A deterministic approximation method in shape optimization under random uncertainties

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Foreword: uncertainties in structural optimization

- Mechanical systems rely on data, e.g. the loads, the properties of a constituent material, or the geometry of the system itself.
- In concrete situations, such data are plagued with uncertainties because:
  - they may be available only through (error-prone) measurements,
  - they may be altered with time (wear) and conditions of the ambient medium.
- The performances of structures are very sensitive to small perturbations of data.

⇒ Need to somehow anticipate uncertainties when designing and optimizing shapes.

A disk brake system

A worn out brake pad

1. Introduction and definitions
   - Foreword
   - The main ideas in an abstract framework

2. Applications in shape optimization
   - Shape optimization of elastic structures
   - Shape optimization under random loads
   - Shape optimization under uncertainties on the elastic material
   - Shape optimization under geometric uncertainties

The main ideas in an abstract framework (I)

- $\mathcal{U}_{ad} \subset \mathcal{H}$ is a set of admissible designs h (e.g. the thickness of a plate, the geometry of a shape).

- $(\mathcal{P}, ||\cdot||)$ is a Banach space of data f (forces, parameters of a material).

- The performances of a design h are evaluated in terms of a cost $\mathcal{C} \equiv \mathcal{C}(f, u_{h,f})$, which involves a state $u_{h,f}$, solution to a physical system:

$$\mathcal{A}(h)u_{h,f} = b(f),$$

  where f acts on the right-hand side for simplicity.

- The data are uncertain, and read:

$$f = f_0 + \widehat{f}(\omega),$$

  where f_0 is a mean value, and ω is an event, in an abstract probability space $(\mathcal{O}, \mathcal{F}, \mathbb{P})$.

The main ideas in an abstract framework (II)

There are two different settings to deal with uncertainties:

- *Worst-case approach:* When only a maximum bound $||\widehat{f}||_{\mathcal{P}} \leq m$ is available on perturbations, one considers the worst-case functional:
$$\mathcal{J}_{wc}(h) = \sup_{||\widehat{f}||_{\mathcal{P}} \leq m} \mathcal{C}(f_0 + \widehat{f}, u_{h, f_0 + \widehat{f}}).$$
Main drawback: Pessimistic approach, which may yield designs with unnecessarily bad nominal performances.

- *Probabilistic approach:* When information is available on the moments of the uncertainties, one may try to minimize the mean value:
$$\mathcal{M}(h) = \int_{\mathcal{O}} \mathcal{C}(f_0 + \widehat{f}(\omega), u_{h, f_0 + \widehat{f}(\omega)})\, \mathbb{P}(d\omega),$$
or a failure probability:
$$\mathcal{P}(h) = \mathbb{P}\left(\left\{ \omega \in \mathcal{O},\, \mathcal{C}\left(f_0 + \widehat{f}(\omega), u_{h, f_0 + \widehat{f}(\omega)} \right) > \alpha \right\} \right).$$

The main ideas in an abstract framework (III)

Working hypotheses:

- Perturbations are small: depending on the context, this may mean:
  - $\widehat{f} \in L^\infty(\mathcal{O}, \mathcal{P})$: all the realizations $\widehat{f}(\omega) \in \mathcal{P}$ are small.
  - $\widehat{f} \in L^p(\mathcal{O}, \mathcal{P})$, for $p < \infty$: $\widehat{f}$ may have unprobably large realizations.
- Perturbations are finite-dimensional:

$$\boxed{\widehat{f}(\omega) = \sum_{i=1}^{N} f_i \xi_i(\omega),}$$

where $f_i \in \mathcal{P}$, and the ξ_i are normalized, uncorrelated random variables:

$$\int_{\mathcal{O}} \xi_i(\omega)\mathbb{P}(d\omega) = 0, \quad \int_{\mathcal{O}} \xi_i(\omega)\xi_j(\omega)\,\mathbb{P}(d\omega) = \delta_{i,j}.$$

Example: $\widehat{f}$ is obtained as a truncated Karhunen-Loève expansion.

