A level-set based mesh evolution method for shape optimization

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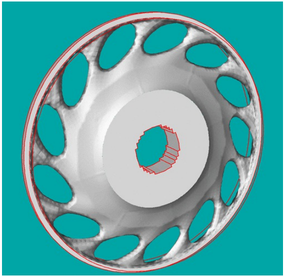
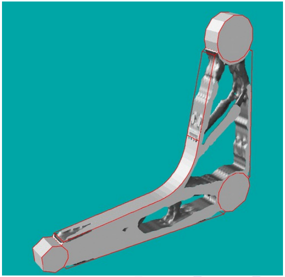
Shape optimization and industrial applications

The increase in the cost of raw materials urges to optimize mechanical parts from the early stages of design.

Shape optimization problems are difficult, partly because they require an accurate description of the various shapes arising in the optimization process.

Automatic techniques (implemented in industrial softwares) have started to replace the traditional trial-and-error methods used by engineers, but still leave room for many developments.

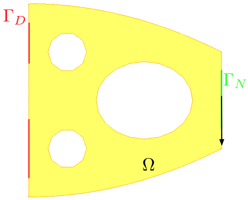




A model problem in linear elasticity

A shape is a bounded domain $\Omega \subset \mathbb{R}^d$, which is

- fixed on a part Γ_D of its boundary,
- submitted to surface loads g, applied on $\Gamma_N \subset \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$.



A 'Cantilever'

The displacement vector field $u_\Omega : \Omega \to \mathbb{R}^d$ is governed by the linear elasticity system:

$$\left\{ \begin{array}{ccll} -\mathrm{div}(Ae(u_\Omega)) & = & 0 & \text{in } \Omega \\ u_\Omega & = & 0 & \text{on } \Gamma_D \\ Ae(u_\Omega)n & = & g & \text{on } \Gamma_N \\ Ae(u_\Omega)n & = & 0 & \text{on } \Gamma := \partial\Omega \setminus (\Gamma_D \cup \Gamma_N) \end{array} \right. ,$$

The deformed cantilever

where $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the strain tensor, and A is the Hooke's law of the material.

A model problem in linear elasticity

Goal: Starting from an initial structure Ω_0, find a new one Ω that minimizes a certain functional of the domain $J(\Omega)$.

Examples:

- The work of the external loads g or compliance $C(\Omega)$ of domain Ω:

$$C(\Omega) = \int_\Omega Ae(u_\Omega) : e(u_\Omega)dx = \int_{\Gamma_N} g.u_\Omega \, ds$$

- A least-square error between u_Ω and a target displacement $u_0 \in H^1(\Omega)^d$ (useful when designing micro-mechanisms):

$$D(\Omega) = \left(\int_\Omega k(x) |u_\Omega - u_0|^\alpha dx \right)^{\frac{1}{\alpha}},$$

where α is a fixed parameter, and $k(x)$ is a weight factor.

A volume constraint may be enforced with a fixed penalty parameter ℓ:

$$\text{Minimize } J(\Omega) := C(\Omega) + \ell \, \mathrm{Vol}(\Omega), \text{ or } D(\Omega) + \ell \, \mathrm{Vol}(\Omega).$$

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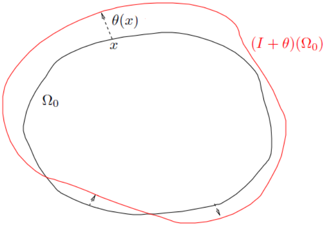
Differentiation with respect to the domain: Hadamard's method

Hadamard's boundary variation method describes variations of a Lipschitz domain Ω_0 of the form:

$$\Omega_0 \to (I + \theta)(\Omega_0),$$

for 'small' $\theta \in W^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right)$.



Definition 1.

A (scalar) function $\Omega \mapsto F(\Omega)$ is shape differentiable at Ω_0 if the function

$$W^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right) \ni \theta \mapsto F((I+\theta)(\Omega_0))$$

is Fréchet-differentiable at 0, i.e. the following expansion holds:

$$F((I+\theta)(\Omega_0)) = F(\Omega_0) + F'(\Omega_0)(\theta) + o\left(||\theta||_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)}\right).$$

Differentiation with respect to the domain: Hadamard's method

- Techniques from optimal control allow to calculate shape gradients; in the case of 'many' functionals $J(\Omega)$, $J'(\Omega)$ has the particular structure:

$$\boxed{J'(\Omega)(\theta) = \int_\Gamma v_\Omega \, \theta \cdot n \, ds,}$$

where v_Ω is a scalar field depending on u_Ω, and possibly on an adjoint p_Ω.

Example: If $J(\Omega) = \int_{\Gamma_N} g \cdot u_\Omega \, ds$ is the compliance, $v_\Omega = -Ae(u_\Omega) : e(u_\Omega)$.

