A deterministic approximation method in shape optimization under random uncertainties

Grégoire Allaire[1], Charles Dapogny[2]

[1] CMAP, UMR 7641 École Polytechnique, Palaiseau, France
[2] Laboratoire Jean Kuntzmann, Université Joseph Fourier, Grenoble, France

11th June, 2015

Foreword: uncertainties in structural optimization

- Mechanical systems rely on data, e.g. the loads, the properties of a constituent material, or the geometry of the system itself.
- In concrete situations, such data are plagued with uncertainties because:
    - they may be available only through (error-prone) measurements,
    - they may be altered with time (wear) and conditions of the ambient medium.
- The performances of structures are very sensitive to small perturbations of data.

⇒ Need to somehow anticipate uncertainties when designing and optimizing shapes.

A disk brake system

A worn out brake pad

The main ideas in an abstract framework (I)

- $\mathcal{U}_{ad} \subset \mathcal{H}$ is a set of admissible designs h (e.g. the thickness of a plate, the geometry of a shape).

- $(\mathcal{P}, ||\cdot||)$ is a Banach space of data f (forces, parameters of a material).

- The performances of a design h are evaluated in terms of a cost $\mathcal{C} \equiv \mathcal{C}(f, u_{h,f})$, which involves a state $u_{h,f}$, solution to a physical system:
$$\mathcal{A}(h)u_{h,f} = b(f),$$
where f acts on the right-hand side for simplicity.

- The data are uncertain, and read:
$$f = f_0 + \widehat{f}(\omega),$$
where f_0 is a mean value, and ω is an event, in an abstract probability space $(\mathcal{O}, \mathcal{F}, \mathbb{P})$.

The main ideas in an abstract framework (II)

There are two different settings to deal with uncertainties:

- *Worst-case approach:* When only a maximum bound $||\widehat{f}||_{\mathcal{P}} \leq m$ is available on perturbations, one considers the worst-case functional:

$$\mathcal{J}_{wc}(h) = \sup_{||\widehat{f}||_{\mathcal{P}} \leq m} \mathcal{C}(f_0 + \widehat{f}, u_{h, f_0 + \widehat{f}}).$$

  Main drawback: Pessimistic approach, which may yield designs with unnecessarily bad nominal performances.

- *Probabilistic approach:* When information is available on the moments of the uncertainties, one may try to minimize the mean value:

$$\mathcal{M}(h) = \int_{\mathcal{O}} \mathcal{C}(f_0 + \widehat{f}(\omega), u_{h, f_0 + \widehat{f}(\omega)}) \, \mathbb{P}(d\omega),$$

  or a failure probability:

$$\mathcal{P}(h) = \mathbb{P}\left(\left\{\omega \in \mathcal{O}, \, \mathcal{C}\left(f_0 + \widehat{f}(\omega), u_{h, f_0 + \widehat{f}(\omega)}\right) > \alpha\right\}\right).$$

The main ideas in an abstract framework (III)

Working hypotheses:

- Perturbations are small: depending on the context, this may mean:
  - $\widehat{f} \in L^\infty(\mathcal{O}, \mathcal{P})$: all the realizations $\widehat{f}(\omega) \in \mathcal{P}$ are small.
  - $\widehat{f} \in L^p(\mathcal{O}, \mathcal{P})$, for $p < \infty$: $\widehat{f}$ may have unprobably large realizations.
- Perturbations are finite-dimensional:

$$\widehat{f}(\omega) = \sum_{i=1}^{N} f_i \xi_i(\omega),$$

where $f_i \in \mathcal{P}$, and the ξ_i are normalized, uncorrelated random variables:

$$\int_{\mathcal{O}} \xi_i(\omega) \mathbb{P}(d\omega) = 0, \quad \int_{\mathcal{O}} \xi_i(\omega) \xi_j(\omega) \, \mathbb{P}(d\omega) = \delta_{i,j}.$$

Example: $\widehat{f}$ is obtained as a truncated Karhunen-Loève expansion.

The main ideas in an abstract framework (IV)

Strategy:

- Calculate approximate functionals $\widetilde{\mathcal{M}}(h)$ and $\widetilde{\mathcal{P}}(h)$, which are
  - deterministic: no random variable or probabilistic integral is involved.
  - consistent with their exact counterparts, i.e. the differences $|\mathcal{M}(h) - \widetilde{\mathcal{M}}(h)|$ and $|\mathcal{P}(h) - \widetilde{\mathcal{P}}(h)|$ are 'small'.
- Calculate their derivatives $\widetilde{\mathcal{M}}'(h)(\widehat{h})$ and $\widetilde{\mathcal{P}}'(h)(\widehat{h})$,
- Minimize the approximate functionals $\widetilde{\mathcal{M}}(h)$ and $\widetilde{\mathcal{P}}(h)$ (under constraints), by using the expressions of their derivatives.