The main ideas in an abstract framework (IV)

Strategy:

- Calculate approximate functionals $\widetilde{\mathcal{M}}(h)$ and $\widetilde{\mathcal{P}}(h)$, which are
  - deterministic: no random variable or probabilistic integral is involved.
  - consistent with their exact counterparts, i.e. the differences $|\mathcal{M}(h) - \widetilde{\mathcal{M}}(h)|$ and $|\mathcal{P}(h) - \widetilde{\mathcal{P}}(h)|$ are 'small'.
- Calculate their derivatives $\widetilde{\mathcal{M}}'(h)(\widehat{h})$ and $\widetilde{\mathcal{P}}'(h)(\widehat{h})$,
- Minimize the approximate functionals $\widetilde{\mathcal{M}}(h)$ and $\widetilde{\mathcal{P}}(h)$ (under constraints), by using the expressions of their derivatives.

  e.g. relying on a steepest-descent algorithm.

The main ideas in an abstract framework (V)

Use the smallness of perturbations to perform a first- or second-order Taylor expansion of the mappings $f \mapsto u_{h,f}$ and $f \mapsto \mathcal{C}(f, u_{h,f})$ around f_0:

$$\boxed{u_{h,f_0+\widehat{f}} \approx u_h + u_h^1(\widehat{f}) + \frac{1}{2} u_h^2(\widehat{f}, \widehat{f}),}$$

$$\text{where} \quad \mathcal{A}(h) u_h^1(\widehat{f}) = \frac{\partial b}{\partial f}(f_0)(\widehat{f}), \text{ and } \mathcal{A}(h) u_h^2(\widehat{f}, \widehat{f}) = \frac{\partial^2 b}{\partial f^2}(f_0)(\widehat{f}, \widehat{f}).$$

$$\boxed{\mathcal{C}(f_0 + \widehat{f}, u_{h,f_0+\widehat{f}}) \approx \mathcal{C}(f_0, u_h) + \mathcal{L}_h(\widehat{f}) + \frac{1}{2} \mathcal{B}_h(\widehat{f}, \widehat{f}),}$$

where the linear and bilinear forms $\mathcal{L}_h$ and $\mathcal{B}_h$ read:

$$\mathcal{L}_h(\widehat{f}) = \frac{\partial \mathcal{C}}{\partial f}(f_0, u_h)(\widehat{f}) + \frac{\partial \mathcal{C}}{\partial u}(f_0, u_h)(u_h^1(\widehat{f})),$$

$$\begin{aligned}\mathcal{B}_h(\widehat{f}, \widehat{f}) = {} & \frac{\partial^2 \mathcal{C}}{\partial f^2}(f_0, u_h)(\widehat{f}, \widehat{f}) + 2 \frac{\partial^2 \mathcal{C}}{\partial f \partial u}(f_0, u_h)(\widehat{f}, u_h^1(\widehat{f})) \\ & + \frac{\partial^2 \mathcal{C}}{\partial u^2}(f_0, u_h)(u_h^1(\widehat{f}), u_h^1(\widehat{f})) + \frac{\partial \mathcal{C}}{\partial u}(f_0, u_h)(u_h^2(\widehat{f}, \widehat{f})).\end{aligned}$$

Approximation of moment functionals

- Replacing the cost with its second-order expansion gives rise to the approximate mean-value functional:

$$\widetilde{\mathcal{M}}(h) = \mathcal{C}(f_0, u_h) + \int_{\mathcal{O}} \mathcal{L}_h(\widehat{f}(\omega))\, \mathbb{P}(d\omega) + \frac{1}{2} \int_{\mathcal{O}} \mathcal{B}_h(\widehat{f}(\omega), \widehat{f}(\omega))\, \mathbb{P}(d\omega).$$

