) \,\mathrm{d}s \;\leq\; \mathrm{cap}^N_{\mathrm{Rob}}(\omega, k).$$

Combining the previous two inequalities, we arrive at

$$\int_\omega \chi \,\mathrm{d}s \;\leq\; C |\omega|^{1/2} \big(\mathrm{cap}^N_{\mathrm{Rob}}(\omega, k) \big)^{1/2}.$$

We now deduce the following estimate

$$\begin{aligned}
\mathrm{cap}^N_{\mathrm{Rob}}(\omega,k) &= k\int_\omega (1-\chi)\,\mathrm{d}s\\
&= k|\omega| - k\int_\omega \chi\,\mathrm{d}s\\
&\geq k|\omega| - Ck|\omega|^{1/2}\big(\mathrm{cap}^N_{\mathrm{Rob}}(\omega,k)\big)^{1/2}.
\end{aligned}$$

We have thus proved that the following quadratic inequality holds true:

$$X^2 + bX - c \ \geq\ 0, \text{ with unknown } X = \big(\mathrm{cap}^N_{\mathrm{Rob}}(\omega,k)\big)^{1/2} \geq 0,$$
$$\text{and coefficients } b = Ck|\omega|^{1/2} > 0,\ \ c = k|\omega| > 0.$$

The latter can only hold if $X \geq \frac{-b+\sqrt{b^2+4c}}{2}$, which yields, after elementary calculations:

$$\begin{aligned}
\mathrm{cap}^N_{\mathrm{Rob}}(\omega,k) &\geq 4k\,|\omega| \frac{1}{\left(\sqrt{4+C^2k}+\sqrt{C^2k}\right)^2},\\
&\geq C\,k|\omega|,
\end{aligned}$$

where the constant C depends on the assumed maximum bound M for k. This completes the proof.

Proof of (iii): Let ω be a fixed, strict subset of Γ_N and let k_ε be a sequence of positive numbers such that $k_\varepsilon \to \infty$. We use the shorthands $\chi_\varepsilon = \chi(\omega,k_\varepsilon)$ and $\chi^\infty = \chi^\infty(\omega)$ for the respective solutions to (2.12) and (2.10). At first, using χ_ε as test function in (2.12), we deduce from (3.2) that

$$\begin{aligned}
\int_\Omega \gamma|\nabla\chi_\varepsilon|^2\,\mathrm{d}x &= k_\varepsilon\int_\omega (1-\chi_\varepsilon)\chi_\varepsilon\,\mathrm{d}s\\
&\leq k_\varepsilon\int_\omega (1-\chi_\varepsilon)\,\mathrm{d}s\\
&= \mathrm{cap}^N_{\mathrm{Rob}}(\omega,k_\varepsilon)\\
&\leq C\mathrm{cap}(\omega),
\end{aligned}\tag{4.12}$$

where the second line follows from (4.4) and the final line is a consequence of (3.2). The sequence χ_ε is therefore uniformly bounded in $H^1(\Omega)$. Hence, up to a subsequence, still denoted by ε, there exists a function $\chi^* \in H^1(\Omega)$ such that $\chi_\varepsilon \to \chi^*$ weakly in $H^1(\Omega)$. Passing to the weak limit in the variational problem (2.13) for χ_ε, it is easy to check that χ^* solves

$$\left\{\begin{array}{rcll}
-\mathrm{div}(\gamma\nabla\chi^*) &=& 0 & \text{in } \Omega,\\
\chi^* &=& 0 & \text{on } \Gamma_D,\\
\gamma\frac{\partial\chi^*}{\partial n} &=& 0 & \text{on } \Gamma_N\setminus\overline{\omega},\\
\chi^* &=& 1 & \text{on } \omega,
\end{array}\right.\tag{4.13}$$

that is, $\chi^* = \chi^\infty$. In addition, the uniqueness of this limit implies that the whole sequence χ_ε weakly converges to χ^∞ in $H^1(\Omega)$. Next, we compute:

$$\begin{aligned}
\mathrm{cap}^N_{\mathrm{Rob}}(\omega,k_\varepsilon) &= \int_{\partial\Omega}\gamma\frac{\partial\chi_\varepsilon}{\partial n}\,\mathrm{d}s\\
&= \int_{\partial\Omega}\gamma\frac{\partial\chi_\varepsilon}{\partial n}\,\chi_\infty\,\mathrm{d}s\\
&= \int_\Omega \gamma\nabla\chi_\varepsilon\cdot\nabla\chi^\infty\,\mathrm{d}x.
\end{aligned}$$

Hence,

$$\mathrm{cap}^N_{\mathrm{Rob}}(\omega,k_\varepsilon) \xrightarrow{\varepsilon\to 0} \int_\Omega \gamma|\nabla\chi^\infty|^2\,\mathrm{d}x,$$

a quantity which is controlled by $\mathrm{cap}(\omega)$ up to constants which are independent of ε, see Proposition 2.1. This terminates the proof of (3.4).

4.4. **Monotonicity of the Robin-Neumann capacity: proof of Proposition 3.1**

Proof of (i): Let ω be a fixed subset of the Neumann region Γ_N, et let $0 < k_1 \leq k_2$. We denote by $\chi_1 := \chi(\omega, k_1)$ and $\chi_2 := \chi(\omega, k_2)$ the equilibrium distributions associated to these two situations, see (2.12). We recall from the pointwise bounds (4.4) in Proposition 4.1 that these functions satisfy

$$0 \leq \chi_1 \leq 1 \text{ and } \; 0 \leq \chi_2 \leq 1 \text{ on } \overline{\Omega}.$$

A simple calculation shows that the difference $r := \chi_2 - \chi_1$ between them is the unique $H^1(\Omega)$ solution to the following boundary value problem:

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\gamma \nabla r) & = & 0 & \text{in } \Omega, \\ r & = & 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial r}{\partial n} & = & 0 & \text{on } \Gamma_N \setminus \overline{\omega}, \\ \gamma \frac{\partial r}{\partial n} + k_2 r & = & (k_2 - k_1)(1 - \chi_1) & \text{on } \omega. \end{array} \right. \tag{4.14}$$

Since $k_1 \leq k_2$ and $(1 - \chi_1)$ is non negative on $\overline{\Omega}$, the term $(k_2 - k_1)(1 - \chi_1)$ at the right-hand side of the Robin boundary condition in (4.14) is non negative. Another application of the pointwise bounds (4.4) shows that:

$$r = \chi_2 - \chi_1 \;\; \geq \;\; 0 \text{ on } \overline{\Omega}. \tag{4.15}$$

Now recalling the definition (2.14) of the Robin-Neumann capacity, we see that:

$$\mathrm{cap}^N_{\mathrm{Rob}}(\omega, k_2) - \mathrm{cap}^N_{\mathrm{Rob}}(\omega, k_1) = \int_\omega \gamma \frac{\partial r}{\partial n} \, \mathrm{d}s.$$

Hence, bringing into play the Dirichlet equilibrium distribution $\chi^\infty \equiv \chi^\infty(\omega)$ defined by (2.10), we obtain:

$$\begin{array}{rcl} \mathrm{cap}^N_{\mathrm{Rob}}(\omega, k_2) - \mathrm{cap}^N_{\mathrm{Rob}}(\omega, k_1) & = & \displaystyle\int_\omega \gamma \frac{\partial r}{\partial n} \chi^\infty \, \mathrm{d}s \\ & = & \displaystyle\int_{\partial\Omega} \gamma \frac{\partial r}{\partial n} \chi^\infty \, \mathrm{d}s \\ & = & \displaystyle\int_{\partial\Omega} \gamma \frac{\partial \chi^\infty}{\partial n} r \, \mathrm{d}s, \\ & = & \displaystyle\int_\omega \gamma \frac{\partial \chi^\infty}{\partial n} r \, \mathrm{d}s, \end{array}$$

where the third line follows after two successive applications of the Green's formula.

Now, since $\chi^\infty \equiv 1$ on ω and $0 \leq \chi^\infty \leq 1$ on $\overline{\Omega}$, the normal derivative $\gamma \frac{\partial \chi^\infty}{\partial n}$ is non negative on ω. Recalling (4.15), this terminates the proof of (i).

Proof of (ii): Let k be a positive real number and let $\omega_1 \subset \omega_2$ be two subsets of the Neumann region Γ_N. We now set $\chi_1 := \chi(\omega_1, k)$ and $\chi_2 := \chi(\omega_2, k)$ the equilibrium distributions, see again the definition (2.12).

In this situation again, an elementary calculation shows that the difference $r := \chi_2 - \chi_1$ between both functions is the unique $H^1(\Omega)$ solution to the following boundary value problem:

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\gamma \nabla r) & = & 0 & \text{in } \Omega, \\ r & = & 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial r}{\partial n} & = & 0 & \text{on } \Gamma_N \setminus \overline{\omega_2}, \\ \gamma \frac{\partial r}{\partial n} + kr & = & k(1 - \chi_1) & \text{on } \omega_2 \setminus \overline{\omega_1}, \\ \gamma \frac{\partial r}{\partial n} + kr & = & 0 & \text{on } \omega_1. \end{array} \right. \tag{4.16}$$

Since the right-hand side of the Robin boundary condition satisfied by r on ω_2 is non negative, a simple adaptation of the proof of the pointwise bounds (4.4) reveals that:

$$r = \chi_2 - \chi_1 \ \geq \ 0 \text{ on } \overline{\Omega}. \tag{4.17}$$

Let us now use the fact that $\gamma \frac{\partial \chi_1}{\partial n} = 0$ on $\omega_2 \setminus \overline{\omega_1}$ to express the difference between the capacities $\text{cap}^N_{\text{Rob}}(\omega_2, k)$ and $\text{cap}^N_{\text{Rob}}(\omega_1, k)$ as:

$$\text{cap}^N_{\text{Rob}}(\omega_2, k) - \text{cap}^N_{\text{Rob}}(\omega_1, k) = \int_{\omega_2} \gamma \frac{\partial r}{\partial n} \, \mathrm{d}s.$$

Like in the proof of point (i), we now introduce the Dirichlet equilibrium distribution $\chi_2^\infty := \chi^\infty(\omega_2)$ defined by (2.10) in this formula and apply Green's formula twice to obtain:

$$\begin{aligned} \text{cap}^N_{\text{Rob}}(\omega_2, k) - \text{cap}^N_{\text{Rob}}(\omega_1, k) &= \int_{\omega_2} \gamma \frac{\partial r}{\partial n} \chi_2^\infty \, \mathrm{d}s \\ &= \int_{\omega_2} \gamma \frac{\partial \chi_2^\infty}{\partial n} r \, \mathrm{d}s \end{aligned}$$

Again, the claimed inequality in (ii) follows from the combination of (4.15) with the non negativity of the normal derivative $\gamma \frac{\partial \chi_2^\infty}{\partial n}$ on ω .

5. Consistency of the expansions when Γ_N is perturbed by a Robin and a Dirichlet boundary condition: Proof of Theorem 3.3

In the previous sections, we have proved that, for a fixed subset ω_ε of the Neumann region Γ_N, the capacity $\text{cap}^N_{\text{Rob}}(\omega, k_\varepsilon)$ behaves as $\text{cap}(\omega)$ when the admittance parameter k_ε tends to ∞. However, this behavior depends on the considered set ω. In this section, we show Theorem 3.3, which claims that, roughly speaking, it persists in the case of a sequence ω_ε of subsets that are vanishingly small, see Theorem 3.3. This indicates in particular that the asymptotic expansion (1.5) associated to a Robin perturbation of the Neumann boundary is "consistent" with that (3.1) associated to a Dirichlet perturbation of this boundary, when the parameter k_ε tends to ∞ "sufficiently fast" in the limit $\varepsilon \to 0$. This result rests on additional assumptions on the sets ω_ε; namely we assume the following:

(5.1)
- If $d = 2$, ω_ε is an arc of length 2ε on $\partial\Omega$,
- If $d = 3$, ω_ε is the image $\Phi(\mathbb{D}_\varepsilon)$ of the planar disk $\mathbb{D}_\varepsilon = \{x = (x_1, x_2, 0) \in \mathbb{R}^3, \quad x_1^2 + x_2^2 < \varepsilon^2\}$ by a given smooth diffeomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$.