- This shape gradient provides a natural descent direction for functional J: *for instance*, defining θ as

$$\theta = -v_\Omega n$$

yields, for $t > 0$ sufficiently small (*to be found numerically*):

$$J((I + t\theta)(\Omega)) = J(\Omega) - t \int_\Gamma v_\Omega^2 ds + o(t) < J(\Omega).$$

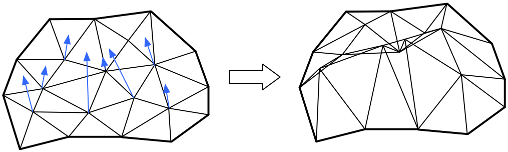
The generic numerical algorithm

Gradient algorithm: For $n = 0, \ldots$ convergence,

1. Compute the solution u_{Ω^n} (and p_{Ω^n}) of the elasticity system on Ω^n.
2. Compute the shape gradient $J'(\Omega^n)$ thanks to the previous formula, and infer a descent direction θ^n for the cost functional.
3. Advect the shape Ω^n according to θ^n, so as to get $\Omega^{n+1} := (I + \theta^n)(\Omega^n)$.

Problem: We need to

- efficiently advect the shape Ω^n at each step
- get a mesh of each shape Ω^n, for finite element computations.



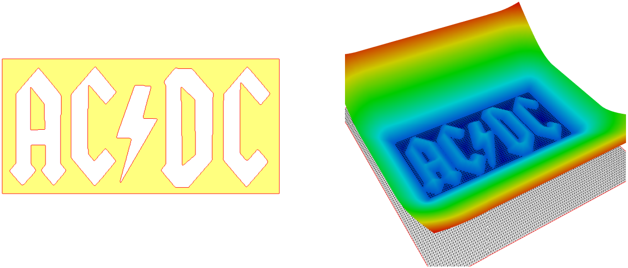
Pushing nodes according to the velocity field may result in an invalid configuration.

A short detour by the Level Set Method

A paradigm: [Osher, Sethian] *the motion of an evolving domain is best described in an implicit way.*

A bounded domain $\Omega \subset \mathbb{R}^d$ is equivalently defined by a function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that:

$$\phi(x) < 0 \quad \text{if } x \in \Omega \quad ; \quad \phi(x) = 0 \quad \text{if } x \in \partial\Omega \quad ; \quad \phi(x) > 0 \quad \text{if } x \in {}^c\overline{\Omega}$$



A bounded domain $\Omega \subset \mathbb{R}^2$ (left); graph of an associated level set function (right).

Surface evolution equations in the level set framework

The motion of an evolving domain $\Omega(t) \subset \mathbb{R}^d$ along a velocity field $v(t,x) \in \mathbb{R}^d$ translates in terms of an associated 'level set function' $\phi(t,.)$ into the level set advection equation:

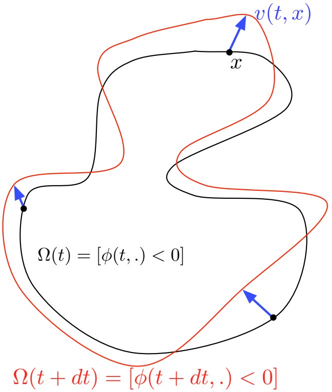
$$\forall t,\ \forall x \in \mathbb{R}^d,\ \frac{\partial \phi}{\partial t}(t,x) + v(t,x).\nabla\phi(t,x) = 0$$

In many applications, the velocity $v(t,x)$ is normal to the boundary $\partial\Omega(t)$:

$$v(t,x) := V(t,x)\frac{\nabla\phi(t,x)}{|\nabla\phi(t,x)|}.$$

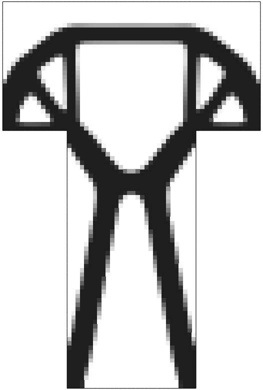
Then the evolution equation rewrites as a Hamilton-Jacobi equation:

$$\forall t,\ \forall x \in \mathbb{R}^d,\ \frac{\partial \phi}{\partial t}(t,x) + V(t,x)|\nabla\phi(t,x)| = 0$$



The level set method of Allaire-Jouve-Toader

- The shapes Ω^n are embedded in a working domain D equipped with a fixed mesh.
- The successive shapes Ω^n are accounted for in the level set framework, i.e. via a function $\phi^n : D \to \mathbb{R}$ which implicitly defines them.
- At each step n, the exact linear elasticity system on Ω^n is approximated by the Ersatz material approach: the void $D \setminus \Omega^n$ is filled with a very 'soft' material, which leads to an approximate system posed on D.
- This approach is very versatile and does not require a mesh of the shapes at each iteration.