- Using the structure of perturbations $\widehat{f}(\omega) = \sum_{i=1}^{N} f_i \xi_i(\omega)$, it comes:

$$\boxed{\widetilde{\mathcal{M}}(h) = \mathcal{C}(f_0, u_h) + \frac{1}{2} \sum_{i=1}^{N} \mathcal{B}_h(f_i, f_i),}$$

a formula which involves the calculation of the $N+2$ 'reduced states':

$$u_h, \;\; u_{h,i} := u_h^1(f_i), \; (i = 1, \dots, N), \; \text{ and } u_h^2 := \sum_{i=1}^{N} u_h^2(f_i, f_i).$$

- This approach can be applied to other moments of $\mathcal{C}$, e.g. its variance:

$$\mathcal{V}(h) = \int_{\mathcal{O}} \left(\mathcal{C}(f_0 + \widehat{f}(\omega), u_{h, f_0 + \widehat{f}(\omega)}) - \mathcal{M}(h) \right)^2 \mathbb{P}(d\omega).$$

Approximation of failure probabilities (I)

Additional hypotheses: The random variables ξ_i are:

- independent,
- Gaussian, i.e. their cumulative distribution function is:

$$\mathbb{P}\left(\{\omega \in \mathcal{O},\ \xi_i(\omega) < \alpha\}\right) = \Phi(\alpha) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{\frac{-\xi^2}{2}}\, d\xi.$$

The (exact) failure probability reads:

$$\mathcal{P}(h) = \frac{1}{(2\pi)^{N/2}} \int_{\mathcal{D}(h)} e^{-\frac{|\xi|^2}{2}}\, d\xi,$$

where the failure region $\mathcal{D}(h)$ is:

$$\mathcal{D}(h) = \left\{ \xi \in \mathbb{R}^N,\ \mathcal{C}\left(f_0 + \sum_{i=1}^{N} f_i \xi_i,\ u_{h, f_0 + \sum_{i=1}^{N} f_i \xi_i} \right) > \alpha \right\}.$$

Idea: Approximate the failure region with:

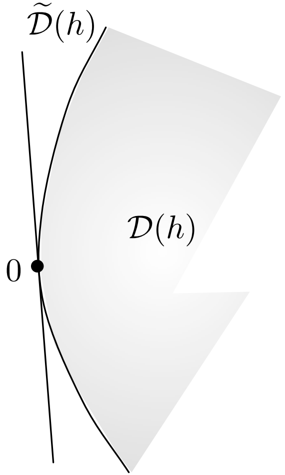
$$\widetilde{\mathcal{D}}(h) = \left\{ \xi \in \mathbb{R}^N,\ \mathcal{C}(f_0, u_h) + \sum_{i=1}^{N} \mathcal{L}_h(f_i)\xi_i > \alpha \right\}.$$

The approximate failure probability

$$\widetilde{\mathcal{P}}(h) = \frac{1}{(2\pi)^{N/2}} \int_{\widetilde{\mathcal{D}}(h)} e^{-\frac{|\xi|^2}{2}}\, d\xi$$

can be calculated in closed form as:

$$\widetilde{\mathcal{P}}(h) = \Phi\left(-\frac{\alpha - \mathcal{C}(f_0, u_h)}{\sqrt{\sum_{i=1}^{N} \mathcal{L}_h(f_i)^2}} \right).$$



The usual linear elasticity setting

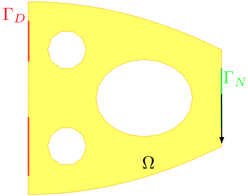
A shape is a bounded domain $\Omega \subset \mathbb{R}^d$, which is

- fixed on a part Γ_D of its boundary,
- submitted to surface loads g, applied on $\Gamma_N \subset \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$.