To simplify the presentation, we limit ourselves to proving this result in the 2d case, in the particular configuration where ω_ε is the segment $(-\varepsilon/2, \varepsilon/2) \times \{0\}$, pertaining to a completely flat part of the security compact $K_N \subset \Gamma_N$, while Ω coincides with the lower half-plane in the neighborhood of 0. Moreover, we assume that the conductivity γ identically equals 1.

Throughout the proof, we use the shorthands χ_ε and χ_ε^∞ for the equilibrium distributions $\chi(\omega_\varepsilon, k_\varepsilon)$ and $\chi^\infty(\omega_\varepsilon)$ defined by (2.10) and (2.12), respectively. We proceed in several steps.

Step 1: We relate the capacities $\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)$ *and* $\text{cap}(\omega_\varepsilon)$*, up to a remainder.*

To this end, let us recall from (2.14) that

$$\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon) \ = \ k_\varepsilon \int_{\omega_\varepsilon} (1 - \chi_\varepsilon) \, \mathrm{d}s \ = \ \int_{\omega_\varepsilon} \frac{\partial \chi_\varepsilon}{\partial n} \, \mathrm{d}s,$$

while the capacity $\text{cap}(\omega_\varepsilon)$ is equivalent to the quantity

$$\int_\Omega |\nabla \chi_\varepsilon^\infty|^2 \, \mathrm{d}x \ = \ \int_{\omega_\varepsilon} \frac{\partial \chi_\varepsilon^\infty}{\partial n} \, \mathrm{d}s, \tag{5.2}$$

up to a multiplicative constant that does not depend on ε, see Proposition 2.1.

We already know from (3.2) that $\text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon) \leq C\,\text{cap}(\omega_\varepsilon)$, and we look for a reciprocal estimate when k_ε is large enough. To achieve this, let us write:

$$\begin{aligned} \text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon) &= \int_{\omega_\varepsilon} \frac{\partial \chi_\varepsilon}{\partial n} \chi_\varepsilon^\infty \, \mathrm{d}s \\ &= \int_\Omega \nabla \chi_\varepsilon \cdot \nabla \chi_\varepsilon^\infty \, \mathrm{d}x \\ &= \int_{\partial\Omega} \frac{\partial \chi_\varepsilon^\infty}{\partial n} \chi_\varepsilon \, \mathrm{d}s \\ &= \int_{\omega_\varepsilon} \frac{\partial \chi_\varepsilon^\infty}{\partial n} \chi_\varepsilon \, \mathrm{d}s. \end{aligned}$$

Let us now bring into play the quantity $\text{cap}(\omega_\varepsilon)$ into this equality:

$$\begin{aligned} \text{cap}^N_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon) &= \int_{\omega_\varepsilon} \frac{\partial \chi_\varepsilon^\infty}{\partial n} \, \mathrm{d}s - \int_{\omega_\varepsilon} \frac{\partial \chi_\varepsilon^\infty}{\partial n}(1-\chi_\varepsilon) \, \mathrm{d}s \\ &= \int_\Omega |\nabla \chi_\varepsilon^\infty|^2 \, \mathrm{d}x - \int_{\omega_\varepsilon} \frac{\partial \chi_\varepsilon^\infty}{\partial n}(1-\chi_\varepsilon) \, \mathrm{d}s \\ &\geq C\text{cap}(\omega_\varepsilon) + \int_{\omega_\varepsilon} \frac{\partial \chi_\varepsilon^\infty}{\partial n} r_\varepsilon \, \mathrm{d}s, \end{aligned} \tag{5.3}$$

where $r_\varepsilon = \chi_\varepsilon - \chi_\varepsilon^\infty$ is the unique $H^1(\Omega)$ solution to the following boundary-value problem:

$$\left\{ \begin{array}{rcll} -\Delta r_\varepsilon &=& 0 & \text{in } \Omega, \\ r_\varepsilon &=& 0 & \text{on } \Gamma_D, \\ \frac{\partial r_\varepsilon}{\partial n} &=& 0 & \text{on } \Gamma_N \setminus \overline{\omega_\varepsilon}, \\ \frac{\partial r_\varepsilon}{\partial n} + k_\varepsilon r_\varepsilon &=& -\frac{\partial \chi_\varepsilon^\infty}{\partial n} & \text{on } \omega_\varepsilon. \end{array} \right. \tag{5.4}$$

To complete the proof, we thus need to estimate the term $R_\varepsilon := \int_{\omega_\varepsilon} \frac{\partial \chi_\varepsilon^\infty}{\partial n} r_\varepsilon \, \mathrm{d}s$. We rewrite the latter by rescaling about the vicinity of the origin $0 \in \omega_\varepsilon$:

$$R_\varepsilon = \int_{\omega_1} \frac{\partial \overline{\chi_\varepsilon^\infty}}{\partial n} \overline{r_\varepsilon} \, \mathrm{d}s, \tag{5.5}$$

where ω_1 is the segment $(-\frac{1}{2}, \frac{1}{2}) \times \{0\}$ and we have set

$$\overline{\chi_\varepsilon^\infty}(z) = \chi_\varepsilon^\infty(\varepsilon z) \text{ and } \overline{r_\varepsilon}(z) = r_\varepsilon(\varepsilon z), \text{ for } z \in \frac{1}{\varepsilon}\Omega.$$

Step 2: We estimate the rescaled function $\overline{\chi_\varepsilon^\infty}$.

Throughout the rest of the proof, we denote by B_r (resp. B_r^-) the ball (resp. the lower half-ball) with center 0 and radius r in $\mathbb{R}^d$. Let us observe that a change of variables yields:

$$\begin{aligned} ||\nabla \overline{\chi_\varepsilon^\infty}||^2_{L^2(B_2^-)^d} &= \varepsilon^2 \int_{B_2^-} |(\nabla \chi_\varepsilon^\infty)(\varepsilon z)|^2 \, \mathrm{d}z \\ &= \int_{B_{2\varepsilon}^-} |\nabla \chi_\varepsilon^\infty(z)|^2 \, \mathrm{d}z \\ &\leq ||\nabla \chi_\varepsilon^\infty||^2_{L^2(\Omega)^d} \\ &\leq C\text{cap}(\omega_\varepsilon). \end{aligned}$$

We now set $m_\varepsilon := \frac{1}{|B_2^-|} \int_{B_2^-} \chi_\varepsilon^\infty \, \mathrm{d}x$, whose value is less than 1, owing to point (i) in Proposition 4.1. By virtue of the Poincaré-Wirtinger inequality in B_2^-, it holds:

$$\begin{aligned} ||\overline{\chi_\varepsilon^\infty} - m_\varepsilon||_{L^2(B_2^-)} &\leq C||\nabla \overline{\chi_\varepsilon^\infty}||_{L^\infty(B_2^-)^d} \\ &\leq C\text{cap}(\omega_\varepsilon)^{1/2}. \end{aligned}$$

Moreover, the function $\overline{\chi_\varepsilon^\infty} - m_\varepsilon$ satisfies the following mixed boundary value problem in B_2^-:

$$\begin{cases} -\Delta\big(\overline{\chi_\varepsilon^\infty} - m_\varepsilon\big) &= & 0 & \text{in } B_2^-, \\ \overline{\chi_\varepsilon^\infty} - m_\varepsilon &= & 1 - m_\varepsilon & \text{on } \omega_1, \\ \frac{\partial\big(\overline{\chi_\varepsilon^\infty} - m_\varepsilon\big)}{\partial n} &= & 0 & \text{on } \omega_2 \setminus \overline{\omega_1}, \\ \big(\overline{\chi_\varepsilon^\infty} - m_\varepsilon\big) &= & g_\varepsilon & \text{on } \partial B_2 \cap B_2^-, \end{cases}$$

where g_ε lies in $H^{1/2}(\partial B_2 \cap B_2^-)$. It follows from the theory of elliptic regularity that $\overline{\chi_\varepsilon^\infty} - m_\varepsilon \in H^s(B_{3/2}^-)$ for any $1 \le s < 3/2$ and that

$$||\overline{\chi_\varepsilon^\infty} - m_\varepsilon||_{H^s(B_{3/2}^-)} \;\le\; C\, ||\overline{\chi_\varepsilon^\infty} - m_\varepsilon||_{H^1(B_2^-)} \;\le\; C\mathrm{cap}(\omega_\varepsilon)^{1/2}, \tag{5.6}$$

with C independent of ε, see [32, 35]. Invoking the trace Theorem, see for instance Theorem 1.5.2.1 in [35], we obtain that $\frac{\partial \overline{\chi_\varepsilon^\infty}}{\partial n} \in H^{-s}(\partial B_{3/2}^-)$ for any $0 < s \le 1/2$, with the estimate:

$$\left|\left|\frac{\partial \overline{\chi_\varepsilon^\infty}}{\partial n}\right|\right|_{H^{-s}(\partial B_{3/2}^-)} \;\le\; C\, ||\overline{\chi_\varepsilon^\infty} - m_\varepsilon||_{H^1(B_2^-)} \;\le\; C\mathrm{cap}(\omega_\varepsilon)^{\frac{1}{2}}.$$

As $\frac{\partial \overline{\chi_\varepsilon^\infty}}{\partial n}$ vanishes outside ω_1, we conclude that $\frac{\partial \overline{\chi_\varepsilon^\infty}}{\partial n} \in \widetilde{H}^{-s}(\omega_1)$ with

$$\left|\left|\frac{\partial \overline{\chi_\varepsilon^\infty}}{\partial n}\right|\right|_{\widetilde{H}^{-s}(\omega_1)} \;\le\; C \left|\left|\frac{\partial \overline{\chi_\varepsilon^\infty}}{\partial n}\right|\right|_{H^{-s}(\partial B_{3/2}^-)} \;\le\; C\mathrm{cap}(\omega_\varepsilon)^{\frac{1}{2}}, \tag{5.7}$$

where C is independent of ε.

Note that it is possible to directly control the $\widetilde{H}^{-s}(\omega_\varepsilon)$ norm of the normal derivative $\frac{\partial \chi_\varepsilon}{\partial n}$ of the unscaled function χ_ε from ω_ε, up to a finer estimate. This result is not needed in the present proof, but since it is interesting in its own right, a statement and proof are included in Appendix A.

Step 3: We estimate the rescaled function $\overline{r_\varepsilon}$.

We first note that

$$\begin{aligned} ||\nabla r_\varepsilon||_{L^2(\Omega)^d} &\le ||\nabla \chi_\varepsilon||_{L^2(\Omega)^d} + ||\nabla \chi_\varepsilon^\infty||_{L^2(\Omega)^d} \\ &\le \left(k_\varepsilon \int_{\omega_\varepsilon} (1-\chi_\varepsilon)\chi_\varepsilon \,\mathrm{ds}\right)^{1/2} + C\mathrm{cap}(\omega_\varepsilon)^{1/2} \\ &\le C\left(\mathrm{cap}^N_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)^{1/2} + \mathrm{cap}(\omega_\varepsilon)^{1/2}\right) \\ &\le C\,\mathrm{cap}(\omega_\varepsilon)^{1/2}, \end{aligned} \tag{5.8}$$

where we have used the pointwise bounds (4.4) to pass from the second line to the third one and (3.2) to arrive at the final line. Rescaling this inequality, we obtain

$$\begin{aligned} ||\nabla \overline{r_\varepsilon}||_{L^2(B_2^-)^d} &= ||\nabla r_\varepsilon||_{L^2(B_{2\varepsilon}^-)^d} \\ &\le C\,\mathrm{cap}(\omega_\varepsilon)^{1/2}. \end{aligned}$$

In addition, we have

$$\begin{aligned} ||\overline{r_\varepsilon}||^2_{H^{1/2}(\omega_1)} &\le ||\overline{r_\varepsilon}||^2_{H^1(B_1^-)} \\ &= ||\overline{r_\varepsilon}||^2_{L^2(B_1^-)} + ||\nabla \overline{r_\varepsilon}||^2_{L^2(B_1^-)^2} \\ &= \varepsilon^{-2}||r_\varepsilon||^2_{L^2(B_\varepsilon^-)} + ||\nabla r_\varepsilon||^2_{L^2(B_\varepsilon^-)^2} \\ &\le \varepsilon^{-2}||r_\varepsilon||^2_{H^1(\Omega)}. \end{aligned}$$