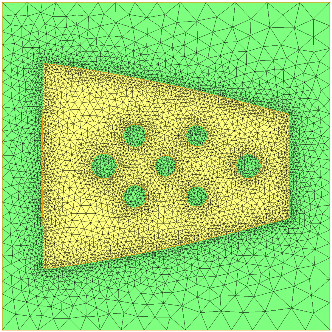
Shape accounted for with a level set description

The proposed method for handling mesh evolution

The mesh $\mathcal{T}^n$ of D is unstructured and changes at each iteration n, so that Ω^n is explicitly discretized in $\mathcal{T}^n$.

- Finite element analyses are held on Ω^n by 'forgetting' the part of $\mathcal{T}^n$ for the void $D \setminus \Omega^n$.
- The advection step $\Omega^n \to \Omega^{n+1}$ is carried out on the whole mesh $\mathcal{T}^n$, using a level set description ϕ^n of Ω^n.

$$(\Omega^n, \mathcal{T}^n) \to (\Omega^{n+1}, \mathcal{T}^{n+1}) \quad \Leftrightarrow \quad \phi^n \to \phi^{n+1}$$



Shape equipped with a mesh, conformally embedded in a mesh of the computational box.

Definition 2.

Let $\Omega \subset \mathbb{R}^d$ a bounded domain. The signed distance function to Ω is the function $\mathbb{R}^d \ni x \mapsto d_\Omega(x)$ defined by:

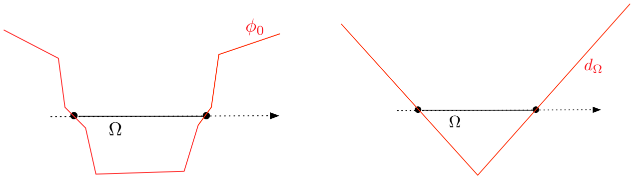
$$d_\Omega(x) = \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \partial\Omega \\ d(x, \partial\Omega) & \text{if } x \in \overline{{}^c\Omega} \end{cases},$$

where $d(\cdot, \partial\Omega)$ is the usual Euclidean distance function.

The signed distance function as the steady state of a PDE

- The signed distance function to a domain $\Omega \subset \mathbb{R}^d$ is the 'canonical' way to initialize a level set function, owing to its unit gradient property:

$$|\nabla d_\Omega(x)| = 1, \quad \text{p.p } x \in \mathbb{R}^d.$$



(left) Any level set function for $\Omega = (0,1) \subset \mathbb{R}$; (right) signed distance function to Ω.

- Many existing approaches: Fast Marching Method [Sethian], Fast Sweeping method [Zhao], mostly on Cartesian grids, or particular unstructured meshes.

The signed distance function as the steady state of a PDE

Another point of view [Chopp]: suppose $\Omega \subset \mathbb{R}^d$ is implicitly known as

$$\Omega = \left\{x \in \mathbb{R}^d ; \phi_0(x) < 0\right\} \text{ and } \partial\Omega = \left\{x \in \mathbb{R}^d ; \phi_0(x) = 0\right\},$$

where ϕ_0 is a function we only suppose continuous. Then the function d_Ω can be considered as the steady state of the so-called unsteady Eikonal equation

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \operatorname{sgn}(\phi_0)(|\nabla \phi| - 1) = 0 & \forall t > 0, x \in \mathbb{R}^d \\ \phi(t = 0, x) = \phi_0(x) & \forall x \in \mathbb{R}^d \end{cases} . \tag{1}$$

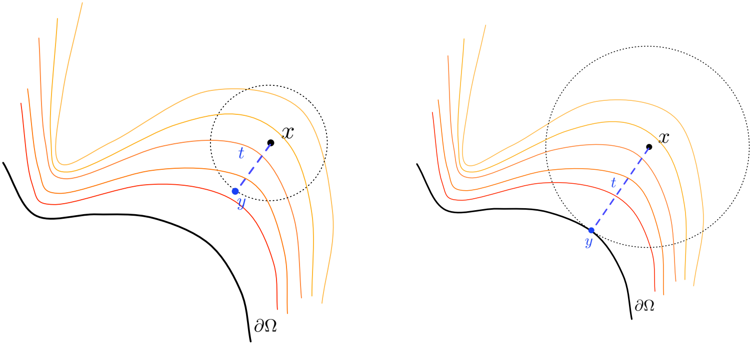
Theorem 1 ([Aubert, Aujol]).

Define that function ϕ, $\forall x \in \mathbb{R}^d$, $\forall t \in \mathbb{R}_+$,

$$\phi(t, x) = \begin{cases} \operatorname{sgn}(\phi_0(x)) \inf\limits_{|y| \le t} (\operatorname{sgn}(\phi_0(x))\phi_0(x + y) + t) & \text{if } t \le d(x, \partial\Omega) \\ \operatorname{sgn}(\phi_0(x)) d(x, \partial\Omega) & \text{if } t > d(x, \partial\Omega) \end{cases}$$

Let $T \in \mathbb{R}_+$. Then ϕ is the unique uniformly continuous viscosity solution of (1) such that, for all $0 \le t \le T$, $\phi(t, x) = 0$ on $\partial\Omega$.