The main ideas in an abstract framework (V)

Use the smallness of perturbations to perform a first- or second-order Taylor expansion of the mappings $f \mapsto u_{h,f}$ and $f \mapsto \mathcal{C}(f, u_{h,f})$ around f_0:

$$\boxed{u_{h,\mathbf{f_0}+\widehat{f}} \approx u_h + u_h^1(\widehat{f}) + \tfrac{1}{2} u_h^2(\widehat{f}, \widehat{f}),}$$

where $\quad \mathcal{A}(h)u_h^1(\widehat{f}) = \dfrac{\partial b}{\partial f}(f_0)(\widehat{f}),$ and $\mathcal{A}(h)u_h^2(\widehat{f}, \widehat{f}) = \dfrac{\partial^2 b}{\partial f^2}(f_0)(\widehat{f}, \widehat{f}).$

$$\boxed{\mathcal{C}(f_0 + \widehat{f}, u_{h,\mathbf{f_0}+\widehat{f}}) \approx \mathcal{C}(f_0, u_h) + \mathcal{L}_h(\widehat{f}) + \tfrac{1}{2}\mathcal{B}_h(\widehat{f}, \widehat{f}),}$$

where the linear and bilinear forms $\mathcal{L}_h$ and $\mathcal{B}_h$ read:

$$\mathcal{L}_h(\widehat{f}) = \frac{\partial \mathcal{C}}{\partial f}(f_0, u_h)(\widehat{f}) + \frac{\partial \mathcal{C}}{\partial u}(f_0, u_h)(u_h^1(\widehat{f})),$$

$$\begin{aligned}\mathcal{B}_h(\widehat{f}, \widehat{f}) = {} & \frac{\partial^2 \mathcal{C}}{\partial f^2}(f_0, u_h)(\widehat{f}, \widehat{f}) + 2\frac{\partial^2 \mathcal{C}}{\partial f \partial u}(f_0, u_h)(\widehat{f}, u_h^1(\widehat{f})) \\ & + \frac{\partial^2 \mathcal{C}}{\partial u^2}(f_0, u_h)(u_h^1(\widehat{f}), u_h^1(\widehat{f})) + \frac{\partial \mathcal{C}}{\partial u}(f_0, u_h)(u_h^2(\widehat{f}, \widehat{f})).\end{aligned}$$

Approximation of moment functionals

- Replacing the cost with its second-order expansion gives rise to the approximate mean-value functional:

$$\widetilde{\mathcal{M}}(h) = \mathcal{C}(f_0, u_h) + \int_{\mathcal{O}} \mathcal{L}_h(\widehat{f}(\omega))\, \mathbb{P}(d\omega) + \frac{1}{2}\int_{\mathcal{O}} \mathcal{B}_h(\widehat{f}(\omega), \widehat{f}(\omega))\, \mathbb{P}(d\omega).$$

- Using the structure of perturbations $\widehat{f}(\omega) = \sum_{i=1}^{N} f_i \xi_i(\omega)$, it comes:

$$\boxed{\widetilde{\mathcal{M}}(h) = \mathcal{C}(f_0, u_h) + \tfrac{1}{2}\sum_{i=1}^{N} \mathcal{B}_h(f_i, f_i),}$$

  a formula which involves the calculation of the $N+2$ 'reduced states':

$$u_h, \;\; u_{h,i} := u_h^1(f_i), \; (i = 1, \dots, N), \; \text{and } u_h^2 := \sum_{i=1}^{N} u_h^2(f_i, f_i).$$

- This approach can be applied to other moments of $\mathcal{C}$, e.g. its variance:

$$\mathcal{V}(h) = \int_{\mathcal{O}} \left(\mathcal{C}(f_0 + \widehat{f}(\omega), u_{h, f_0 + \widehat{f}(\omega)}) - \mathcal{M}(h)\right)^2 \mathbb{P}(d\omega).$$

Approximation of failure probabilities (I)

Additional hypotheses: The random variables ξ_i are:

- independent,
- Gaussian, i.e. their cumulative distribution function is:

$$\mathbb{P}\left(\{\omega \in \mathcal{O},\ \xi_i(\omega) < \alpha\}\right) = \Phi(\alpha) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{\frac{-\xi^2}{2}}\, d\xi.$$

The (exact) failure probability reads:

$$\mathcal{P}(h) = \frac{1}{(2\pi)^{N/2}} \int_{\mathcal{D}(h)} e^{-\frac{|\xi|^2}{2}}\, d\xi,$$

where the failure region $\mathcal{D}(h)$ is:

$$\mathcal{D}(h) = \left\{ \xi \in \mathbb{R}^N,\ \mathcal{C}\left(f_0 + \sum_{i=1}^{N} f_i \xi_i,\ u_{h, f_0 + \sum_{i=1}^{N} f_i \xi_i} \right) > \alpha \right\}.$$

Approximation of failure probabilities (II)

Idea: Approximate the failure region with:

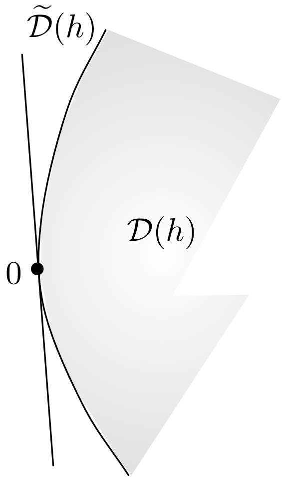
$$\widetilde{\mathcal{D}}(h) = \left\{ \xi \in \mathbb{R}^N,\ \mathcal{C}(f_0, u_h) + \sum_{i=1}^{N} \mathcal{L}_h(f_i)\xi_i > \alpha \right\}.$$

The approximate failure probability

$$\widetilde{\mathcal{P}}(h) = \frac{1}{(2\pi)^{N/2}} \int_{\widetilde{\mathcal{D}}(h)} e^{-\frac{|\xi|^2}{2}}\, d\xi$$

can be calculated in closed form as:

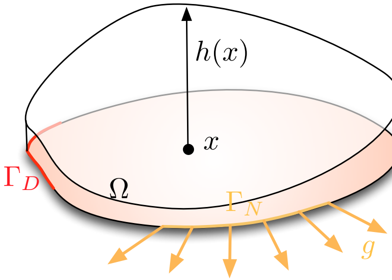
$$\widetilde{\mathcal{P}}(h) = \Phi\left(-\frac{\alpha - \mathcal{C}(f_0, u_h)}{\sqrt{\sum_{i=1}^{N} \mathcal{L}_h(f_i)^2}} \right).$$



The parametric optimization setting (I)

- The thickness of a plate with (smooth) cross-section $\Omega \subset \mathbb{R}^d$ is optimized.
- $\mathcal{U}_{ad} \subset L^\infty(\Omega)$ is a set of admissible thickness functions:

$$\mathcal{U}_{ad} = \{h \in L^\infty(\Omega),\ h_{min} \leq h(x) \leq h_{max},\ \text{a.e. in } \Omega\}.$$



Setting of the parametric optimization problem.

The parametric optimization setting (II)

- The plate is clamped on a part of its boundary $\Gamma_D \subset \partial\Omega$.

- Surface loads $g \in L^2(\Gamma_N)^d$ are applied on the complementary part $\Gamma_N := \Omega \setminus \overline{\Gamma_D}$, as well as body forces $f \in L^2(\Omega)^d$.

- The elastic displacement of the plate is the unique solution

$$u_h \in H^1_{\Gamma_D}(\Omega)^d := \left\{ u \in H^1(\Omega)^d,\ \ u = 0 \text{ on } \Gamma_D \right\}.$$

  to the linear elasticity system:

$$\begin{cases} -\mathrm{div}(hAe(u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ hAe(u)n = g & \text{on } \Gamma_N \end{cases}.$$

  Here $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the strain tensor, and A is the Hooke's law:

$$\forall e \in \mathcal{S}(\mathbb{R}^d),\ \ Ae = 2\mu e + \lambda \mathrm{tr}(e) I,$$

  where λ, μ are the Lamé coefficients of the material.

Parametric optimization under random loads

- We consider perturbations on the body forces; $\mathcal{P} = L^2(\Omega)^d$, and:

$$f(x) = f_0(x) + \widehat{f}(x,\omega), \text{ where } \widehat{f}(x,\omega) = \sum_{i=1}^{N} f_i(x)\,\xi_i(\omega) \in L^2(\mathcal{O}, L^2(\Omega)^d).$$

- The cost function is of the form:

$$\mathcal{C}(f, h) = \int_\Omega j(u_{h,f})\,dx,$$

where $j : \mathbb{R}^d \to \mathbb{R}$ is smooth, satisfies growth conditions and:

$$\left\{ \begin{array}{cl} -\text{div}(hAe(u_{h,f})) = f & \text{in } \Omega \\ u_{h,f} = 0 & \text{on } \Gamma_D \\ hAe(u_{h,f})n = g & \text{on } \Gamma_N \end{array} \right. .$$

- The objective function to approximate is the failure probability:

$$\mathcal{P}(h) = \mathbb{P}\left(\left\{\omega \in \mathcal{O},\ \mathcal{C}(h, f_0 + \widehat{f}(\omega)) > \alpha\right\}\right),$$

where α is a given safety threshold.

Probability failure under random loads (I)

Assumption: The random variables ξ_i are independent and Gaussian.

The approximate failure probability $\widetilde{\mathcal{P}}(h)$ reads:

$$\widetilde{\mathcal{P}}(h) = \Phi\left(-\frac{\alpha - b_h}{|a_h|}\right),$$

where b_h and the entries of $a_h := (a_{h,1}, ..., a_{h,N})$ are:

$$b_h = \int_\Omega j(f_0, u_h)\, dx, \;\; a_{h,i} = \int_\Omega \left(\nabla_f j(f_0, u_h) \cdot f_i + \nabla_u j(f_0, u_h) \cdot u^1_{h,i}\right) dx,$$

the $u^1_{h,i}$ being the solutions of: $\left\{\begin{array}{cl} -\mathrm{div}(hAe(u)) = f_i & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ hAe(u)n = 0 & \text{on } \Gamma_N \end{array}\right.$.