Recalling that $|\omega_\varepsilon| = 2\varepsilon$, it follows from (5.8) and the Poincaré's inequality (4.1) that

$$||\overline{r_\varepsilon}||_{H^{1/2}(\omega_1)} \;\le\; \frac{C}{|\omega_\varepsilon|}\,\mathrm{cap}(\omega_\varepsilon)^{1/2}. \tag{5.9}$$

On the other hand, multiplying (5.4) by r_ε and integrating, we obtain

$$\int_\Omega |\nabla r_\varepsilon|^2 \,\mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} r_\varepsilon^2 \,\mathrm{d}s = -\int_{\partial\Omega} \frac{\partial \chi_\varepsilon^\infty}{\partial n} r_\varepsilon \,\mathrm{d}s,$$

and so:

$$(5.10)\qquad \begin{aligned} ||r_\varepsilon||_{L^2(\omega_\varepsilon)} &\leq k_\varepsilon^{-1/2} \left|\left| \frac{\partial \chi_\varepsilon^\infty}{\partial n} \right|\right|_{H^{-1/2}(\partial\Omega)}^{1/2} ||r_\varepsilon||_{H^{1/2}(\partial\Omega)}^{1/2} \\ &\leq C\, k_\varepsilon^{-1/2} \, ||\chi_\varepsilon^\infty||_{H^1(\Omega)}^{1/2} \, ||\nabla r_\varepsilon||_{L^2(\Omega)}^{1/2} \\ &\leq C\, k_\varepsilon^{-1/2} \mathrm{cap}(\omega_\varepsilon)^{1/2}, \end{aligned}$$

where we have used (5.8) to obtain the final inequality. A simple change of variables now yields:

$$||\overline{r_\varepsilon}||_{L^2(\omega_1)} \leq \frac{C}{|\omega_\varepsilon|^{1/2} k_\varepsilon^{1/2}} \mathrm{cap}(\omega_\varepsilon)^{1/2}.$$

By interpolation, we deduce from (5.9) and (5.10) that $\overline{r_\varepsilon} \in H^{1/4}(\omega_1)$ with:

$$(5.11)\qquad ||\overline{r_\varepsilon}||_{H^{1/4}(\omega_1)} \leq \frac{C}{|\omega_\varepsilon|^{3/4} k_\varepsilon^{1/4}} \,\mathrm{cap}(\omega_\varepsilon)^{1/2}.$$

Step 4: Conclusion.

It follows from the definition (5.5) of R_ε and from the estimates (5.7)and (5.11) that:

$$\begin{aligned} R_\varepsilon &\leq \left|\left| \frac{\partial \overline{\chi_\varepsilon^\infty}}{\partial n} \right|\right|_{\widetilde{H}^{-1/4}(\omega_1)} ||\overline{r_\varepsilon}||_{H^{1/4}(\omega_1)} \\ &\leq \frac{C}{|\omega_\varepsilon|^{3/4} k_\varepsilon^{1/4}} \,\mathrm{cap}(\omega_\varepsilon). \end{aligned}$$

Hence, if k_ε tends to infinity so fast that $k_\varepsilon |\omega_\varepsilon|^3 \to \infty$, the remainder R_ε tends to 0. This terminates the proof.

6. Proofs of the results about Robin perturbation of the Dirichlet boundary Γ_D

In this section, we consider the situation where the perturbation of the background potential u_0 in (2.1) is induced by a replacement of the Dirichlet boundary condition by a Robin boundary condition on a subset ω_ε of the region $\Gamma_D \subset \partial\Omega$: the perturbed potential $u_\varepsilon^{D,k_\varepsilon} \in H^1(\Omega)$, denoted by u_ε throughout this section, solves the boundary-valued problem (2.6), so that the difference $r_\varepsilon = u_\varepsilon - u_0$ is the $H^1(\Omega)$ solution to the following problem:

$$(6.1)\qquad \left\{ \begin{array}{rcll} -\mathrm{div}(\gamma \nabla r_\varepsilon) &=& 0 & \text{in } \Omega, \\ r_\varepsilon &=& 0 & \text{on } \Gamma_D \setminus \omega_\varepsilon, \\ \gamma \frac{\partial r_\varepsilon}{\partial n} &=& 0 & \text{on } \Gamma_N, \\ \gamma \frac{\partial r_\varepsilon}{\partial n} + k_\varepsilon r_\varepsilon &=& -\gamma \frac{\partial u_0}{\partial n} & \text{on } \omega_\varepsilon. \end{array} \right.$$

In this setting, we aim to derive an asymptotic expansion of r_ε as $\varepsilon \to 0$.

6.1. **Preliminary estimates**

Let ω_ε be a sequence of subsets of the Dirichlet region Γ_D that are strictly contained in the latter, in the sense that they all belong to a fixed compact $K_D \subset \Gamma_D$, i.e. (2.5) holds true; let also $k_\varepsilon > 0$ be a sequence of positive numbers. In order to estimate the difference r_ε in (6.1), we first consider, more generally, the solution v_ε to the following boundary-value problem:

$$(6.2)\qquad \left\{ \begin{array}{rcll} -\mathrm{div}(\gamma \nabla v_\varepsilon) &=& 0 & \text{in } \Omega, \\ v_\varepsilon &=& 0 & \text{on } \Gamma_D \setminus \omega_\varepsilon, \\ \gamma \frac{\partial v_\varepsilon}{\partial n} &=& 0 & \text{on } \Gamma_N, \\ \gamma \frac{\partial v_\varepsilon}{\partial n} + k_\varepsilon v_\varepsilon &=& g & \text{on } \omega_\varepsilon, \end{array} \right.$$

where g is a given function in $\mathcal{C}^\infty(\mathbb{R}^d)$. Equivalently, this function satisfies the following variational problem:

$$\text{Find } v_\varepsilon \in H^1_{\Gamma_D}(\Omega) \text{ s.t. } \forall v \in H^1_{\Gamma_D}(\Omega), \quad \int_\Omega \gamma \nabla v \cdot \nabla v \, \mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} v_\varepsilon v \, \mathrm{d}s = \int_{\omega_\varepsilon} gv \, \mathrm{d}s. \tag{6.3}$$

Before starting, let us recall the following version of the Poincaré's inequality for functions vanishing on $\Gamma_D \setminus K_D$: there exists a constant $C > 0$ depending on Ω, Γ_N, Γ_D, K_D and the bounds γ^-, γ^+ on the conductivity γ in (1.2) such that:

$$\forall v \in H^1_{\Gamma_D \setminus K_D}(\Omega), \quad \frac{1}{C}||v||^2_{H^1(\Omega)} \leq \int_\Omega \gamma |\nabla v|^2 \, \mathrm{d}x \leq ||v||^2_{H^1(\Omega)}. \tag{6.4}$$

We now state a lemma containing preliminary estimates about v_ε.

Lemma 6.1. *Let ω_ε be a sequence of subsets of Γ_D satisfying the separation assumption (2.5) and let k_ε be any sequence of positive real numbers; let $g \in \mathcal{C}^\infty(\mathbb{R}^d)$, and let v_ε be the unique solution to (6.2). Then,*

(i) The following pointwise bounds hold true:

$$\frac{1}{k_\varepsilon} \min\left(0, \inf_{x \in \overline{\Omega}} g(x)\right) \leq v_\varepsilon(x) \leq \frac{1}{k_\varepsilon} \max\left(0, \sup_{x \in \overline{\Omega}} g(x)\right) \text{ for a.e. } x \in \overline{\Omega}. \tag{6.5}$$

(ii) The following H^1 estimate holds true:

$$||v_\varepsilon||_{H^1(\Omega)} \leq C||g||_\infty \operatorname{cap}^D_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)^{1/2}, \tag{6.6}$$

where the constant C depends on Ω, Γ_N, Γ_D and the compact K_D, but not on ω_ε or k_ε.

(iii) The following improved L^2 estimate holds true:

$$||v_\varepsilon||_{L^2(\Omega)} \leq C||g||_\infty \operatorname{cap}^D_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)^{3/4}, \tag{6.7}$$

where, again, C depends only Ω, Γ_N, Γ_D and the compact K_D.

Proof. Proof of (i). The proof is similar to that of (i) in Proposition 4.1. Let M be the quantity at the right-hand side of (6.5); setting $\psi_\varepsilon(x) = \max(v_\varepsilon(x) - M, 0)$ as test function in the variational problem (6.3) satisfied by v_ε, we see that:

$$\int_\Omega \gamma |\nabla v_\varepsilon|^2 \mathbb{1}_{v_\varepsilon \geq M} \, \mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} v_\varepsilon (v_\varepsilon - M) \mathbb{1}_{v_\varepsilon \geq M} \, \mathrm{d}s = \int_{\omega_\varepsilon} g(v_\varepsilon - M) \mathbb{1}_{v_\varepsilon \geq M} \, \mathrm{d}s,$$

and so:

$$\int_\Omega \gamma |\nabla v_\varepsilon|^2 \mathbb{1}_{v_\varepsilon \geq M} \, \mathrm{d}x = \int_{\omega_\varepsilon} (g - k_\varepsilon v_\varepsilon)(v_\varepsilon - M) \mathbb{1}_{v_\varepsilon \geq M} \, \mathrm{d}s \leq 0,$$

as follows from the very definition of M. Hence, $v_\varepsilon(x) \leq M$ for a.e. $x \in \overline{\Omega}$, which is exactly the right-hand inequality in (6.5). The left-hand inequality in this statement follows by a similar argument.

Proof of (ii). We actually prove the stronger estimate

$$\int_\Omega \gamma |\nabla v_\varepsilon|^2 \, \mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} v_\varepsilon^2 \, \mathrm{d}s \leq C||g||_\infty \operatorname{cap}^D_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon), \tag{6.8}$$

which directly implies (6.6) upon application of the Poincaré's inequality (6.4) for functions vanishing on $\Gamma_D \setminus K_D$.

To achieve this, we first take $v = v_\varepsilon$ as test function in the variational problem (6.3) for v_ε, which yields:

$$\begin{aligned} \int_\Omega \gamma |\nabla v_\varepsilon|^2 \, \mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} v_\varepsilon^2 \, \mathrm{d}s &= \int_{\omega_\varepsilon} g v_\varepsilon \, \mathrm{d}s \\ &\leq ||g||_{\mathcal{C}(\overline{\Omega})} \int_{\omega_\varepsilon} |v_\varepsilon| \, \mathrm{d}s. \end{aligned} \tag{6.9}$$

Next, bringing into play the Robin-Dirichlet equilibrium distribution $\xi_\varepsilon := \xi(\omega_\varepsilon, k_\varepsilon)$ defined by (2.15), a simple calculation yields:

$$\begin{aligned}\int_{\omega_\varepsilon} |v_\varepsilon| \, \mathrm{d}s &= \int_{\omega_\varepsilon} \gamma \frac{\partial \xi_\varepsilon}{\partial n} |v_\varepsilon| \, \mathrm{d}s + \int_{\omega_\varepsilon} k_\varepsilon \xi_\varepsilon |v_\varepsilon| \, \mathrm{d}s \\ &= \int_{\partial\Omega} \gamma \frac{\partial \xi_\varepsilon}{\partial n} |v_\varepsilon| \, \mathrm{d}s + \int_{\omega_\varepsilon} k_\varepsilon \xi_\varepsilon |v_\varepsilon| \, \mathrm{d}s \\ &= \int_{\Omega} \gamma \nabla \xi_\varepsilon \cdot \nabla |v_\varepsilon| \, \mathrm{d}x + \int_{\omega_\varepsilon} k_\varepsilon \xi_\varepsilon |v_\varepsilon| \, \mathrm{d}s.\end{aligned}$$

The Cauchy-Schwarz inequality then implies that:

$$\int_{\omega_\varepsilon} |v_\varepsilon| \, \mathrm{d}s \;\leq\; \left(\int_\Omega \gamma |\nabla \xi_\varepsilon|^2 \, \mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} \xi_\varepsilon^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \left(\int_\Omega \gamma |\nabla v_\varepsilon|^2 \, \mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} v_\varepsilon^2 \, \mathrm{d}s \right)^{\frac{1}{2}}.$$

Inserting this inequality into (6.9) and rearranging, we arrive at:

$$\int_\Omega \gamma |\nabla v_\varepsilon|^2 \, \mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} v_\varepsilon^2 \, \mathrm{d}s \;\leq\; ||g||^2_{\mathcal{C}(\overline{\Omega})} \left(\int_\Omega \gamma |\nabla \xi_\varepsilon|^2 \, \mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} \xi_\varepsilon^2 \, \mathrm{d}s \right).$$

Eventually, recalling the definition (2.16) of the Robin-Dirichlet capacity $\mathrm{cap}^D_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)$, this proves the desired result (6.8), and thus (6.6).