The signed distance function as the steady state of a PDE



Some level sets of function ϕ_0; (left): computation of $\phi(t,x) = \phi_0(y) + t$ for small t; (right): computation of $\phi(t,x) = \phi_0(y) + t = d(x, \partial\Omega)$ at $t = d(x, \partial\Omega)$.

The proposed algorithm

Basic idea: *Compute iteratively the solution $\phi(t,x)$, using the exact formula.*

Let dt be a time step, and $t^n = ndt$.
The continuous formula for ϕ can be made iterative: denoting $\phi^n(x) = \phi(t^n, x)$, we have, for $n = 0, \dots$

$$\forall x \in {}^c\Omega,\ \phi^{n+1}(x) = \inf_{|y| \leq dt} \phi^n(x+y) + dt$$

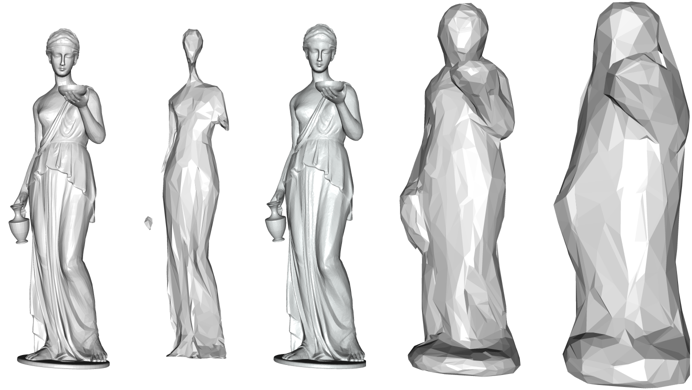
$$\forall x \in \Omega,\ \phi^{n+1}(x) = \sup_{|y| \leq dt} \phi^n(x+y) - dt$$

and, dt being small enough, the above infimum and supremum are evaluated by taking y in the gradient direction; at a vertex x of the computational mesh $\mathcal{T}$:

$$\forall x \in {}^c\Omega,\ \phi^{n+1}(x) \approx \inf_{T \in Ball(x)} \phi^n \left(x - dt \frac{\nabla \phi^n|_T}{|\nabla \phi_n|_T|} \right) + dt$$

$$\forall x \in \Omega,\ \phi^{n+1}(x) \approx \sup_{T \in Ball(x)} \phi^n \left(x + dt \frac{\nabla \phi^n|_T}{|\nabla \phi_n|_T|} \right) - dt.$$

A 3d example... the 'Aphrodite'.

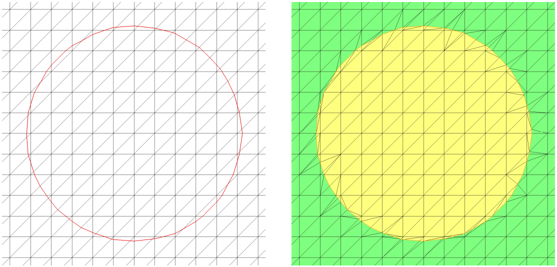


(a) (b) (c) (d) (e)

Isosurfaces of the signed distance function to the 'Aphrodite' (a): (b): isosurface -0.01*, (c): isosurface* 0*, (d): isosurface* 0.02*, (e): isosurface* 0.05*.*

Meshing the negative subdomain of a level set function

Discretizing explicitely the 0 level set of a function $\phi : D \to \mathbb{R}$ defined at the vertices of a simplicial mesh $\mathcal{T}$ of a box D is fairly easy, using patterns.



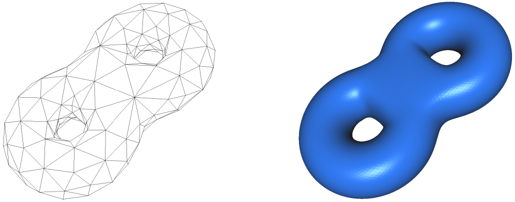
(left) 0 *level set of a scalar function defined over a mesh; (right) explicit discretization in the mesh.*

However, doing so is bound to produce a very low-quality mesh, on which finite element computations will prove slow, inaccurate, not to say impossible.

$\Rightarrow$ Need to improve the quality of a mesh, while retaining its geometric features.