Proposition 1.

There exists a constant C (uniform with respect to $h \in \mathcal{U}_{ad}$) such that:

$$||\widehat{f}||_{L^2(\mathcal{O}, L^2(\Omega)^d)} \leq \varepsilon \Rightarrow |\widetilde{\mathcal{P}}(h) - \mathcal{P}(h)| \leq C\varepsilon^2 |\log \varepsilon|^{\frac{N+1}{2}}.$$

Probability failure under random loads (II)

Theorem 2.

The function $\widetilde{\mathcal{P}}(h)$ is Fréchet-differentiable at any $h \in \mathcal{U}_{ad}$, and its derivative is:

$$\forall \widehat{h} \in L^\infty(\Omega), \;\; \widetilde{\mathcal{P}}'(h)(\widehat{h}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\alpha - b_h}{|a_h|}\right)^2} \int_\Omega \widehat{h} \; \mathcal{D}_h \, dx,$$

where the integrand $\mathcal{D}_h$ reads:

$$\mathcal{D}_h = \frac{1}{|a_h|} Ae(u_h) : e(p_h^0) + \frac{\alpha - b_h}{|a_h|^3} \left(Ae(u_h) : e(p_h^1) + \sum_{i=1}^N a_{h,i} Ae(u_{h,i}^1) : e(p_h^0) \right),$$

and the adjoint states $p_h^0, p_h^1 \in H^1_{\Gamma_D}(\Omega)^d$ are defined by: $\forall v \in H^1_{\Gamma_D}(\Omega)^d$,

$$\int_\Omega hAe(p_h^0) : e(v) \, dx = -\int_\Omega \nabla_u j(f_0, u_h) \cdot v \, dx,$$

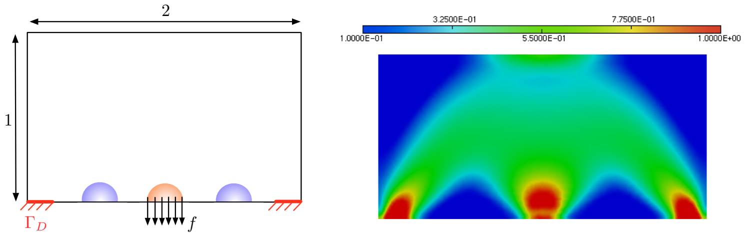
$$\int_\Omega hAe(p_h^1) : e(v) \, dx = -\sum_{i=1}^N a_{h,i} \int_\Omega (\nabla^2_{f,u} j(f_0, u_h)(f_i, v) + \nabla^2_u j(f_0, u_h)(u_{h,i}^1, v)) dx.$$

Probability failure under random loads: numerical example (I)

- The unperturbed forces $f_0 = (0, -10)$ apply on the red spot, and the two perturbation scenarii $f_1, f_2 = (0, -10)$ are supported on the blue spots.
- The cost function is the compliance of the plate:

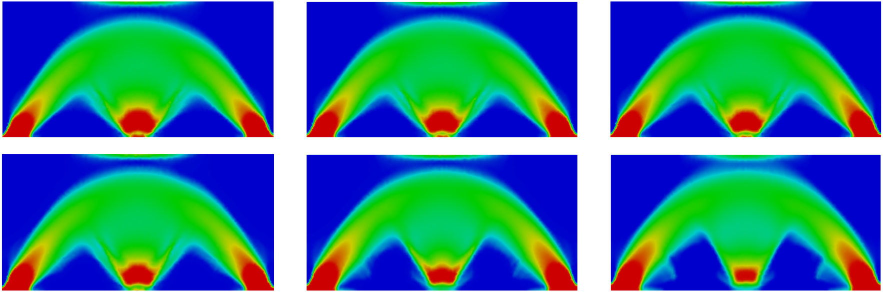
$$\mathcal{C}(h, f) = \int_\Omega hAe(u_{h,f}) : e(u_{h,f})\, dx = \int_\Omega f \cdot u_{h,f}\, dx.$$

- A volume constraint $\mathrm{Vol}(h) = \int_\Omega h = V_T$ is imposed owing to an augmented Lagrangian algorithm.



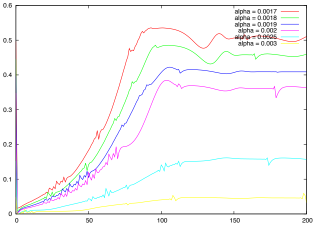
(Left) Description of the test-case, (right) optimal shape without uncertainties. The value of the compliance is 0.001729.