For further reference, note that we have proved in passing the following estimate:

$$\int_{\omega_\varepsilon} |v_\varepsilon| \, \mathrm{d}s \;\leq\; C ||g||_{\mathcal{C}(\overline{\Omega})} \mathrm{cap}^D_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon). \tag{6.10}$$

Proof of (iii). We adapt the duality argument in the proof of point (iii) in Proposition 4.1 to the present situation. Let w_ε be the $H^1(\Omega)$ solution to the following boundary-value problem:

$$\left\{ \begin{array}{cl} -\mathrm{div}(\gamma \nabla w_\varepsilon) = -v_\varepsilon & \text{in } \Omega, \\ w_\varepsilon = 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial w_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N. \end{array} \right. \tag{6.11}$$

We introduce two open sets $\mathcal{O}_1 \Subset \mathcal{O}_2$ such that $K_D \subset \mathcal{O}_1$ and $\overline{\mathcal{O}_2} \cap \overline{\Gamma_N} = \emptyset$, as well as a smooth cut-off function η satisfying $\eta \equiv 1$ on $\mathcal{O}_1$ and $\eta \equiv 0$ on $\mathbb{R}^d \setminus \mathcal{O}_2$. Here again, classical elliptic regularity theory and the Sobolev embedding theorem imply that $w_\varepsilon \in H^3(\mathcal{O}_2 \cap \overline{\Omega})$ with

$$||w_\varepsilon||_{\mathcal{C}^1(\mathcal{O}_2 \cap \overline{\Omega})} \leq C ||w_\varepsilon||_{H^3(\mathcal{O}_2 \cap \Omega)} \leq C ||v_\varepsilon||_{H^1(\Omega)} \leq C \, ||g||_\infty \mathrm{cap}^D_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)^{1/2}. \tag{6.12}$$

Now, repeated integrations by parts yield:

$$\begin{aligned}\int_\Omega v_\varepsilon^2 \, \mathrm{d}x &= \int_\Omega \mathrm{div}(\gamma \nabla w_\varepsilon) v_\varepsilon \, \mathrm{d}x \\ &= \int_{\partial\Omega} \gamma \frac{\partial w_\varepsilon}{\partial n} v_\varepsilon \, \mathrm{d}s - \int_\Omega \gamma \nabla w_\varepsilon \cdot \nabla v_\varepsilon \, \mathrm{d}x \\ &= \int_{\partial\Omega} \gamma \frac{\partial w_\varepsilon}{\partial n} v_\varepsilon \, \mathrm{d}s - \int_{\partial\Omega} \gamma \frac{\partial v_\varepsilon}{\partial n} w_\varepsilon \, \mathrm{d}s,\end{aligned}$$

and so, using the boundary conditions satisfied by v_ε and w_ε, see (6.2) and (6.11):

$$\int_\Omega v_\varepsilon^2 \, \mathrm{d}x = \int_{\omega_\varepsilon} \gamma \frac{\partial w_\varepsilon}{\partial n} v_\varepsilon \, \mathrm{d}s.$$

Next, we estimate:

$$\begin{aligned}\int_\Omega v_\varepsilon^2 \, \mathrm{d}x &\leq ||w_\varepsilon||_{\mathcal{C}^1(\mathcal{O}_2 \cap \overline{\Omega})} \int_{\omega_\varepsilon} |v_\varepsilon| \, \mathrm{d}s \\ &\leq C ||g||_{\mathcal{C}^1(\overline{\Omega})} \mathrm{cap}^D_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)^{\frac{3}{2}},\end{aligned}$$

where we have used the $\mathcal{C}^1$ estimate (6.12) about w_ε and the intermediate result (6.10) from the proof of (ii) in passing from the first line to the second one. This terminates the proof of the desired estimate (6.7).

□

6.2. **Proof of Theorem 3.4: asymptotic expansion of** r_ε

The goal of this section is to prove the general asymptotic formula (3.5) for the perturbed voltage potential u_ε by the replacement of the homogeneous Dirichlet boundary condition by a Robin boundary condition with arbitrary parameter k_ε on a subset $\omega_\varepsilon \subset \Gamma_D$ of arbitrary shape.

Proof of Theorem 3.4. We follow again the approach of [17], and we first establish a representation formula for the error $r_\varepsilon = u_\varepsilon - u_0$ between the perturbed and background potentials, solution to (6.1) at a fixed point $x \in \Omega$. It follows from the definition (2.2) of the Green's function $G(x,y)$ and two successive integration by parts that:

$$\begin{aligned} r_\varepsilon(x) &= -\int_\Omega \mathrm{div}_y(\gamma(y)\nabla_y G(x,y)) r_\varepsilon(y)\,\mathrm{d}y \\ &= \int_\Omega \gamma(y)\nabla_y G(x,y)\cdot\nabla r_\varepsilon(y)\,\mathrm{d}y \;-\; \int_{\partial\Omega}\gamma(y)\frac{\partial G}{\partial n_y}(x,y) r_\varepsilon(y)\,\mathrm{d}s(y) \\ &= -\int_{\partial\Omega}\gamma(y)\frac{\partial G}{\partial n_y}(x,y) r_\varepsilon(y)\,\mathrm{d}s(y) \;+\; \int_{\partial\Omega}\gamma(y)\frac{\partial r_\varepsilon}{\partial n}(y) G(x,y)\,\mathrm{d}s(y). \end{aligned}$$

In view of the boundary conditions satisfied by $G(x,\cdot)$ and r_ε, the above equation reduces to

$$r_\varepsilon(x) = -\int_{\omega_\varepsilon}\gamma(y)\frac{\partial G}{\partial n_y}(x,y) r_\varepsilon(y)\,\mathrm{d}s(y). \tag{6.13}$$

Observing that the function $y \mapsto \gamma(y)\frac{\partial G}{\partial n_y}(x,y)$ is smooth on ω_ε, we introduce a smooth function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ with compact support, which coincides with $\gamma(\cdot)G(x,\cdot)$ on K_D and vanishes on a neighborhood of Γ_N that does not intersection K_D.

We now insert the Robin-Dirichlet equilibrium distribution $\xi_\varepsilon = \xi(\omega_\varepsilon, k_\varepsilon)$ defined in (2.15) into (6.13) via its Robin boundary condition on ω_ε. An elementary calculation based on repeated applications of the Green's formula then yields:

$$\begin{aligned} r_\varepsilon(x) &= -\int_{\omega_\varepsilon}\phi(y) r_\varepsilon(y)\,\mathrm{d}s(y) \\ &= -\int_{\omega_\varepsilon}\gamma\frac{\partial\xi_\varepsilon}{\partial n}\phi r_\varepsilon\,\mathrm{d}s - k_\varepsilon\int_{\omega_\varepsilon}\xi_\varepsilon\phi r_\varepsilon\,\mathrm{d}s \\ &= -\int_{\partial\Omega}\gamma\frac{\partial\xi_\varepsilon}{\partial n}\phi r_\varepsilon\,\mathrm{d}s - k_\varepsilon\int_{\omega_\varepsilon}\xi_\varepsilon\phi r_\varepsilon\,\mathrm{d}s \\ &= -\int_\Omega \gamma\nabla\xi_\varepsilon\cdot\nabla(\phi r_\varepsilon)\,\mathrm{d}x - k_\varepsilon\int_{\omega_\varepsilon}\xi_\varepsilon\phi r_\varepsilon\,\mathrm{d}s \\ &= -\int_\Omega \gamma\nabla(\phi\xi_\varepsilon)\cdot\nabla r_\varepsilon\,\mathrm{d}x - k_\varepsilon\int_{\omega_\varepsilon}\xi_\varepsilon\phi r_\varepsilon\,\mathrm{d}s + R_\varepsilon \\ &= -\int_{\omega_\varepsilon}\gamma\frac{\partial r_\varepsilon}{\partial n}\phi\xi_\varepsilon\,\mathrm{d}s - k_\varepsilon\int_{\omega_\varepsilon}\xi_\varepsilon\phi r_\varepsilon\,\mathrm{d}s + R_\varepsilon, \end{aligned}$$

and so:

$$r_\varepsilon(x) = \int_{\omega_\varepsilon}\gamma\frac{\partial u_0}{\partial n}\phi\xi_\varepsilon\,\mathrm{d}s + R_\varepsilon. \tag{6.14}$$

Here we have used the fact that r_ε and ξ_ε are solutions to (6.1) and (2.15) respectively, and we have set:

$$R_\varepsilon = -\int_\Omega(\gamma\nabla\xi_\varepsilon\cdot\nabla\phi)\, r_\varepsilon\,\mathrm{d}x + \int_\Omega(\gamma\nabla r_\varepsilon\cdot\nabla\phi)\,\xi_\varepsilon\,\mathrm{d}x.$$

Now applying the "classical" H^1 and improved L^2 estimates about r_ε and ξ_ε contained in Lemma 6.1 to control the remainder R_ε, we obtain:

$$\begin{aligned} |R_\varepsilon| &\leq ||\phi||_{\mathcal{C}^1(\mathbb{R}^d)} \Big(||\nabla \xi_\varepsilon||_{L^2(\Omega)^d} ||r_\varepsilon||_{L^2(\Omega)} + ||\nabla r_\varepsilon||_{L^2(\Omega)^d} ||\xi_\varepsilon||_{L^2(\Omega)} \Big) \\ &\leq C \, ||\phi||_{\mathcal{C}^1(\mathbb{R}^d)} \operatorname{cap}^D_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)^{\frac{5}{4}}. \end{aligned} \tag{6.15}$$

On a different note, as another consequence of Lemma 6.1, the following estimate holds true: for any $\psi \in \mathcal{C}^1(\Gamma_D)$

$$\forall \psi \in \mathcal{C}(\partial\Omega), \quad \frac{1}{\operatorname{cap}^D_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)} \int_{\omega_\varepsilon} \xi_\varepsilon \psi \, \mathrm{d}s \ \leq \ C ||\psi||_{\mathcal{C}(\partial\Omega)}.$$

Invoking the Banach-Alaoglu Theorem, there exists a subsequence of the ε (which is not relabelled for simplicity) and a non negative Radon measure μ supported in K_D such that for any $\phi \in \mathcal{C}^1_0(\Gamma_D)$,

$$\text{For any function} \quad \phi \in \mathcal{C}^1(\overline{\Gamma_D}), \quad \frac{1}{\operatorname{cap}^D_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)} \int_{\omega_\varepsilon} \xi_\varepsilon \phi \ \xrightarrow{\varepsilon \to 0} \ \langle \mu, \phi \rangle \, .$$

Combining (6.14) with (6.15), we therefore obtain:

$$r_\varepsilon(x) = \operatorname{cap}^D_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon) \left\langle \mu, \gamma \frac{\partial u_0}{\partial n} \gamma \frac{\partial G}{\partial n_y}(x, \cdot) \right\rangle \ + \ \mathrm{o}\left(\operatorname{cap}^D_{\text{Rob}}(\omega_\varepsilon, k_\varepsilon)\right),$$

which terminates the proof.

□

6.3. Proof of Theorem 3.5

We now turn to analyze the behavior of the Robin-Dirichlet capacity $\operatorname{cap}^D_{\text{Rob}}(\omega, k)$ depending on the value of the admittance parameter k. In particular, in the regime where $k \to 0$, we compare this quantity with that $e(\omega)$ measuring the smallness of the support ω of a replacement of the homogeneous Dirichlet condition $u = 0$ with a Neumann condition, in line with the intuition that the Robin condition $\gamma \frac{\partial u}{\partial n} + ku = 0$ formally reduces to a change with the Neumann condition $\gamma \frac{\partial u}{\partial n} = 0$ in this regime.