Local remeshing in $3d$

- Let $\mathcal{T}$ be an initial - valid, yet potentially ill-shaped - tetrahedral mesh. $\mathcal{T}$ carries a surface mesh $\mathcal{S}_{\mathcal{T}}$, whose triangles are faces of tetrahedra of $\mathcal{T}$.
- $\mathcal{T}$ is intended as an approximation of an ideal domain $\Omega \subset \mathbb{R}^3$, and $\mathcal{S}_{\mathcal{T}}$ as an approximation of its boundary $\partial\Omega$.

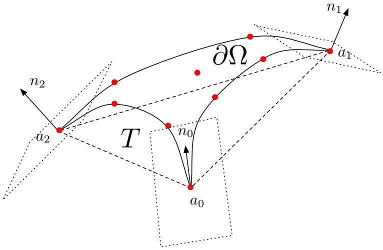


Poor geometric approximation (left) of a domain with smooth boundary (right)

Thanks to local mesh operations, we aim at getting a new, well-shaped mesh $\widetilde{\mathcal{T}}$, whose corresponding surface mesh $\mathcal{S}_{\widetilde{\mathcal{T}}}$ is a good approximation of $\partial\Omega$.

Local remeshing in $3d$: definition of an ideal domain

- In realistic cases, the underlying ideal domain Ω of $\mathcal{T}$ is unknown.
- However, from the knowledge of $\mathcal{T}$ (and $\mathcal{S}_\mathcal{T}$), one can reconstruct geometric features of Ω or $\partial\Omega$: normal vectors at regular points of $\partial\Omega$,...
- These features allow to set rules for the creation of a local parametrization of $\partial\Omega$ around a surface triangle $T \in \mathcal{S}_\mathcal{T}$, e.g. as a Bézier surface.

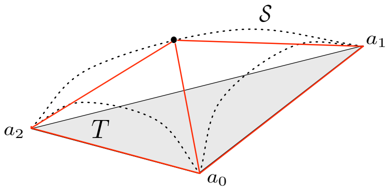


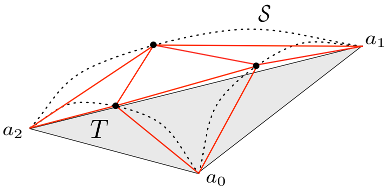
Generation of a cubic Bézier parametrization for the piece of $\partial\Omega$ associated to triangle T, from the approximated geometrical features (normal vectors at nodes).

Local mesh operators: edge splitting

If an edge pq is too long, insert its midpoint m, then split it into two.

- If pq belongs to a surface triangle $T \in \mathcal{S}_{\mathcal{T}}$, the midpoint m is inserted as the midpoint on the local piece of $\partial\Omega$ computed from T. Else, it is merely inserted as the midpoint of p and q.
- An edge may be 'too long' because it is too long when compared to the prescribed size, or because it causes a bad geometric approximation of $\partial\Omega$,...



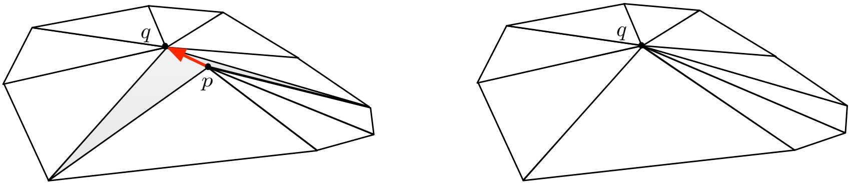


Splitting of one (left) or three (right) edges of triangle T, positioning the three new points on the ideal surface $\mathcal{S}$ (dotted).

Local mesh operators: edge collapse

If an edge *pq* is too short, merge its two endpoints.

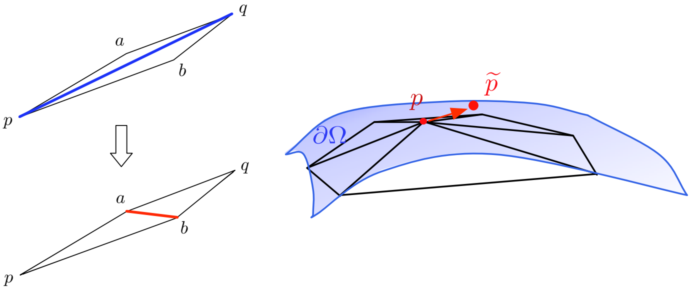
- This operation may deteriorate the geometric approximation of $\partial\Omega$, and even invalidate some tetrahedra: some checks have to be performed to ensure the validity of the resulting configuration.
- An edge may be ‘too short’ because it is too long when compared to the prescribed size, or because it proves unnecessary to a nice geometric approximation of $\partial\Omega$,...



Collapse of point p over q.