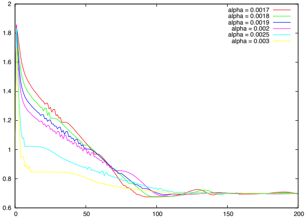
Probability failure under random loads: numerical example (II)



Minimization of the failure probability: optimal thickness distributions for the values $\alpha = 0.0017, 0.0018, 0.0019, 0.002, 0.0025, 0.003$.

Probability failure under random loads: numerical example (III)





Evolutions of the approximate failure probability (left), and of the volume (right).

Preliminaries: the usual linear elasticity setting (I)

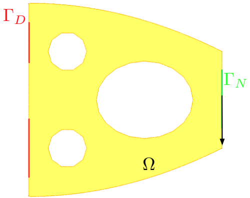
A shape is a bounded domain $\Omega \subset \mathbb{R}^d$, which is

- fixed on a part Γ_D of its boundary,
- submitted to surface loads g, applied on $\Gamma_N \subset \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$.

The displacement vector field $u_\Omega \in H^1_{\Gamma_D}(\Omega)^d$ is governed by the linear elasticity system:

$$\left\{ \begin{array}{rcll} -\mathrm{div}(Ae(u_\Omega)) & = & f & \text{in } \Omega \\ u_\Omega & = & 0 & \text{on } \Gamma_D \\ Ae(u_\Omega)n & = & g & \text{on } \Gamma_N \\ Ae(u_\Omega)n & = & 0 & \text{on } \Gamma := \partial\Omega \setminus (\Gamma_D \cup \Gamma_N) \end{array} \right. ,$$

where $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the strain tensor, and A is the Hooke's law of the material.



A 'Cantilever'

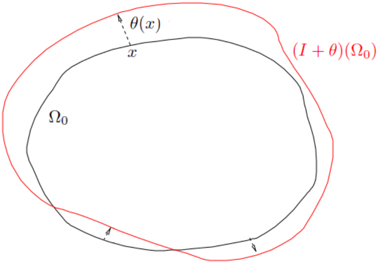
The deformed cantilever

Differentiation with respect to the domain: Hadamard's method (I)

Hadamard's boundary variation method describes variations of a reference, Lipschitz domain Ω of the form:

$$\Omega \to \Omega_\theta := (I + \theta)(\Omega),$$

for 'small' $\theta \in W^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right)$.



In practice:

- We restrict to a set of admissible shapes:

$$\mathcal{U}_{ad} := \left\{\Omega \subset \mathbb{R}^d \text{ is open, bounded and Lipschitz, } \Gamma_D \cup \Gamma_N \subset \partial\Omega\right\}.$$

- Deformations θ are assumed within the admissible set:

$$\Theta_{ad} := \left\{\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \text{ such that } \theta = 0 \text{ on } \Gamma_D \cup \Gamma_N\right\}.$$

Definition 1.

Given a smooth domain Ω*, a functional* $J(\Omega)$ *of the domain is shape differentiable at* Ω *if the function*

$$W^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right) \ni \theta \mapsto J(\Omega_\theta)$$

is Fréchet-differentiable at 0*, i.e. the following expansion holds around* 0*:*

$$J(\Omega_\theta) = J(\Omega) + J'(\Omega)(\theta) + o\left(||\theta||_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)}\right).$$

Shape derivatives can be computed using techniques from optimal control; in the case of 'many' functions of the domain $J(\Omega)$, they enjoy the structure:

$$\boxed{J'(\Omega)(\theta) = \int_\Gamma v_\Omega \, \theta \cdot n \, ds,}$$

where v_Ω is a scalar field depending on u_Ω, and possibly on an adjoint state p_Ω.

Shape optimization under random loads (I)

- We consider uncertainties on the body forces f ($\mathcal{P} = L^2(\mathbb{R}^d)^d$):

$$f(x) = f_0(x) + \widehat{f}(x,\omega), \text{ where } \widehat{f}(x,\omega) = \sum_{i=1}^{N} f_i(x)\, \xi_i(\omega) \in L^2(\mathcal{O}, L^2(\mathbb{R}^d)^d).$$

- The cost function is of the form:

$$\mathcal{C}(f,\Omega) = \int_\Omega j(f, u_{\Omega,f})\, dx,$$

where $j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is smooth, satisfies growth conditions, and $u_{\Omega,f} \in H^1_{\Gamma_D}(\Omega)^d$ solves:

$$\begin{cases} -\text{div}(Ae(u_\Omega)) & = & f & \text{in } \Omega \\ u_\Omega & = & 0 & \text{on } \Gamma_D \\ Ae(u_\Omega)n & = & 0 & \text{on } \Gamma_N \\ Ae(u_\Omega)n & = & 0 & \text{on } \Gamma \end{cases}.$$

Shape optimization under random loads (II)