Proof of (i): Recalling the definitions (2.11) and (2.15) of the equilibrium distributions $\xi^0 = \xi^0(\omega)$ and $\xi = \xi(\omega, k)$ and that (2.16) of the Robin-Dirichlet capacity, we see that:

$$\begin{aligned} \operatorname{cap}^D_{\text{Rob}}(\omega, k) &= \int_\omega \xi \, \mathrm{d}s \\ &= \int_\omega \gamma \frac{\partial \xi^0}{\partial n} \xi \, \mathrm{d}s \\ &= \int_\Omega \gamma \nabla \xi^0 \cdot \nabla \xi \, \mathrm{d}x \\ &\leq ||\nabla \xi^0||_{L^2(\Omega)^d} ||\nabla \xi||_{L^2(\Omega)^d} \end{aligned}$$

and the desired estimate (3.6) follows from Proposition 2.1 and Lemma 6.1.

Proof of (ii): Let $\omega \Subset \Gamma_D$ and $k > 0$ be given. A simple application of the Cauchy-Schwarz inequality reveals that:

$$\operatorname{cap}^D_{\text{Rob}}(\omega, k) \ = \ \int_\omega \xi \, \mathrm{d}s \ \leq \ |\omega|^{1/2} \, ||\xi||_{L^2(\omega)}.$$

On a different note, the very definition (2.16) of the Robin-Dirichlet capacity shows that the estimate

$$k ||\xi||^2_{L^2(\omega)} \ \leq \ \operatorname{cap}^D_{\text{Rob}}(\omega, k).$$

Combining both inequalities and rearranging, we arrive at:

$$\operatorname{cap}^D_{\text{Rob}}(\omega, k) \leq \frac{1}{k} |\omega|,$$

as desired.

Proof of (iii): Let ω_ε be a sequence of subsets of the fixed compact $K_D \subset \Gamma_D$, and let k_ε be a sequence of positive real numbers such that $k_\varepsilon \to 0$ as $\varepsilon \to 0$. Let $\xi^0_\varepsilon = \xi^0(\omega_\varepsilon, k_\varepsilon)$ and $\xi_\varepsilon = \xi(\omega_\varepsilon, k_\varepsilon)$ be the equilibrium distributions characterized by (2.11) and (2.15), respectively. Then, we calculate:

$$\begin{aligned} e(\omega_\varepsilon) &\leq C \int_\Omega \gamma |\nabla \xi^0_\varepsilon|^2 \,\mathrm{d}s \\ &= C \int_{\omega_\varepsilon} \xi^0_\varepsilon \,\mathrm{d}s \\ &= C \int_{\omega_\varepsilon} \Big(\gamma \frac{\partial \xi_\varepsilon}{\partial n} + k_\varepsilon \xi_\varepsilon \Big) \xi^0_\varepsilon \,\mathrm{d}s \\ &= C \Big(\int_{\partial\Omega} \gamma \frac{\partial \xi_\varepsilon}{\partial n} \xi^0_\varepsilon \,\mathrm{d}s + \int_{\omega_\varepsilon} k_\varepsilon \xi_\varepsilon \xi^0_\varepsilon \,\mathrm{d}s \Big). \end{aligned}$$

Now using the Green's formula and the Cauchy-Schwarz inequality, we arrive at:

$$\begin{aligned} e(\omega_\varepsilon) &\leq \Big(\int_\Omega \gamma |\nabla \xi^0_\varepsilon|^2 \,\mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} |\xi^0_\varepsilon|^2 \,\mathrm{d}s \Big)^{1/2} \Big(\int_\Omega \gamma |\nabla \xi_\varepsilon|^2 \,\mathrm{d}x + k_\varepsilon \int_{\omega_\varepsilon} |\xi_\varepsilon|^2 \,\mathrm{d}s \Big)^{1/2} \\ &\leq C\,(1 + k_\varepsilon)^{1/2}\, ||\xi^0_\varepsilon||_{H^1(\Omega)} \mathrm{cap}^D_{\mathrm{Rob}}(\omega_\varepsilon, k_\varepsilon)^{1/2}, \end{aligned}$$

where we have used the continuity of the trace on $H^1_{\Gamma_D}$ to pass form the first line to the second one. Invoking again Proposition 2.1 together with (3.6), we obtain the desired estimate.

6.4. **Monotonicity of the Robin-Dirichlet capacity: proof of Proposition 3.2**

Proof of (i): Let $\xi_1 := \xi(\omega, k_1)$ and $\xi_2 := \xi(\omega, k_2)$ be the versions of the equilibrium distribution in (2.15) associated to the Robin conditions parametrized by k_1 and k_2, respectively. The difference $r := \xi_2 - \xi_1$ between both functions is the unique $H^1(\Omega)$ solution to the following boundary value problem:

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\gamma \nabla r) &=& 0 & \text{in } \Omega, \\ r &=& 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial r}{\partial n} &=& 0 & \text{on } \Gamma_N \setminus \overline{\omega}, \\ \gamma \frac{\partial r}{\partial n} + k_2 r &=& (k_1 - k_2)\xi_1 & \text{on } \omega. \end{array} \right. \tag{6.16}$$

The ordering $k_1 \leq k_2$ and the non negativity of ξ_1 imply that the term $(k_1 - k_2)\chi_1$ at the right-hand side of the Robin boundary condition in (6.16) is non positive. Invoking the pointwise bounds (6.5), we obtain:

$$r = \xi_2 - \xi_1 \;\leq\; 0 \text{ on } \overline{\Omega}.$$

We now readily infer from the definition (2.16) of the Robin-Dirichlet capacity that:

$$\mathrm{cap}^D_{\mathrm{Rob}}(\omega, k_2) - \mathrm{cap}^D_{\mathrm{Rob}}(\omega, k_1) \;=\; \int_\omega (\xi_2 - \xi_1) \,\mathrm{d}s \;\leq\; 0,$$

which is the expected conclusion.

Proof of (ii): Let now k be a positive real number and let $\omega_1 \subset \omega_2$ be two subsets of the Dirichlet region Γ_D. We set $\xi_1 := \xi(\omega_1, k)$ and $\xi_2 := \xi(\omega_2, k)$. An elementary calculation shows that the difference $r := \xi_2 - \xi_1$ between both functions is the unique $H^1(\Omega)$ solution to the following boundary value problem:

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\gamma \nabla r) &=& 0 & \text{in } \Omega, \\ \gamma \frac{\partial r}{\partial n} &=& 0 & \text{on } \Gamma_N, \\ r &=& 0 & \text{on } \Gamma_D \setminus \overline{\omega_2}, \\ r &=& \xi_2 & \text{on } \omega_2 \setminus \overline{\omega_1}, \\ \gamma \frac{\partial r}{\partial n} + k r &=& 0 & \text{on } \omega_1. \end{array} \right. \tag{6.17}$$

Since $\xi_2 \geq 0$ on $\overline{\Omega}$, a straightforward adaptation of the proof of the pointwise bounds (4.4) shows that:

$$r = \xi_2 - \xi_1 \;\geq\; 0 \text{ on } \overline{\Omega}. \tag{6.18}$$

Let us eventually calculate:

$$\begin{aligned}\mathrm{cap}^D_{\mathrm{Rob}}(\omega_2,k)-\mathrm{cap}^D_{\mathrm{Rob}}(\omega_1,k) &= \int_{\omega_2}\xi_2\,\mathrm{d}s-\int_{\omega_1}\xi_1\,\mathrm{d}s\\ &= \int_{\omega_2\setminus\overline{\omega_1}}\xi_2\,\mathrm{d}s+\int_{\omega_1}(\xi_2-\xi_1)\,\mathrm{d}s.\end{aligned}$$

The first term in the above right-hand side is non negative since $\xi_2\geq 0$, while the second term in there is also non negative because of (6.18). This terminates the proof of (ii).

7. Intrinsic definitions of $\mathrm{cap}^N_{\mathrm{Rob}}(\omega,k)$ and $\mathrm{cap}^D_{\mathrm{Rob}}(\omega,k)$

In this section, we introduce quantities that are equivalent to the capacities $\mathrm{cap}^D_{\mathrm{Rob}}(\omega,k)$ and $\mathrm{cap}^N_{\mathrm{Rob}}(\omega,k)$, but that are perhaps more natural, as they do not depend on the conductivity γ, the domain Ω or the regions Γ_D and Γ_N.

As in the previous sections, Ω denotes a smooth bounded open subset of $\mathbb{R}^d$. Let ω be a relatively open subset of $\partial\Omega$ and let $n(x)$ be the unit normal vector to $\partial\Omega$ at $x\in\omega$, pointing outward Ω. For a quantity α which is smooth from either side of ω but possibly discontinuous across ω, we denote by

$$\forall x\in\partial\Omega,\quad \alpha^{\pm}(x):=\lim_{t\to 0^+}\alpha(x\pm tn(x)),$$

the one-sided limits of α at x.

We introduce two additional auxiliary functions, $z,w\in H^1(\mathbb{R}^d\setminus\overline{\omega})$, defined via the following boundary-value problems:

$$\left\{\begin{array}{rcll}-\Delta z+z&=&0&\text{in }\mathbb{R}^d\setminus\overline{\omega},\\ \frac{\partial z^+}{\partial n}-kz^+&=&1&\text{on }\omega,\\ \frac{\partial z^-}{\partial n}+kz^-&=&1&\text{on }\omega.\end{array}\right.\qquad \left\{\begin{array}{rcll}-\Delta w+w&=&0&\text{in }\mathbb{R}^d\setminus\overline{\omega},\\ \frac{\partial w^+}{\partial n}-kw^+&=&-k&\text{on }\omega,\\ \frac{\partial w^-}{\partial n}+kw^-&=&k&\text{on }\omega.\end{array}\right. \tag{7.1}$$

The associated variational formulations read:

$$\text{For all } v\in H^1(\mathbb{R}^d\setminus\overline{\omega}),\quad \int_{\mathbb{R}^d\setminus\overline{\omega}}\nabla w\cdot\nabla v\,\mathrm{d}x+\int_{\mathbb{R}^d}wv\,\mathrm{d}x\;+\;k\int_\omega(w^-v^-+w^+v^+)\,\mathrm{d}s=k\int_\omega(v^-+v^+)\,\mathrm{d}s, \tag{7.2}$$

and

$$\text{For all } v\in H^1(\mathbb{R}^d\setminus\overline{\omega}),\quad \int_{\mathbb{R}^d\setminus\overline{\omega}}\nabla z\cdot\nabla v\,\mathrm{d}x+\int_{\mathbb{R}^d}zv\,\mathrm{d}x\;+\;k\int_\omega(z^-v^-+z^+v^+)\,\mathrm{d}s=\int_\omega(v^--v^+)\,\mathrm{d}s.$$

Their well-posedness is immediately ensured by the Lax Milgram theorem.

Let now ω be an oriented Lipschitz open surface in $\mathbb{R}^d$, and let $k>0$. We define the intrinsic Robin-Neumann capacity $C^N(\omega,k)$ of ω with parameter k by the formula:

$$C^N(\omega,k)=-\int_\omega\left[\frac{\partial w}{\partial n}\right]\,\mathrm{d}s. \tag{7.3}$$

Using the Robin boundary conditions featured in the problem (7.1), this quantity rewrites:

$$C^N(\omega,k)=k\int_\omega\Big((1-w^-)+(1-w^+)\Big)\,\mathrm{d}s.$$

Alternatively, invoking the variational form (7.2) for w, we obtain yet another expression of $C^N(\omega,k)$:

(7.4)

$$\begin{aligned}C^N(\omega,k) &= \int_{\mathbb{R}^d\setminus\overline{\omega}}|\nabla w|^2\,\mathrm{d}x+\int_{\mathbb{R}^d}w^2\,\mathrm{d}x\;+\;k\int_\omega\Big((w^-)^2+(w^+)^2\Big)\,\mathrm{d}s\\ &\qquad -k\int_\omega\Big(w^-+w^+\Big)\,\mathrm{d}s\;+\;k\int_\omega\Big((1-w^-)+(1-w^+)\Big)\,\mathrm{d}s\\ &= \int_{\mathbb{R}^d\setminus\overline{\omega}}|\nabla w|^2\,\mathrm{d}x+\int_{\mathbb{R}^d}w^2\,\mathrm{d}x\;+\;k\int_\omega\Big((1-w^-)^2+(1-w^+)^2\Big)\,\mathrm{d}s.\end{aligned}$$

In a similar spirit, we define the intrinsic Robin-Dirichlet capacity $C^D(\omega,k)$ of ω with parameter k as:

$$C^D(\omega,k) \;=\; \int_\omega (z^- - z^+)\,\mathrm{d}s, \tag{7.5}$$

which alternatively equals:

$$C^D(\omega,k) \;=\; \int_{\mathbb{R}^d\setminus\overline{\omega}} |\nabla z|^2\,\mathrm{d}x + \int_{\mathbb{R}^d} z^2\,\mathrm{d}x \;+\; k\int_\omega \Big((z^-)^2+(z^+)^2\Big)\,\mathrm{d}s. \tag{7.6}$$

Our main result concerning the equivalence between the quantities $\mathrm{cap}^D_{\mathrm{Rob}}(\omega,k)$ and the intrinsic version $C^D(\omega,k)$ is the following.