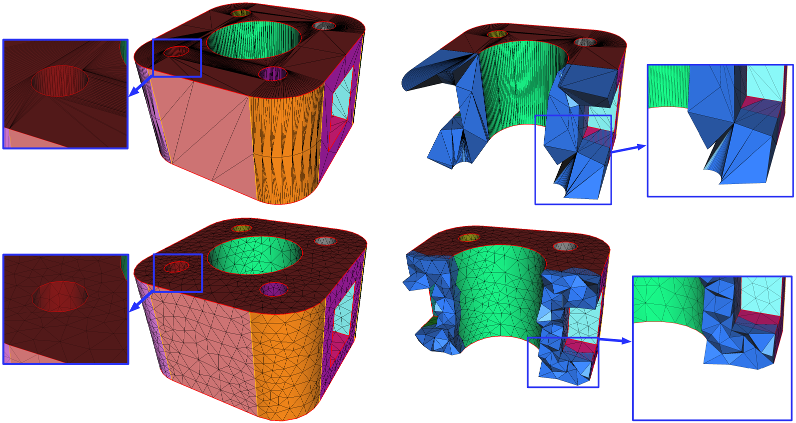
Local mesh operators: edge swap, node relocation

For the sake of enhancement of the global quality of the mesh (or the geometrical approximation of $\partial\Omega$), some connectivities can be swapped, and some nodes can be slightly moved.



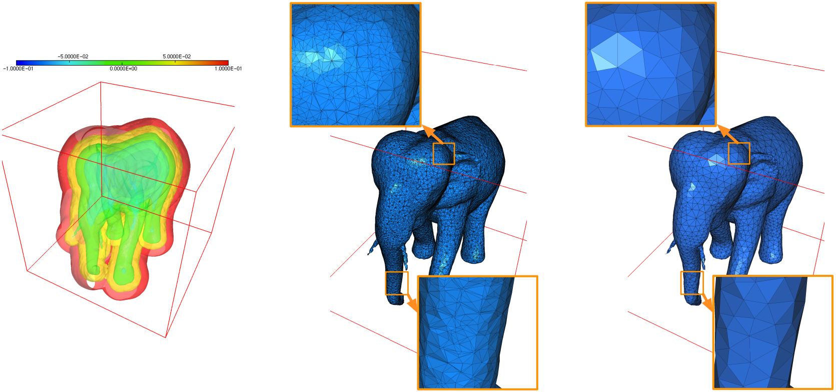
(left) 2d swap of edge pq, creating edge ab ; (right) relocation of node x to $\widetilde{x}$, along the surface.

Local remeshing in $3d$: numerical examples



Mechanical part before (left) and after (right) remeshing.

Local remeshing in $3d$: numerical examples



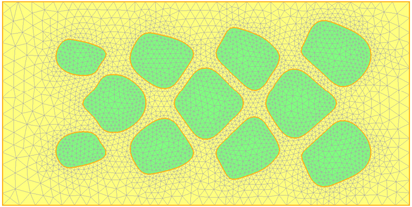
(left) Some isosurfaces of an implicit function defined in a cube, (centre) result after rough discretization in the ambient mesh, (right) result after local remeshing.

Numerical implementation

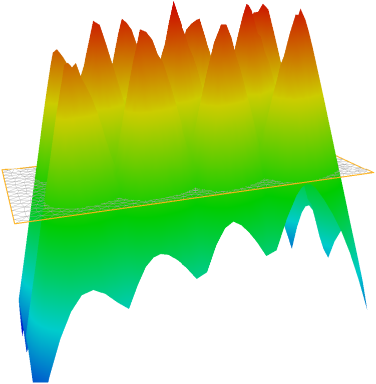
- At each iteration, the shape Ω^n is endowed with an unstructured mesh $\mathcal{T}^n$ of a larger, fixed, bounding box D, in which a mesh of Ω^n explicitly appears as a submesh.
- When dealing with finite element computations on Ω^n, the part of $\mathcal{T}^n$, exterior to Ω^n is simply 'forgotten'.
- When dealing with the advection step, a level set function ϕ^n is generated on the whole mesh $\mathcal{T}^n$, and the level set advection equation is solved on this mesh, to get ϕ^{n+1}.
- From the knowledge of ϕ^{n+1}, a new unstructured mesh $\mathcal{T}^{n+1}$, in which the new shape Ω^{n+1} explicitly appears, is recovered.

The algorithm in motion...

Step 1: Start with the actual shape Ω^n, and generate its signed distance function d_{Ω^n} over D, equipped with the mesh $\mathcal{T}^n$.



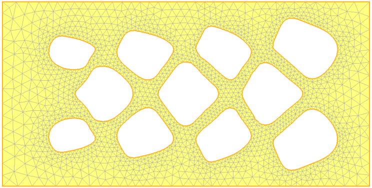
(a) The initial shape



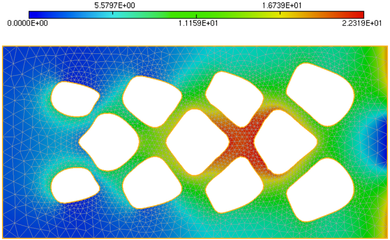
(b) Graph of d_{Ω^n}

The algorithm in motion...