The approximate mean value functional reads:

$$\widetilde{\mathcal{M}}(\Omega) = \int_\Omega j(f_0, u_\Omega)\, dx + \frac{1}{2}\sum_{i=1}^{N}\int_\Omega \nabla_f^2 j(f_0, u_\Omega)(f_i, f_i)\, dx$$
$$+ \sum_{i=1}^{N}\int_\Omega \nabla_f \nabla_u j(f_0, u_\Omega)(f_i, u^1_{\Omega,i})\, dx + \frac{1}{2}\sum_{i=1}^{N}\int_\Omega \nabla_u^2 j(f_0, u_\Omega)(u^1_{\Omega,i}, u^1_{\Omega,i})\, dx,$$

the $u^1_{\Omega,i}$ being the solutions of: $\left\{ \begin{array}{rl} -\mathrm{div}(Ae(u)) = f_i & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ Ae(u)n = 0 & \text{on } \Gamma_N \\ Ae(u)n = 0 & \text{on } \Gamma \end{array} \right. .$

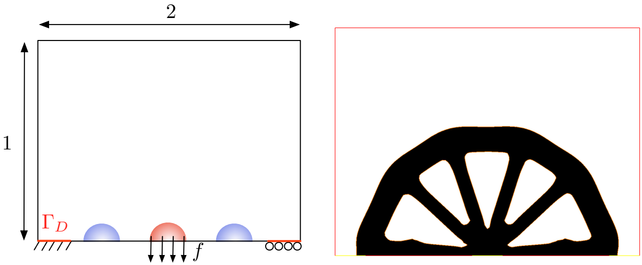
Proposition 3.

Under the additional assumption that $\widehat{f} \in L^3(\mathcal{O}, L^3(\mathbb{R}^d)^d)$, there exists a constant $C > 0$ (depending on Ω) such that:

$$|\widetilde{\mathcal{M}}(\Omega) - \mathcal{M}(\Omega)| \leq C||\widehat{f}||^3_{L^3(\mathcal{O}, L^3(\mathbb{R}^d)^d)}.$$

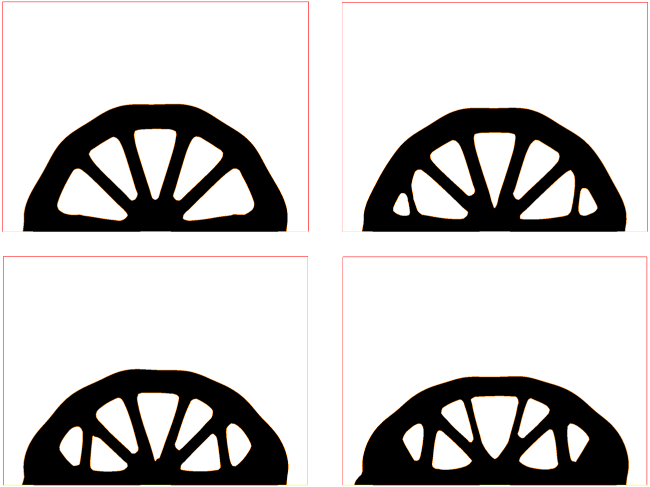
Optimization of a bridge under random loads (I)

- Two load scenarii $f_1, f_2 = (0, -m)$ are supported in the blue spots.
- The considered objective function is: $\mathcal{L}(\Omega) = \widetilde{\mathcal{M}}(\Omega) + \delta\sqrt{\widetilde{\mathcal{V}}(\Omega)}$.
- A volume constraint $\text{Vol}(\Omega) = V_T$ is imposed owing to an augmented Lagrangian algorithm.



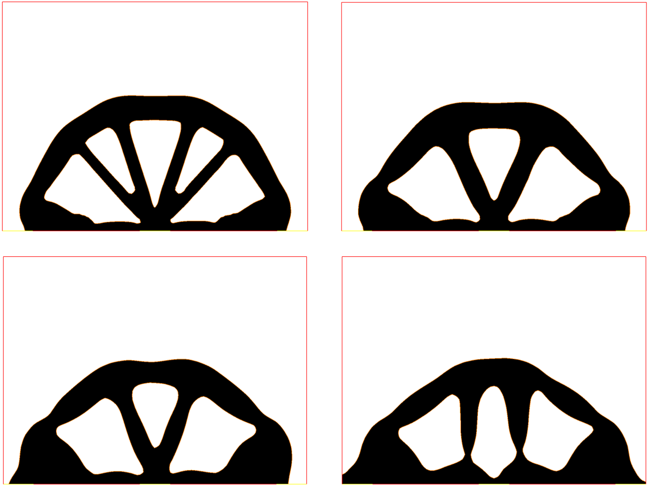
(Left) The bridge test case, (right) optimal shape in the unperturbed situation.

Optimization of a bridge under random loads (II)



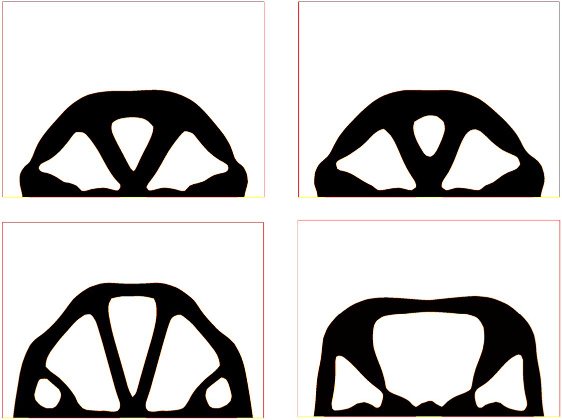
Optimal shapes for $\delta = 0$ *and* $m = 1, 2, 5, 10$.