Proposition 7.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^d$, whose boundary $\partial\Omega$ is the disjoint reunion of a Neumann and a Dirichlet region Γ_N, Γ_D, respectively. Let $\omega\subset\Gamma_D$ be a Lipschitz open subset which is well-inside Γ_D, in the sense that (2.5) holds true. Let also $k>0$ be a given positive real number. Then there exists a constant C which depends on Ω, Γ_N, Γ_D and the compact set K_D, but not on ω and k such that:*

$$C^{-1}\,C^D(\omega,k) \;\leq\; \mathrm{cap}^D_{\mathrm{Rob}}(\omega,k) \;\leq\; C\,C^D(\omega,k), \tag{7.7}$$

Proof. Throughout the proof, we denote by $C>0$ a constant, that may change from one line to the other and that depends on Ω, Γ_N, Γ_D, the bounds γ^- and γ^+ on the conductivity γ and the compact set K_D, but that is in any case independent of ω and k.

We first prove the left-hand inequality in (7.7). Using the definition (7.5) of $C^D(\omega,k)$ and bringing into play the Robin-Dirichlet equilibrium distribution $\xi\equiv\xi(\omega,k)$ defined by (2.15) via its Robin boundary condition, we obtain:

$$\begin{aligned} C^D(\omega,k) &= \int_\omega (z^- - z^+)\,\mathrm{d}s \\ &= \int_\omega \left(\gamma\frac{\partial\xi}{\partial n}+k\xi\right)(z^- - z^+)\,\mathrm{d}s \\ &= \int_{\partial\Omega} \left(\gamma\frac{\partial\xi}{\partial n}+k\xi\right)(z^- - z^+)\,\mathrm{d}s, \end{aligned}$$

where the final line follows from the continuity of z across $\partial\Omega\setminus\overline{\omega}$.

Let us now introduce the extension operator $E_{\mathrm{int}}\;:H^1(\mathbb{R}^d\setminus\overline{\Omega})\to H^1(\Omega)$ which to $v\in H^1(\mathbb{R}^d\setminus\overline{\Omega})$ associates the unique $H^1(\Omega)$ solution $E_{\mathrm{int}}v$ to the boundary-value problem:

$$\left\{\begin{array}{rcll} -\mathrm{div}(\gamma\nabla E_{\mathrm{int}}v) &=& 0 & \text{in } \Omega,\\ E_{\mathrm{int}}v &=& v & \text{on } \partial\Omega. \end{array}\right. \tag{7.8}$$

It is immediate to see that there exists a constant $C>0$ such that:

$$\forall v\in H^1(\mathbb{R}^d\setminus\overline{\Omega}),\quad ||E_{\mathrm{int}}v||_{H^1(\Omega)} \;\leq\; C||v||_{H^1(\mathbb{R}^d\setminus\overline{\Omega})}.$$

We thus obtain, with the help of the Green's formula:

$$\begin{aligned} C^D(\omega,k) &= \int_{\partial\Omega}\gamma\frac{\partial\xi}{\partial n}\Big(z^- - E_{\mathrm{int}}z^+\Big)\,\mathrm{d}s + \int_\omega k\xi(z^- - z^+)\,\mathrm{d}s \\ &= \int_\Omega \gamma\nabla\xi\cdot\nabla(z^- - E_{\mathrm{int}}z^+)\,\mathrm{d}x \;+\; \int_\omega k\xi(z^- - z^+)\,\mathrm{d}s. \end{aligned}$$

Now using the Cauchy-Schwarz inequality, we arrive at:

$$\begin{aligned} C^D(\omega,k) &\leq ||\nabla\xi||_{L^2(\Omega)^d}||\nabla(z^- - E_{\mathrm{int}}z^+)||_{L^2(\Omega)^d} \;+\; k||\xi||_{L^2(\omega)}\,||z^- - z^+||_{L^2(\omega)} \\ &\leq \left(\int_\Omega \gamma|\nabla\xi|^2\,\mathrm{d}x + k\int_\omega \xi^2\,\mathrm{d}s\right)^{1/2}\left(\int_\Omega |\nabla(z^- - E_{\mathrm{int}}z^+)|^2\,\mathrm{d}x + k\int_\omega (z^- - z^+)^2\,\mathrm{d}s\right)^{1/2} \\ &\leq C\,\mathrm{cap}^D_{\mathrm{Rob}}(\omega,k)^{1/2}\left(\int_{\mathbb{R}^d\setminus\overline{\omega}} |\nabla z|^2\,\mathrm{d}x + k\int_\omega \Big((z^+)^2+(z^-)^2\Big)\,\mathrm{d}s\right)^{1/2}, \end{aligned}$$

where we have used the energy estimate (6.6) to obtain the final line. Now using the alternative expression (7.6) for $C^D(\omega,k)$, we arrive at:

$$C^D(\omega,k) \;\leq\; C\text{cap}^D_{\text{Rob}}(\omega,k),$$

as desired.

We now turn to the right-hand inequality in (7.7), recalling the definition (2.15) of the Robin-Dirichlet equilibrium potential ξ, and that (2.16) of the corresponding capacity. Let us introduce a smooth cut-off function $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ which equals 1 on K_D and 0 on Γ_N. We calculate:

$$\begin{aligned}
\int_\omega \xi \,\mathrm{d}s &= \int_\omega \xi \left(\frac{\partial z^-}{\partial n} + kz^-\right) \mathrm{d}s \\
&= \int_{\partial\Omega} \phi\xi \left(\frac{\partial z^-}{\partial n} + kz^-\right) \mathrm{d}s \\
&= \int_\Omega \phi\xi\, \Delta z \,\mathrm{d}x \;+\; \int_\Omega \nabla(\phi\xi)\cdot\nabla z \,\mathrm{d}x \;+\; k\int_{\partial\Omega} \phi\xi z^- \,\mathrm{d}s \\
&= \int_\Omega \Big(\nabla(\phi\xi)\cdot\nabla z + \phi\xi z\Big)\,\mathrm{d}x \;+\; k\int_\omega \phi\xi z^- \,\mathrm{d}s,
\end{aligned}$$

where the final line follows from integration by parts and the problem (7.1) satisfied by z.

Combining now the Cauchy-Schwarz inequality and the Poincaré's inequality, we obtain:

$$\begin{aligned}
\int_\omega \xi \,\mathrm{d}s &\leq C\Big(||\xi||_{H^1(\Omega)}\,||z||_{H^1(\Omega)} \;+\; k||\xi||_{L^2(\omega)}\,||z^-||_{L^2(\omega)}\Big) \\
&\leq C\left(\int_\Omega \gamma|\nabla\xi|^2\,\mathrm{d}x \;+\; k\int_\omega \xi^2\,\mathrm{d}s\right)^{1/2} \left(||z||^2_{H^1(\mathbb{R}^d\setminus\overline{\omega})} \;+\; k\int_\omega \Big((z^+)^2 + (z^-)^2\Big)\,\mathrm{d}s\right)^{1/2}.
\end{aligned}$$

Now using again the formula (2.16) for $\text{cap}^D_{\text{Rob}}(\omega,k)$ as well as the alternative expression (7.6) for $C^D(\omega,k)$, we obtain:

$$\text{cap}^D_{\text{Rob}}(\omega,k) \;\leq\; CC^D(\omega,k),$$

which is the expected result. □

The corresponding result about the equivalence between Robin-Neumann capacities $\text{cap}^N_{\text{Rob}}(\omega,k)$ and $C^N(\omega,k)$ is the following.

Proposition 7.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^d$, whose boundary $\partial\Omega$ is the disjoint reunion of a Neumann and a Dirichlet region Γ_N, Γ_D, respectively. Let $\omega \subset \Gamma_D$ be a Lipschitz open subset which is well-inside Γ_D, in the sense that (2.3) holds true. Let also $k > 0$ be a given positive real number. Then there exists a constant C which depends on Ω, Γ_N, Γ_D, the conductivity γ and the compact set K_N, but not on ω and k such that:*

$$C^{-1}\,C^N(\omega,k) \;\leq\; \text{cap}^N_{\text{Rob}}(\omega,k) \;\leq\; C\,C^N(\omega,k), \tag{7.9}$$

Proof. The proof is quite similar to that of Proposition 7.1 and for brevity, we only provide a sketch. We henceforth denote by $\chi \equiv \chi(\omega,k)$ the Robin-Neumann equilibrium distribution, solution to (2.12). Again, throughout, $C > 0$ is a constant that may change from one line to the other and that depends on Ω, Γ_N, Γ_D, γ and K_N, but that is in any case independent of ω and k.

Let us first prove the left-hand inequality in (7.9). To achieve this, we insert the Robin boundary condition satisfied by χ on ω into the definition of $C^5\omega,k)$:

$$\begin{aligned}
C^N(\omega,k) &= k\int_\omega \Big((1-w^-) + (1-w^+)\Big)\,\mathrm{d}s \\
&= \int_\omega \left(\gamma\frac{\partial\chi}{\partial n} + k\chi\right)\Big((1-w^-) + (1-w^+)\Big)\,\mathrm{d}s \\
&= 2\int_\omega \gamma\frac{\partial\chi}{\partial n}\,\mathrm{d}s - \int_\omega \gamma\frac{\partial\chi}{\partial n}(w^+ + w^-)\,\mathrm{d}s + \int_\omega k\chi\Big((1-w^-) + (1-w^+)\Big)\,\mathrm{d}s \\
&= 2\int_\omega \gamma\frac{\partial\chi}{\partial n}\,\mathrm{d}s - \int_\omega \gamma\frac{\partial\chi}{\partial n}(w^+ + w^-)\,\mathrm{d}s - \int_\omega \left[\frac{\partial w}{\partial n}\right]\chi\,\mathrm{d}s.
\end{aligned}$$

Introducing a smooth cut-off function ϕ which equals 1 on the compact set K_N and 0 on Γ_D, and using the definition (2.14) of $\text{cap}^N_{\text{Rob}}(\omega,k)$, we obtain:

$$C^N(\omega,k) = 2\text{cap}^N_{\text{Rob}}(\omega,k) - \int_{\partial\Omega} \gamma \frac{\partial \chi}{\partial n}\phi(w^+ + w^-)\,\mathrm{d}s - \int_{\partial\Omega}\left[\frac{\partial w}{\partial n}\right]\chi\phi\,\mathrm{d}s. \tag{7.10}$$

Let us now recall the definition (7.11) of the extension operator $E_{\text{int}} : H^1(\mathbb{R}^d\setminus\overline{\Omega}) \to H^1(\Omega)$ of a function defined outside Ω to a function defined inside Ω. Likewise, introducing a fixed bounded domain $\mathcal{O}\subset\mathbb{R}^d$ such that $\Omega \Subset \mathcal{O}$, we define the operator $E_{\text{ext}} : H^1(\Omega) \to H^1(\mathbb{R}^d\setminus\overline{\Omega})$ which to $v\in H^1(\Omega)$ associates the unique solution $E_{\text{ext}}v \in H^1_0(\mathcal{O})$ to the boundary-value problem:

$$\left\{\begin{array}{rcll} -\text{div}(\gamma\nabla E_{\text{ext}}v) &=& 0 & \text{in } \mathcal{O}\setminus\overline{\Omega},\\ E_{\text{ext}}v &=& v & \text{on } \partial\Omega\\ E_{\text{ext}}v &=& 0 & \text{on } \partial\mathcal{O}.\end{array}\right. \tag{7.11}$$

Again, it is immediate to see that there exists a constant $C>0$ such that:

$$\forall v\in H^1(\Omega),\quad ||E_{\text{ext}}v||_{H^1(\mathbb{R}^d\setminus\overline{\Omega})} \leq C||v||_{H^1(\Omega)}.$$