The displacement vector field $u_\Omega \in H^1_{\Gamma_D}(\Omega)^d$ is governed by the linear elasticity system:

$$\left\{ \begin{array}{rcll} -\mathrm{div}(Ae(u_\Omega)) & = & f & \text{in } \Omega \\ u_\Omega & = & 0 & \text{on } \Gamma_D \\ Ae(u_\Omega)n & = & g & \text{on } \Gamma_N \\ Ae(u_\Omega)n & = & 0 & \text{on } \Gamma := \partial\Omega \setminus (\Gamma_D \cup \Gamma_N) \end{array} \right. ,$$

where $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the strain tensor, and A is the Hooke's law of the material.



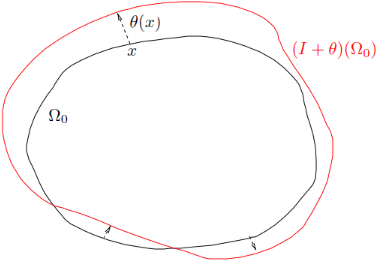
A 'Cantilever'

The deformed cantilever

Hadamard's boundary variation method describes variations of a reference, Lipschitz domain Ω of the form:

$$\Omega \to \Omega_\theta := (I + \theta)(\Omega),$$

for 'small' $\theta \in W^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right)$.



In practice:

- We restrict to a set of admissible shapes:

$$\mathcal{U}_{ad} := \left\{\Omega \subset \mathbb{R}^d \text{ is open, bounded and Lipschitz, } \Gamma_D \cup \Gamma_N \subset \partial\Omega\right\}.$$

- Deformations θ are assumed within the admissible set:

$$\Theta_{ad} := \left\{\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d), \text{ such that } \theta = 0 \text{ on } \Gamma_D \cup \Gamma_N\right\}.$$

Definition 1.

Given a smooth domain Ω, a functional $J(\Omega)$ of the domain is shape differentiable at Ω if the function

$$W^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right) \ni \theta \mapsto J(\Omega_\theta)$$

is Fréchet-differentiable at 0*, i.e. the following expansion holds around* 0*:*

$$J(\Omega_\theta) = J(\Omega) + J'(\Omega)(\theta) + o\left(||\theta||_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)}\right).$$

Shape derivatives can be computed using techniques from optimal control; in the case of 'many' functions of the domain $J(\Omega)$, they enjoy the structure:

$$\boxed{J'(\Omega)(\theta) = \int_\Gamma v_\Omega \, \theta \cdot n \, ds,}$$

where v_Ω is a scalar field depending on u_Ω, and possibly on an adjoint state p_Ω.

The generic numerical algorithm

This shape gradient provides a natural descent direction for functional J: *for instance*, defining θ as

$$\theta = -v_\Omega n$$

yields, for $t > 0$ sufficiently small (*to be found numerically*):

$$J((I + t\theta)(\Omega)) = J(\Omega) - t\int_\Gamma v_\Omega^2 ds + o(t) < J(\Omega)$$

Gradient algorithm: For $n = 0, \dots$ convergence,

1. Compute the solution u_{Ω^n} (and p_{Ω^n}) of the elasticity system on Ω^n.
2. Compute the shape gradient $J'(\Omega^n)$ thanks to the previous formula, and infer a descent direction θ^n for the cost functional.
3. Advect the shape Ω^n according to θ^n, so as to get $\Omega^{n+1} := (I + \theta^n)(\Omega^n)$.

Shape optimization under random loads (I)

- We consider uncertainties on the body forces f ($\mathcal{P} = L^2(\mathbb{R}^d)^d$):

$$f(x) = f_0(x) + \widehat{f}(x,\omega), \text{ where } \widehat{f}(x,\omega) = \sum_{i=1}^{N} f_i(x)\, \xi_i(\omega) \in L^2(\mathcal{O}, L^2(\mathbb{R}^d)^d).$$