Step 2: "Forget" the exterior of the shape $D \setminus \Omega^n$, and perform the computation of the shape gradient $J'(\Omega^n)$ on (the mesh of) Ω^n.



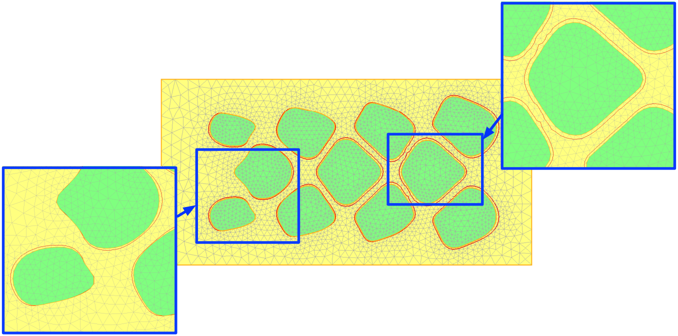
(a) The "interior mesh"



(b) Computation of $J'(\Omega^n)$

The algorithm in motion...

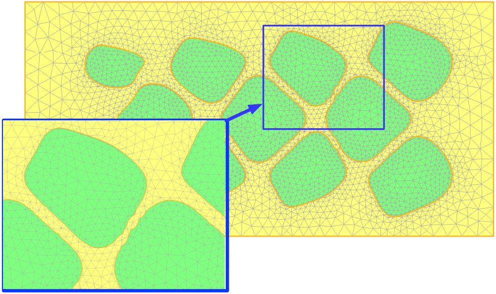
Step 3: "Remember" the whole mesh $\mathcal{T}^n$ of D. Extend the velocity field $J'(\Omega^n)$ to the whole mesh, and advect d_{Ω^n} along $J'(\Omega^n)$ for a (small) time step τ^n. A new level set function ϕ^{n+1} is obtained on $\mathcal{T}^n$, corresponding to the new shape Ω^{n+1}.



The shape Ω^n, discretized in the mesh (in yellow), and the "new", advected 0-level set (in red).

The algorithm in motion...

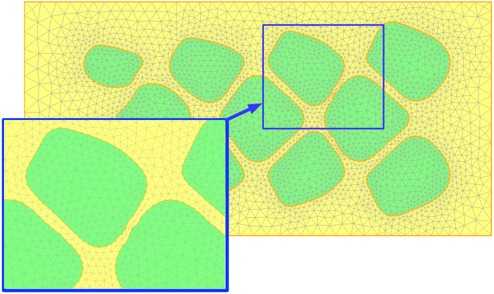
Step 4: To close the loop, the 0 level set of ϕ^{n+1} is explicitly discretized in mesh $\mathcal{T}^n$. As expected, roughly "breaking" this line generally yields a very ill-shaped mesh.



Rough discretization of the 0 level set of ϕ^{n+1} into $\mathcal{T}^n$; the resulting mesh of D is ill-shaped.

The algorithm in motion...

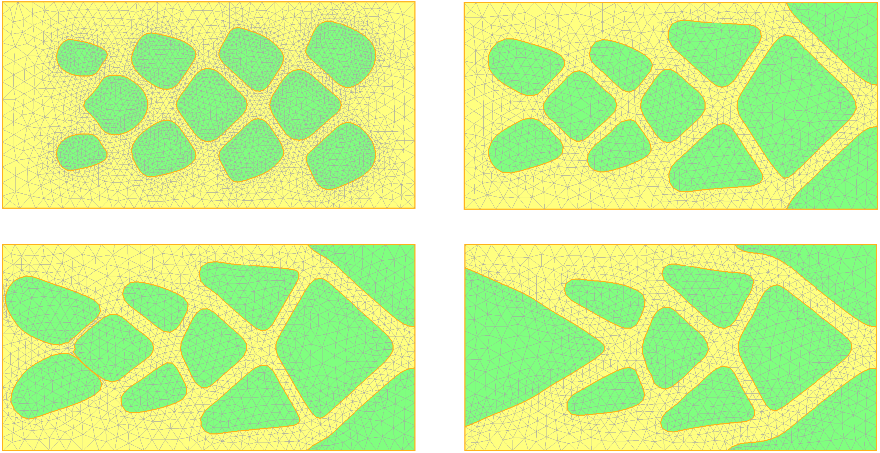
The mesh modification step is then performed, so as to enhance the overall quality of the mesh according to the geometry of the shape. $\mathcal{T}^{n+1}$ is eventually obtained.