We now rewrite (7.10) as:

$$C^N(\omega,k) = 2\text{cap}^N_{\text{Rob}}(\omega,k) - \int_{\partial\Omega}\gamma\frac{\partial\chi}{\partial n}\phi(E_{\text{int}}w^+ + w^-)\,\mathrm{d}s - \int_{\partial\Omega}\gamma\frac{\partial w^+}{\partial n}E_{\text{ext}}(\chi\phi)\,\mathrm{d}s + \int_{\partial\Omega}\gamma\frac{\partial w^-}{\partial n}\chi\phi\,\mathrm{d}s.$$

An application of the Green's formula now yields:

$$C^N(\omega,k) = 2\text{cap}^N_{\text{Rob}}(\omega,k) - \int_\Omega \gamma\nabla\chi\cdot\nabla(\phi(E_{\text{int}}w^+ + w^-))\,\mathrm{d}x + \int_{\mathbb{R}^d\setminus\overline{\Omega}}\nabla w\cdot\nabla(E_{\text{ext}}(\chi\phi))\,\mathrm{d}x + \int_\Omega \nabla w\cdot\nabla(\chi\phi)\,\mathrm{d}x.$$

Now applying the Cauchy-Schwarz inequality, we arrive at:

$$\begin{array}{rcl} C^N(\omega,k) &\leq& 2\text{cap}^N_{\text{Rob}}(\omega,k) + C||\chi||_{H^1(\Omega)}||w||_{H^1(\mathbb{R}^d\setminus\overline{\omega})}\\ &\leq& 2\text{cap}^N_{\text{Rob}}(\omega,k) + C\text{cap}^N_{\text{Rob}}(\omega,k)^{\frac12}C^N(\omega,k)^{\frac12},\end{array}$$

where the second line follows from (4.5) and (7.4). Solving this quadratic inequality for $C^N(\omega,k)^{1/2}$, we see that there exists a constant $C>0$ such that:

$$C^N(\omega,k) \leq C\text{cap}^N_{\text{Rob}}(\omega,k),$$

which is the desired result.

Let us now turn to the right-hand inequality in (7.9). Like in the previous argument, we start with the definition (2.14) of the Robin-Neumann capacity $\text{cap}^N_{\text{Rob}}(\omega,k)$, in which we bring into play the function w in (7.1) via its boundary condition on ω:

$$\begin{array}{rcl} \text{cap}^N_{\text{Rob}}(\omega,k) &=& k\displaystyle\int_\omega(1-\chi)\,\mathrm{d}s\\ &=& -\dfrac12\displaystyle\int_\omega\left[\frac{\partial w}{\partial n}\right](1-\chi)\,\mathrm{d}s + \frac12\int_\omega k(w^+ + w^-)(1-\chi)\,\mathrm{d}s\\ &=& \dfrac12 C^N(\omega,k) + \dfrac12\displaystyle\int_\omega\left[\frac{\partial w}{\partial n}\right]\chi\,\mathrm{d}s + \frac12\int_\omega\gamma\frac{\partial\chi}{\partial n}(w^+ + w^-)\,\mathrm{d}s,\end{array}$$

where we have used the definition (7.3) of $C^N(\omega,k)$ to obtain the last line. Now introducing a smooth cut-off function ϕ which equals 1 on Γ_N and 0 on Γ_D, and using again the extension operators $E_{\text{int}} : H^1(\mathbb{R}^d\setminus\overline{\Omega}) \to H^1(\Omega)$, $E_{\text{ext}} : H^1(\Omega) \to H^1(\mathbb{R}^d\setminus\overline{\Omega})$, we obtain:

$$\begin{array}{rcl} \text{cap}^N_{\text{Rob}}(\omega,k) &=& \dfrac12 C^N(\omega,k) + \dfrac12\displaystyle\int_{\partial\Omega}\left[\frac{\partial w}{\partial n}\right]\phi\chi\,\mathrm{d}s + \frac12\int_{\partial\Omega}\gamma\frac{\partial\chi}{\partial n}(w^+ + w^-)\phi\,\mathrm{d}s\\ &=& \dfrac12 C^N(\omega,k) + \dfrac12\displaystyle\int_{\partial\Omega}\frac{\partial w^+}{\partial n}\phi E_{\text{ext}}\chi\,\mathrm{d}s - \frac12\int_{\partial\Omega}\frac{\partial w^-}{\partial n}\phi\chi\,\mathrm{d}s + \frac12\int_{\partial\Omega}\gamma\frac{\partial\chi}{\partial n}(E_{\text{int}}w^+ + w^-)\phi\,\mathrm{d}s.\end{array}$$

Now applying the Green's formula and the Cauchy-Schwarz inequality, we obtain:

$$\text{cap}_{\text{Rob}}^N(\omega, k) \leq \frac{1}{2} C^N(\omega, k) + C ||w||_{H^1(\mathbb{R}^d \setminus \overline{\omega})} ||\chi||_{H^1(\Omega)},$$

and so, considering again (4.5) and (7.4):

$$\text{cap}_{\text{Rob}}^N(\omega, k) \leq \frac{1}{2} C^N(\omega, k) + C \text{cap}_{\text{Rob}}^N(\omega, k)^{\frac{1}{2}} C^N(\omega, k)^{\frac{1}{2}}.$$

Solving this quadratic inequality for $\text{cap}_{\text{Rob}}^N(\omega, k)^{1/2}$, we see that there exists a constant $C > 0$ such that:

$$\text{cap}_{\text{Rob}}^N(\omega, k) \leq C C^N(\omega, k),$$

as expected.

□

8. Conclusion and perspectives

In this article, we have investigated the asymptotic effect of a particular class of perturbations of the boundary conditions attached to the conductivity equation for the voltage potential: its Neumann or Dirichlet conditions are replaced with a Robin boundary condition with arbitrary admittance parameter k_ε on a subset ω_ε of the relevant region of the physical domain which vanishes at the limit $\varepsilon \to 0$. In both situations, we have derived a general asymptotic formula for the voltage potential u_ε resulting from such perturbations: the amplitude of the perturbation is quantified by a new quantity depending on ω_ε and k_ε – the Robin-Neumann and Robin-Dirichlet capacity, respectively. We have proved that, in the regimes where k_ε is either "very small" or "very large", these quantities are consistent with the asymptotic rates obtained in our previous work, where a Neumann boundary condition of the conductivity equation was replaced with a Dirichlet condition on a small subset of $\partial\Omega$, and vice-versa. An interesting perspective of this work consists in deriving explicit versions of these asymptotic expansions in the particular physical configuration where the vanishing support ω_ε of the Robin boundary condition is a curvilinear segment of length 2ε in 2d, or a surface disk of radius ε in 3d. Of particular interest are asymptotic formulas that would hold true uniformly with respect to the admittance parameter k_ε: these would allow to appraise how the asymptotic rates of convergence of the voltage potential to its background counterpart transition from one extreme behavior (of the order $k_\varepsilon|\omega_\varepsilon|$ when $k_\varepsilon \ll 1$ in the case where the perturbation affects the Neumann boundary) to the other (of the order $\text{cap}(\omega_\varepsilon)$ when $k_\varepsilon \gg 1$ in the same configuration).

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Appendix A. Control of the extension of a distribution in $H^{-s}(\omega), 0 < s < 1$

This appendix provides a proof of the fact (5.7) used during the proof of Theorem 3.3, whereby the function $\frac{\overline{\chi_\varepsilon^\infty}}{\partial n}$ lies in $\widetilde{H}^{-s}(\omega_1)$ with an estimate of its norm. Interestingly, the proof indicates how the estimate may depend on the size of ω. Although this degree of precision was not needed in the proof of Theorem 3.3, we believe that the result and its proof are interesting in their own right.

In this section, we assume that Ω is a smooth bounded open set in $\mathbb{R}^d, d = 2, 3$. Its boundary is made of two disjoint open Dirichlet and Neumann regions Γ_D and Γ_N with non-empty interiors:

$$\partial\Omega \;\; = \;\; \overline{\Gamma_D} \cup \overline{\Gamma_N}, \text{ where } \Gamma_D \cap \Gamma_N = \emptyset.$$

For simplicity, we assume that Γ_N contains the origin 0 and that it is completely flat in a fixed neighborhood of 0. The general situation follows from application of a smooth diffeomorphism to this flat configuration.

Lemma A.1. *Let $\omega \Subset \Gamma_N \subset \partial\Omega$ be either a segment of length $2r$ around 0 if $d = 2$, or a flat disk with center 0 and radius r if $d = 3$, such that*

$$\operatorname{dist}(\omega, \Gamma_D) \ \geq \ Cr, \tag{A.1}$$

for a constant $C > 0$ which does not depend on r. Let $0 < s < 1$, and let g be an element in $H^{-s}(\partial\Omega)$ such that $g = 0$ on $\Gamma_N \setminus \overline{\omega}$. Then g belongs to $\widetilde{H}^{-s}(\omega)$ and there exists a constant $C > 0$ independent of r such that

$$||g||_{\widetilde{H}^{-s}(\omega)} \ \leq \ \frac{C}{r^s} ||g||_{H^{-s}(\partial\Omega)}. \tag{A.2}$$

Proof. For simplicity of the exposition, we only provide the proof when the space dimension equals $d = 2$: $\omega \subset \Gamma_N$ is then the straight segment of length $2r$ around 0, while Γ_N contains the segment of length 2 around 0:

$$\omega = (-r, r) \times \{0\} \ \subset \ (-1, 1) \times \{0\} \ \subset \ \Gamma_N.$$

With a small abuse of notation, we identify a point $(x, 0) \in \omega$ with its horizontal coordinate $x \in (-r, r)$, and denote by C a positive constant that does not depend on r. Note that, for any Lipschitz subset $\sigma \subset \partial\Omega$, an element $g \in H^{-s}(\partial\Omega)$ naturally induces an element in $H^{-s}(\sigma)$, still denoted by g, with

$$\langle g, v \rangle_{H^{-s}(\sigma), \widetilde{H}^s(\sigma)} \quad = \quad \langle g, \widetilde{v} \rangle_{H^{-s}(\partial\Omega), H^s(\partial\Omega)},$$

where $\widetilde{v}$ denotes the extension by 0 of a function $v \in \widetilde{H}^s(\sigma)$. The argument proceeds in two steps.

Step 1: We construct a suitable extension operator from $H^s(\omega)$ to $\widetilde{H}^s(\Gamma_N)$.

To achieve this, let $\rho \ : \mathbb{R}^+ \to [0, 1]$ denote a smooth function such that

$$\rho(x) \quad = \quad \begin{cases} 1 & \text{if} \quad 0 \leq x \leq r/2, \\ 0 & \text{if} \quad x \geq r, \end{cases}$$

together with the estimate:

$$\forall x \geq 0, \quad |\rho'(x)| \ \leq \ \frac{4}{r}. \tag{A.3}$$

Let $0 < s < 1$. For any function $v \in \mathcal{C}^\infty(\overline{\omega})$, we define:

$$Rv(x) \quad = \quad \begin{cases} v(x) & \text{if} \quad 0 < x < r, \\ \rho(-x)v(-x) & \text{if} \quad x \in \Gamma_N, \quad -r < x < 0, \\ \rho_(x - r)v(2r - x) & \text{if} \quad x \in \Gamma_N, \quad r < x < 2r, \\ 0 & \text{otherwise,} \end{cases}$$

Note that Rv is supported in Γ_N, due to (A.1), see Fig. 1 for an illustration.