- The cost function is of the form:

$$\mathcal{C}(f,\Omega) = \int_\Omega j(f, u_{\Omega,f})\, dx,$$

where $j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is smooth, satisfies growth conditions, and $u_{\Omega,f} \in H^1_{\Gamma_D}(\Omega)^d$ is the solution u of:

$$\left\{ \begin{array}{rcll} -\text{div}(Ae(u)) & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \Gamma_D \\ Ae(u)n & = & 0 & \text{on } \Gamma_N \\ Ae(u)n & = & 0 & \text{on } \Gamma \end{array} \right. .$$

Shape optimization under random loads (II)

The approximate mean value functional reads:

$$\widetilde{\mathcal{M}}(\Omega) = \int_\Omega j(f_0, u_\Omega)\, dx + \frac{1}{2}\sum_{i=1}^{N}\int_\Omega \nabla_f^2 j(f_0, u_\Omega)(f_i, f_i)\, dx$$

$$+ \sum_{i=1}^{N}\int_\Omega \nabla_f \nabla_u j(f_0, u_\Omega)(f_i, u^1_{\Omega,i})\, dx + \frac{1}{2}\sum_{i=1}^{N}\int_\Omega \nabla_u^2 j(f_0, u_\Omega)(u^1_{\Omega,i}, u^1_{\Omega,i})\, dx,$$

the $u^1_{\Omega,i}$ being the solutions of: $\left\{\begin{array}{cl} -\mathrm{div}(Ae(u)) = f_i & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ Ae(u)n = 0 & \text{on } \Gamma_N \\ Ae(u)n = 0 & \text{on } \Gamma \end{array}\right. .$

Proposition 1.

Under the additional assumption that $\widehat{f} \in L^3(\mathcal{O}, L^3(\mathbb{R}^d)^d)$, there exists a constant $C > 0$ (depending on Ω) such that:

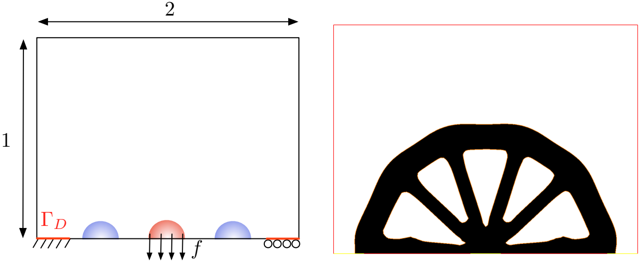
$$|\widetilde{\mathcal{M}}(\Omega) - \mathcal{M}(\Omega)| \leq C||\widehat{f}||^3_{L^3(\mathcal{O}, L^3(\mathbb{R}^d)^d)}.$$

Optimization of a bridge under random loads (I)

- The cost function is the compliance of shapes:

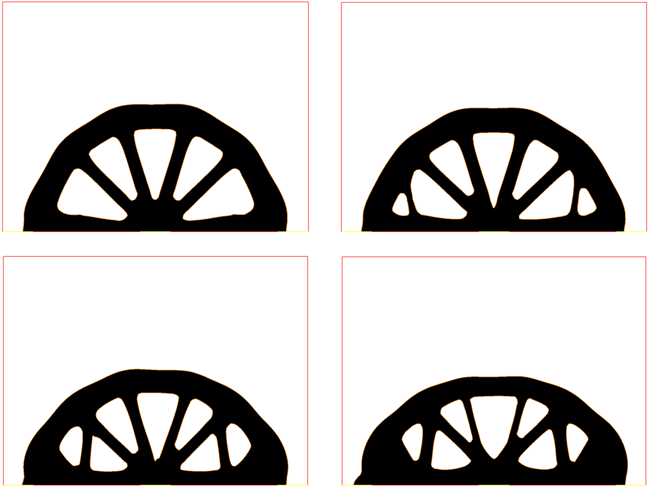
$$\mathcal{C}(f,\Omega) = \int_\Omega f \cdot u_{\Omega,f}\, dx = \int_\Omega Ae(u_{\Omega,f}) : e(u_{\Omega,f})\, dx.$$

- Two load scenarii $f_1, f_2 = (0, -m)$ are supported in the blue spots.
- The considered objective function is: $\mathcal{L}(\Omega) = \widetilde{\mathcal{M}}(\Omega) + \delta\sqrt{\widetilde{\mathcal{V}}(\Omega)}$.
- A constraint $\mathrm{Vol}(\Omega) = V_T$ is enforced by an augmented Lagrangian algorithm.