Optimization of a bridge under random loads (III)



Optimal shapes for $\delta = 3$ *and* $m = 1, 2, 5, 10$.

Comparison with the worst-case approach



Optimal shapes for the linearized worst-case design approach with $m = 1, 2, 5, 10$.

Observation: The optimal shapes for the probabilistic functionals show systematically better nominal performances than their worst-case counterparts.

Optimization under material uncertainties

- Perturbations over the Young's modulus E of the material are considered:

$$E = E_0 + \widehat{E}(x,\omega), \text{ where } \widehat{E}(x,\omega) = \sum_{i=1}^{N} E_i(x)\xi_i(\omega) \in L^\infty(\mathcal{O}, L^\infty(\mathbb{R}^d)).$$

- The cost function is of the form $\mathcal{C}(\Omega, E) = \int_\Omega j(u_{\Omega,E})\, dx$, where:

$$\begin{cases} -\mathrm{div}(A(E)e(u_\Omega)) & = & 0 & \text{in } \Omega \\ u_\Omega & = & 0 & \text{on } \Gamma_D \\ A(E)e(u_\Omega)n & = & g & \text{on } \Gamma_N \\ A(E)e(u_\Omega)n & = & 0 & \text{on } \Gamma \end{cases}.$$

- Minimization of the approximate mean value of $\mathcal{C}$:

$$\widetilde{\mathcal{M}}(\Omega) = \int_\Omega j(u_\Omega)\, dx + \frac{1}{2}\sum_{i=1}^{N}\int_\Omega \nabla^2 j(u_\Omega)(u^1_{\Omega,i}, u^1_{\Omega,i})\, dx + \frac{1}{2}\int_\Omega \nabla j(u_\Omega)\cdot u^2_\Omega\, dx,$$

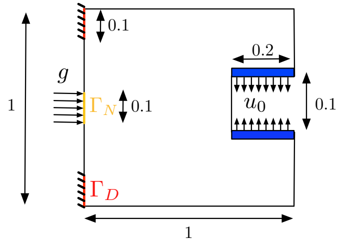
where the $u_{h,i}$, $i = 1, \ldots, N$, and u_h^2 are the reduced states.

Optimization of a grip under material uncertainties (I)

- The cost function is $\mathcal{C}(\Omega, E) = \int_\Omega k(x)|u_{\Omega,E} - u_0|^2\, dx$, where k is a localization factor, and u_0 is a target displacement.
- The two-point correlation of $\widehat{E}(x,\omega)$ is known:

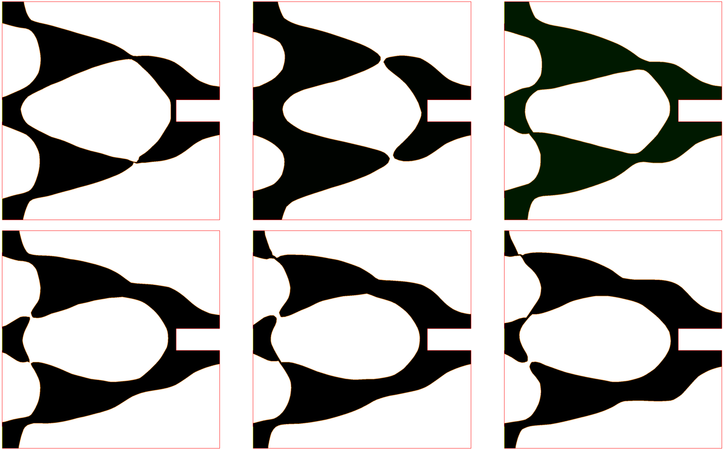
$$\mathrm{Cor}(\widehat{E})(x,y) := \int_{\mathcal{O}} \widehat{E}(x,\omega)\widehat{E}(y,\omega)\, \mathbb{P}(d\omega) = \beta^2 e^{-\frac{|x-y|}{l}}.$$

- A Karhunen-Loève expansion of $\widehat{E}$ is performed and truncated after the $N = 5^{\text{th}}$ term.



Setting of the gripping mechanism example.

Optimization of a grip under material uncertainties (II)



Optimal shapes associated to values of $\beta = 0, 0.5, 1, 1.5, 2, 2.5$.