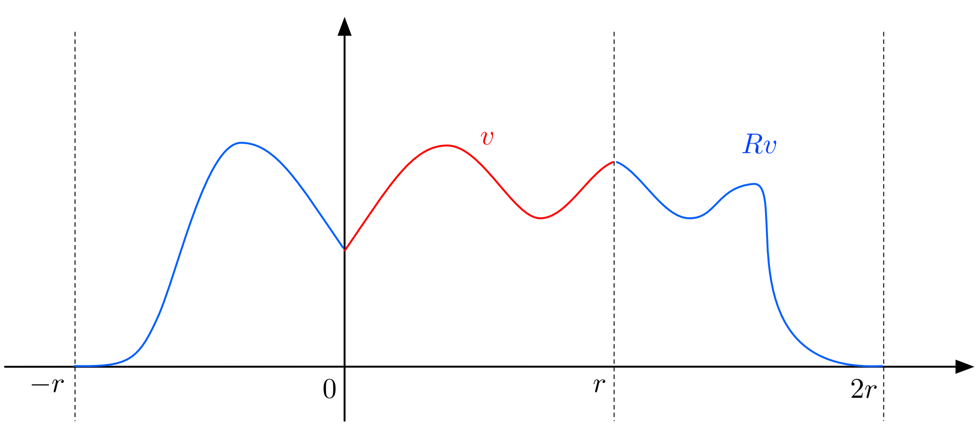

FIGURE 1. *Illustration of the extension operator R defined in Appendix A.*

Recalling the definitions of fractional Sobolev norms on a subset of the boundary of a domain from Section 2.2, for an arbitrary $v \in \mathcal{C}^\infty(\overline{\omega})$, we next estimate the norm $||Rv||_{\widetilde{H}^s(\Gamma_N)}$ in terms of $||v||_{H^s(\omega)}$:

$$||v||^2_{H^s(\omega)} = \int_\omega |v(x)|^2 \,\mathrm{d}s(x) + \int_\omega\int_\omega \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y),$$

and, in view of the definition of Rv,

$$\begin{aligned} ||Rv||^2_{\widetilde{H}^s(\Gamma_N)} &= \int_{\partial\Omega} Rv^2 \,\mathrm{d}s(x) + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|Rv(x)-Rv(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \\ &= \int_{\Gamma_N} Rv^2 \,\mathrm{d}s(x) + \int_{\Gamma_N}\int_{\Gamma_N} \frac{|Rv(x)-Rv(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y). \end{aligned} \tag{A.4}$$

Let us first estimate

$$\begin{aligned} \int_{\Gamma_N} Rv(x)^2\,\mathrm{d}s(x) &= \int_{-r}^0 Rv(x)^2\,\mathrm{d}s(x) + \int_0^r Rv(x)^2\,\mathrm{d}s(x) + \int_r^{2r} Rv(x)^2\,\mathrm{d}s(x) \\ &\leq 2\int_0^r \rho(x)^2 v(x)^2\,\mathrm{d}s(x) + \int_0^r v(x)^2\,\mathrm{d}s(x) \\ &\leq 3||v||^2_{L^2(\omega)}, \end{aligned} \tag{A.5}$$

where we have used the definition of $Rv(x)$ and the fact that $0 \leq \rho(x) \leq 1$.

Let us now turn to estimate the $\widetilde{H}^s(\Gamma_N)$ semi-norm of Rv; it holds:

$$\begin{aligned} \int_{\Gamma_N}\int_{\Gamma_N} \frac{|Rv(x)-Rv(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) &= \int_\omega\int_\omega \frac{|Rv(x)-Rv(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \\ &\quad + \int_{\Gamma_N\setminus\overline{\omega}}\int_{\Gamma_N\setminus\overline{\omega}} \frac{|Rv(x)-Rv(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \\ &\quad + 2\int_{\Gamma_N\setminus\overline{\omega}}\int_\omega \frac{|Rv(x)-Rv(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \\ &=: T_1 + T_2 + 2T_3, \end{aligned} \tag{A.6}$$

with obvious notations.

We first estimate the term T_1. By construction, $Rv(x) = v(x)$ for $x \in \omega$, and so:

$$T_1 = \int_\omega\int_\omega \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) = |v|^2_{H^s(\omega)}. \tag{A.7}$$

We next turn to T_2. According to our simplifying assumptions regarding the geometry of ω, we have:

$$\begin{aligned} T_2 = &\int_{y=-r}^0\int_{x=-r}^0 \frac{|\rho(-x)v(-x)-\rho(-y)v(-y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \\ &+ \int_{y=r}^{2r}\int_{x=r}^{2r} \frac{|\rho(x-2r)v(2r-x)-\rho(y-2r)v(2r-y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \\ &+ 2\int_{y=r}^{2r}\int_{x=-r}^0 \frac{|\rho(-x)v(-x)-\rho(y-2r)v(2r-y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y). \end{aligned}$$

We now denote by S_1, S_2 and S_3 the three integrals in the above right-hand side. The first term S_1 can be estimated as follows:

$$
\begin{aligned}
S_1 &\leq C \int_{y=-r}^{0}\int_{x=-r}^{0} \frac{\rho(-x)^2|v(-x)-v(-y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \\
&\qquad\qquad\qquad + C \int_{y=-r}^{0}\int_{x=-r}^{0} \frac{|\rho(-x)-\rho(-y)|^2 v(-y)^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \\
&\leq C \int_\omega\int_\omega \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \;+\; C||\rho'||^2_{L^\infty(\mathbb{R}_+)} \int_\omega\int_\omega |x-y|^{1-2s}|v(y)|^2\,\mathrm{d}x\mathrm{d}y \\
&\leq C \int_\omega\int_\omega \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \;+\; C||\rho'||^2_{L^\infty(\mathbb{R}_+)} \int_\omega\int_\omega (2r)^{1-2s}|v(y)|^2\,\mathrm{d}s(x)\mathrm{d}s(y) \\
&\leq C \int_\omega\int_\omega \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \;+\; C\,r^{-2s} \int_\omega v(y)^2\,\mathrm{d}s(y) \\
&\leq C\,r^{-2s}\,||v||^2_{H^s(\omega)},
\end{aligned}
\tag{A.8}
$$

where the penultimate line follows from (A.3). The same argument applies to S_2 and shows that:

$$
S_2 \leq C r^{-2s} ||v||^2_{H^s(\omega)}. \tag{A.9}
$$

As far as S_3 is concerned, we have:

$$
\begin{aligned}
S_3 &\leq \frac{C}{r^{1+2s}} \int_{y=r}^{2r}\int_{x=-r}^{0} \left(|\rho(-x)v(-x)-\rho(y-2r)v(2r-y)|^2\right)\mathrm{d}s(x)\mathrm{d}s(y) \\
&\leq \frac{C}{r^{2s}} \int_{x=0}^{r} |v(x)|^2\,\mathrm{d}s(x)\mathrm{d}s(y) \\
&= C\,r^{-2s}\,||v||^2_{L^2(\omega)}.
\end{aligned}
\tag{A.10}
$$

Combining the estimates (A.8) to (A.10), we obtain:

$$
T_2 \;\leq\; C\,r^{-2s}\,||v||^2_{H^s(\omega)}. \tag{A.11}
$$

Finally, let us decompose the last term T_3 in (A.6) as follows:

$$
\begin{aligned}
T_3 &= \int_{\Gamma_N\setminus\overline{\omega}}\int_\omega \frac{|Rv(x)-Rv(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y), \\
&=: R_1 + R_2,
\end{aligned}
\tag{A.12}
$$

where

$$
R_1 = \int_{y=-r}^{0}\int_0^r \frac{|v(x)-Rv(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \text{ and } R_2 = \int_{y=r}^{2r}\int_0^r \frac{|v(x)-Rv(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y).
$$

In view of the definition of ρ, and since $|x-y|>r/2$ when $-r<y<-r/2<0<x$, we see that

$$
\begin{aligned}
R_1 &= \int_{y=-r}^{-r/2}\int_0^r \frac{|v(x)-Rv(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \;+\; \int_{y=-r/2}^{0}\int_0^r \frac{|v(x)-\rho(-y)v(-y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \\
&\leq C \int_{y=-r}^{-r/2}\int_0^r \frac{|v(x)-Rv(y)|^2}{r^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \;+\; \int_{y=-r/2}^{0}\int_0^r \frac{|v(x)-v(-y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y),
\end{aligned}
$$

where we have used the definition of ρ to obtain the last line. Rearranging this expression, we obtain:

$$
\begin{aligned}
R_1 &\leq C \int_{y=-r}^{-r/2}\int_0^r \frac{1}{r^{1+2s}}\left(|v(x)|^2+|v(-y)|^2\right)\mathrm{d}s(x)\mathrm{d}s(y) \;+\; \int_{y=-r/2}^{0}\int_0^r \frac{|v(x)-v(-y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y) \\
&\leq C r^{-2s}\,||v||^2_{L^2(\omega)} + \int_\omega\int_\omega \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}s(x)\mathrm{d}s(y),
\end{aligned}
$$

where we have used the fact that $|x+y|<|x-y|$ when $-r/2<y<0<x$ to pass from the first line to the second one.

The second term R_2 in the expression (A.12) of T_3 can be estimated in the same manner, which finally yields:

$$T_3 \;\leq\; C\, r^{-2s}\, ||v||^2_{H^s(\omega)}. \tag{A.13}$$

Gathering (A.5)-(A.13) and noticing that $2r = |\omega|$, we have eventually proved that:

$$\forall v \in \mathcal{C}^\infty(\overline{\omega}), \quad ||Rv||_{\widetilde{H}^s(\Gamma_N)} \;\leq\; C\, |\omega|^{-s}\, ||v||_{H^s(\omega)}. \tag{A.14}$$

By density of $\mathcal{C}^\infty(\overline{\omega})$ in $H^s(\omega)$, this inequality allows to extend R as a bounded linear mapping from $H^s(\omega)$ to $\widetilde{H}^s(\Gamma_N)$, which still satisfies the estimate (A.14).

Step 2: We estimate the $\widetilde{H}^{-s}(\omega)$ norm of g by duality.

Let $0 < s < 1$, and let g denote a function in $H^{-s}(\partial\Omega)$ such that

$$g = 0 \quad \text{on } \Gamma_N \setminus \overline{\omega}.$$

Furthermore, let $v \neq 0$ be a given function in $H^s(\omega)$. Using the operator R constructed in step 1, we may extend v to a function Rv defined on the whole of $\partial\Omega$ such that $Rv = 0$ on Γ_D. It follows that:

$$\begin{aligned} \langle g, v \rangle_{\widetilde{H}^{-s}(\omega), H^s(\omega)} &= \langle g, Rv \rangle_{H^{-s}(\Gamma_N), \widetilde{H}^s(\Gamma_N)} \\ &= ||Rv||_{\widetilde{H}^s(\Gamma_N)} \left\langle g, \frac{Rv}{||Rv||_{\widetilde{H}^s(\Gamma_N)}} \right\rangle_{H^{-s}(\Gamma_N), \widetilde{H}^s(\Gamma_N)} \\ &\leq ||R||_{\mathcal{L}(H^s(\omega), \widetilde{H}^s(\Gamma_N))}\, ||v||_{H^s(\omega)}\, \langle g, w \rangle_{H^{-s}(\Gamma_N), \widetilde{H}^s(\Gamma_N)}, \end{aligned}$$

where $w := \dfrac{Rv}{||Rv||_{\widetilde{H}^s(\Gamma_N)}}$. Taking the supremum with respect to $w \in \widetilde{H}^s(\Gamma_N)$, then with respect to $v \in H^s(\omega)$, we obtain

$$\begin{aligned} \sup_{\substack{v \in H^s(\omega), \\ ||v||_{H^s(\omega)} = 1}} \langle g, v \rangle_{H^{-s}(\partial\Omega), H^s(\partial\Omega)} &\leq ||R||_{\mathcal{L}(H^s(\omega), \widetilde{H}^s(\Gamma_N))} \sup_{\substack{w \in \widetilde{H}^s(\Gamma_N), \\ ||w||_{\widetilde{H}^s(\Gamma_N)} = 1}} \langle g, w \rangle_{H^{-s}(\Gamma_N), \widetilde{H}^s(\Gamma_N)} \\ &\leq ||R||_{\mathcal{L}(H^s(\omega), \widetilde{H}^s(\Gamma_N))} \sup_{\substack{w \in \widetilde{H}^s(\partial\Omega), \\ ||w||_{\widetilde{H}^s(\partial\Omega)} = 1}} \langle g, w \rangle_{H^{-s}(\partial\Omega), H^s(\partial\Omega)}. \end{aligned}$$

In other words, g belongs to $\widetilde{H}^{-s}(\omega)$ and we have:

$$||g||_{\widetilde{H}^{-s}(\omega)} \;\leq\; ||R||_{\mathcal{L}(H^s(\omega), \widetilde{H}^s(\Gamma_N))}\, ||g||_{H^{-s}(\partial\Omega)}.$$

Recalling the estimate (A.14), we arrive at the desired conclusion (A.2), which terminates the proof. $\square$

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