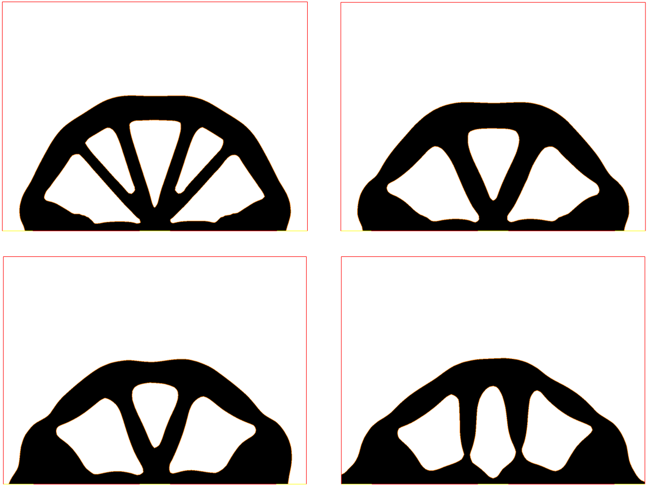
(Left) The bridge test case, (right) optimal shape in the unperturbed situation.

Optimization of a bridge under random loads (II)



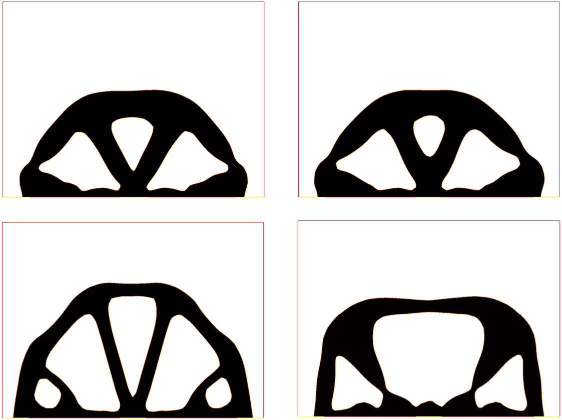
Optimal shapes for $\delta = 0$ *and* $m = 1, 2, 5, 10$.

Optimization of a bridge under random loads (III)



Optimal shapes for $\delta = 3$ *and* $m = 1, 2, 5, 10$.

Comparison with the worst-case approach



Optimal shapes for the linearized worst-case design approach with $m = 1, 2, 5, 10$.

Observation: The optimal shapes for the probabilistic functionals show systematically better nominal performances than their worst-case counterparts.

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Optimization under material uncertainties

- Perturbations over the Young's modulus E of the material are considered:

$$E = E_0 + \widehat{E}(x,\omega), \text{ where } \widehat{E}(x,\omega) = \sum_{i=1}^{N} E_i(x)\xi_i(\omega) \in L^\infty(\mathcal{O}, L^\infty(\mathbb{R}^d)).$$

- The cost function is of the form $\mathcal{C}(\Omega, E) = \int_\Omega j(u_{\Omega,E})\, dx$, where:

$$\left\{ \begin{array}{rcl} -\text{div}(A(E)e(u_\Omega)) & = & 0 \quad \text{in } \Omega \\ u_\Omega & = & 0 \quad \text{on } \Gamma_D \\ A(E)e(u_\Omega)n & = & g \quad \text{on } \Gamma_N \\ A(E)e(u_\Omega)n & = & 0 \quad \text{on } \Gamma \end{array} \right. .$$