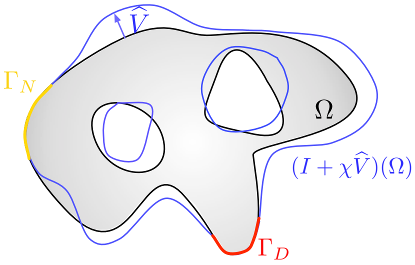
Modelling geometric uncertainties

Perturbations of a shape $\Omega \in \mathcal{U}_{ad}$ are considered with the structure:

$$\Omega \longmapsto (I + \chi(x)\widehat{v}(x,\omega)n_\Omega(x))(\Omega),$$

where:

- χ is a cutoff function, vanishing on $\Gamma_D \cup \Gamma_N$,
- n_Ω is (an extension of) the normal vector to $\partial\Omega$,
- The scalar field $\widehat{v} \in L^\infty(\mathcal{O}, \mathcal{C}^{2,\infty}(\mathbb{R}^d))$ arises as $\widehat{v}(x,\omega) = \sum_{i=1}^N v_i(x)\xi_i(\omega)$.



Perturbation of Γ by a vector field $\widehat{V}$.

Optimization of a L-beam under geometric uncertainties

- The cost function is of the form:

$$\mathcal{C}(\Omega) = \int_\Omega j(\sigma(u_\Omega))\, dx,$$

where $\sigma(u) = Ae(u)$ is the stress tensor.

- The approximate variance functional reads:

$$\widetilde{\mathcal{V}}(\Omega) = \sum_{i=1}^{N} a_{\Omega,i}^2 \text{ with } a_{\Omega,i} = \int_\Gamma \left(j(\sigma(u_\Omega)) + Ae(u_\Omega) : e(p_\Omega) - f \cdot p_\Omega \right) v_i\, ds, \tag{1}$$

and the adjoint state $p_\Omega \in H^1_{\Gamma_D}(\Omega)^d$ is the solution of:

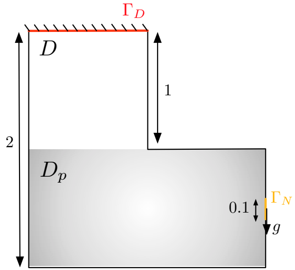
$$\begin{cases} -\mathrm{div}(Ae(p)) = \mathrm{div}(A\frac{\partial j}{\partial \sigma}(\sigma(u_\Omega))) & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_D, \\ Ae(p)n = -A\frac{\partial j}{\partial \sigma}(\sigma(u_\Omega))n & \text{on } \Gamma \cup \Gamma_N. \end{cases}$$

Optimization of a L-beam under geometric uncertainties

- Perturbations occur on a subregion $D_p \subset D$; their correlation function is:

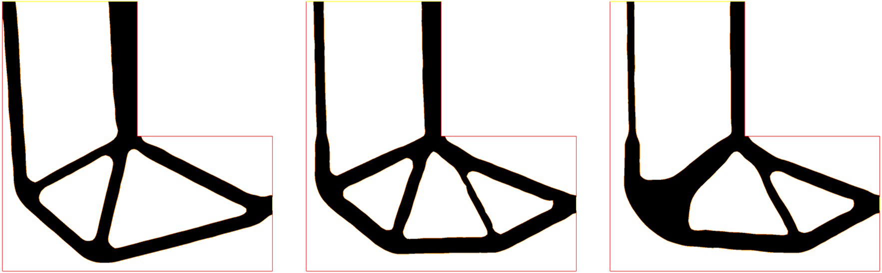
$$\mathrm{Cor}(\widehat{v})(x,\omega) = \beta^2 e^{-\frac{|x-y|}{l}}.$$

- The cost function is $\mathcal{C}(\Omega) = \int_\Omega ||\sigma(u_\Omega)||^5 \, dx$, and the objective $\mathcal{C}(\Omega) + \delta\sqrt{\widetilde{\mathcal{V}}(\Omega)}$ is minimized under a volume constraint.



Details of the L-shaped beam test-case.

Optimization of a L-beam under geometric uncertainties



Optimal shapes in the minimization of the stress-based criterion, where the parameter δ equals (from the left to the right) $0, 0.5, 2$.

Thank you for your attention!

References I

[AlDa1]G. Allaire and C. Dapogny, *A linearized approach to worst-case design in parametric and geometric shape optimization*, M3AS Vol. 24, No. 11 (2014) 2199-2257.

[AlDa2]G. Allaire and C. Dapogny, *A deterministic approximation method in shape optimization under random uncertainties*, submitted, (2015).

[AlJouToa] G. Allaire, F. Jouve and A.M. Toader, *Structural optimization using shape sensitivity analysis and a level-set method*, J. Comput. Phys., 194 (2004) pp. 363–393.

[HenPi] A. Henrot and M. Pierre, *Variation et optimisation de formes, une analyse géométrique*, Mathématiques et Applications 48, Springer, Heidelberg, (2005).

[Mau] K. Maute, *Topology Optimization under uncertainty*, in Topology Optimization in Structural and Continuum Mechanics, CISM International Centre for Mechanical Sciences Volume 549, (2014), pp. 457–471.

References II

[OSe] S. J. Osher and J.A. Sethian, *Front propagating with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations*, J. Comp. Phys. **78** (1988) pp. 12-49