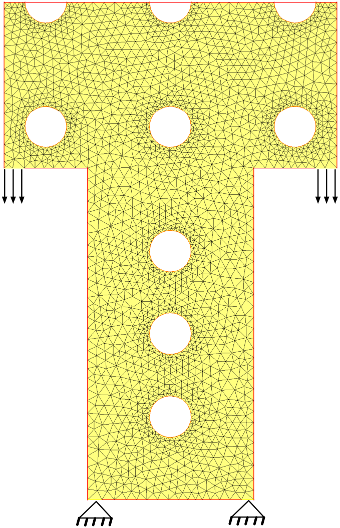
Quality-oriented remeshing of the previous mesh ends with the new, well-shaped mesh $\mathcal{T}^{n+1}$ of D in which Ω^{n+1} is explicitly discretized.

The algorithm in motion...

Go on as before, until convergence (discretize the 0-level set in the computational mesh, clean the mesh,...).



Numerical results: $2d$ optimal mast



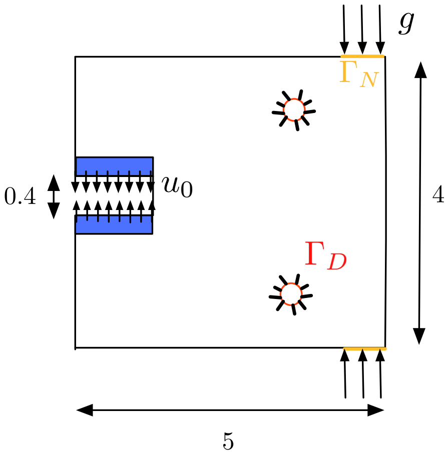
The 'benchmark' two-dimensional optimal mast test case.

- Minimization of the compliance

$$C(\Omega) = \int_{\Omega} Ae(u_{\Omega}) : e(u_{\Omega})\, dx.$$

- A volume constraint is enforced by means of a fixed Lagrange multiplier.

Numerical results: 2d gripping mechanism



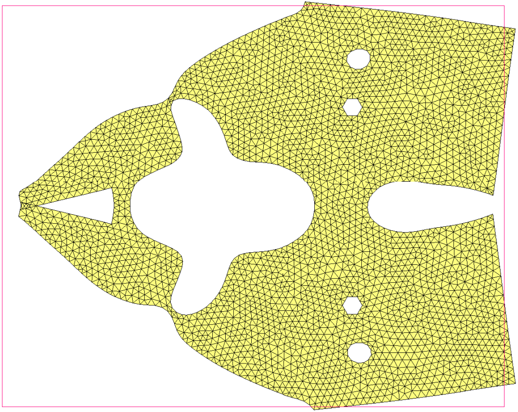
Device of a gripping mechanism.

The least-square criterion is minimized:

$$D(\Omega) = \int_\Omega k(x) ||u_\Omega - u_0||^2 \, dx,$$

where k is a the characteristic function of a region near the jaws, and u_0 is cooked so that the jaws close.

Numerical results: deformation of the optimal grip



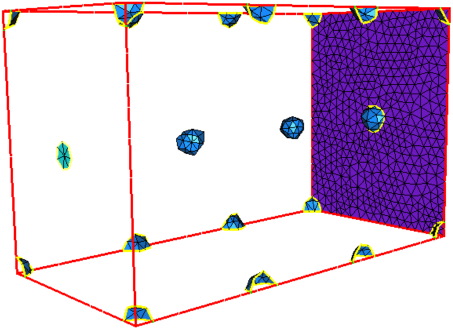
Numerical results: 3d cantilever

The ‘benchmark’ three-dimensional cantilever test case.

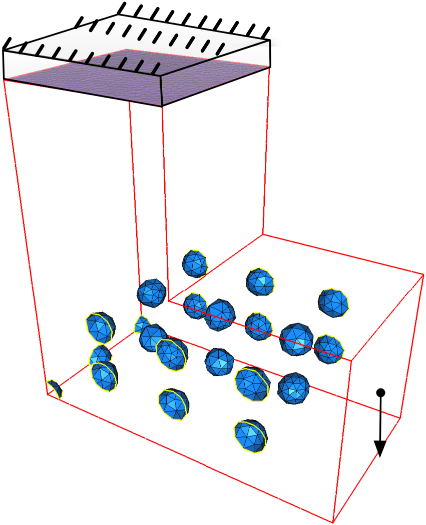
- Minimization of the compliance

$$C(\Omega) = \int_\Omega Ae(u_\Omega) : e(u_\Omega)\, dx.$$

- A volume constraint is enforced by means of a fixed Lagrange multiplier.



Numerical results: $3d$ L-Beam



Optimal design of a $3d$ L-shaped beam.

- Minimization of a stress-based criterion

$$S(\Omega) = \int_{\Omega} k(x) ||\sigma(u_{\Omega})||^2 \, dx,$$

where k is a weight factor, and $\sigma(u) = Ae(u)$ is the stress tensor.

- A volume constraint is enforced by means of a fixed Lagrange multiplier.

Thank you !