- Minimization of the approximate mean value of $\mathcal{C}$:

$$\widetilde{\mathcal{M}}(\Omega) = \int_\Omega j(u_\Omega)\, dx + \frac{1}{2}\sum_{i=1}^{N}\int_\Omega \nabla^2 j(u_\Omega)(u^1_{\Omega,i}, u^1_{\Omega,i})\, dx + \frac{1}{2}\int_\Omega \nabla j(u_\Omega)\cdot u^2_\Omega\, dx,$$

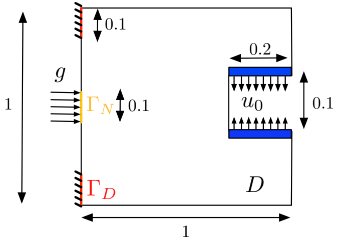
where the $u^1_{\Omega,i}$, $i = 1, ..., N$, and u^2_Ω are the reduced states.

Optimization of a grip under material uncertainties (I)

The cost function is

$$\mathcal{C}(\Omega, E) = \int_{\Omega} k(x)|u_{\Omega,E} - u_0|^2 \, dx,$$

where k is a localization factor, and u_0 is a target displacement, cooked so that the jaws close.



Setting of the gripping mechanism example.

Optimization of a grip under material uncertainties (II)

- The perturbations $\widehat{E}(x,\omega)$ are known via their two-point correlation function:

$$\mathrm{Cor}(\widehat{E})(x,y) := \int_{\mathcal{O}} \widehat{E}(x,\omega)\widehat{E}(y,\omega)\,\mathbb{P}(d\omega) = \beta^2 e^{-\frac{|x-y|}{\ell}},$$

where β is a scaling factor, and ℓ is a characteristic length.

- A Karhunen-Loève expansion of $\widehat{E}$ is performed, then truncated:

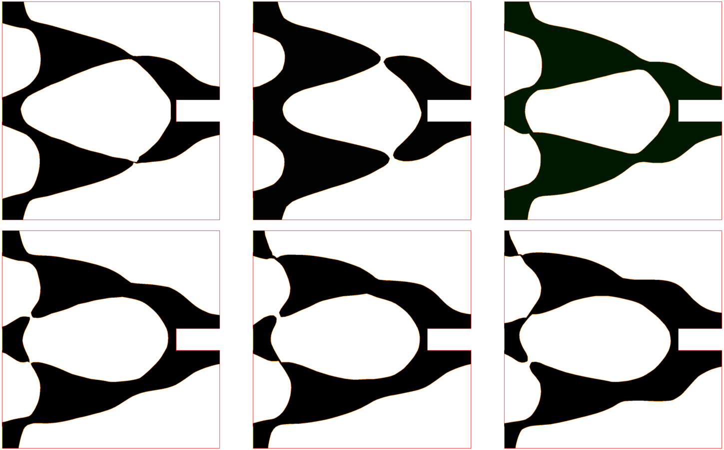
$$\widehat{E}(x,\omega) \approx \sum_{i=1}^{N} \sqrt{\lambda_i} f_i(x)\xi_i(\omega),$$

where the (λ_i, f_i) are the eigenpairs of the Hilbert-schmidt operator

$$L^2(D) \ni f \mapsto \int_D \mathrm{Cor}(\widehat{E})(x,y) f(y)\,dx \in L^2(D),$$

and the $\xi_i(\omega) = \int_D \widehat{E}(x,\omega) f_i(x)\,dx$ are normalized and uncorrelated random variables.

Optimization of a grip under material uncertainties (III)



Optimal shapes associated to values of $\beta = 0, 0.5, 1, 1.5, 2, 2.5$.

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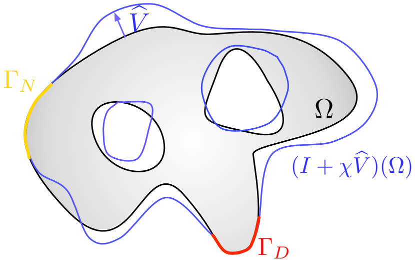
Modelling geometric uncertainties

Perturbations of a shape $\Omega \in \mathcal{U}_{ad}$ are considered with the structure:

$$\Omega \longmapsto (I + \chi(x)\widehat{v}(x,\omega)n_\Omega(x))(\Omega),$$

where:

- χ is a cutoff function, vanishing on $\Gamma_D \cup \Gamma_N$,
- n_Ω is (an extension of) the normal vector to $\partial\Omega$,
- The scalar field $\widehat{v} \in L^\infty(\mathcal{O}, \mathcal{C}^{2,\infty}(\mathbb{R}^d))$ arises as $\widehat{v}(x,\omega) = \sum_{i=1}^N v_i(x)\xi_i(\omega)$.



Perturbation of Γ by a vector field $\widehat{V}$.

Optimization of a L-beam under geometric uncertainties

- The cost function is of the form:

$$\mathcal{C}(\Omega) = \int_\Omega j(\sigma(u_\Omega))\, dx,$$

where $\sigma(u) = Ae(u)$ is the stress tensor.

- The approximate variance functional reads:

$$\widetilde{\mathcal{V}}(\Omega) = \sum_{i=1}^{N} a_{\Omega,i}^2 \text{ with } a_{\Omega,i} = \int_\Gamma \left(j(\sigma(u_\Omega)) + Ae(u_\Omega) : e(p_\Omega) - f \cdot p_\Omega \right) v_i \, ds, \tag{1}$$

and the adjoint state $p_\Omega \in H^1_{\Gamma_D}(\Omega)^d$ is the solution of:

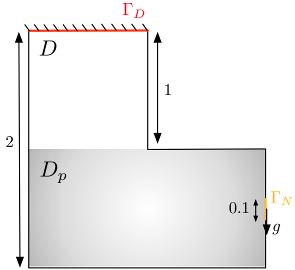
$$\begin{cases} -\mathrm{div}(Ae(p)) = \mathrm{div}(A\frac{\partial j}{\partial \sigma}(\sigma(u_\Omega))) & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_D, \\ Ae(p)n = -A\frac{\partial j}{\partial \sigma}(\sigma(u_\Omega))n & \text{on } \Gamma \cup \Gamma_N. \end{cases}$$

Optimization of a L-beam under geometric uncertainties

- Perturbations occur on a subregion $D_p \subset D$; their correlation function is:

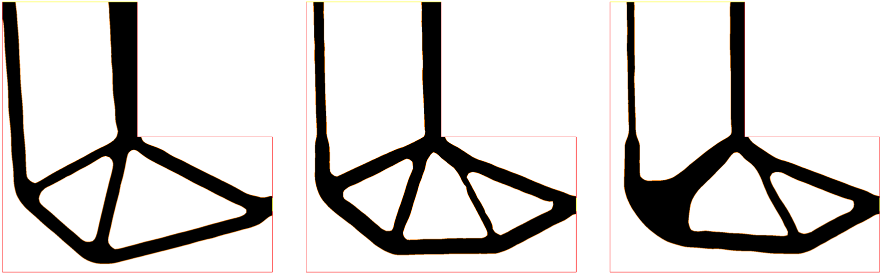
$$\mathrm{Cor}(\widehat{v})(x,\omega) = \beta^2 e^{-\frac{|x-y|}{\ell}}.$$

- The cost function is $\mathcal{C}(\Omega) = \int_\Omega ||\sigma(u_\Omega)||^5 \, dx$, and the objective $\mathcal{C}(\Omega) + \delta\sqrt{\widetilde{\mathcal{V}}(\Omega)}$ is minimized under a volume constraint.



Details of the L-shaped beam test-case.

Optimization of a L-beam under geometric uncertainties



Optimal shapes in the minimization of the stress-based criterion, where the parameter δ *equals (from the left to the right)* $0, 0.5, 2$.

Thank you !

Thank you for your attention!

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