An introduction to shape and topology optimization

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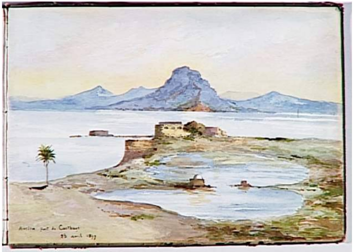
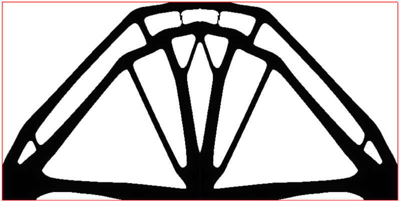
Foreword

- Shape optimization is about the minimization of an objective function $J(\Omega)$, depending on a shape Ω of $\mathbb{R}^2$ or $\mathbb{R}^3$, under certain constraints.
- Such problems have come up early in the history of sciences, and they are ubiquitous in nature.
- Nowadays, they arouse a tremendous enthusiasm in engineering.
- They are at the interface between mathematics, physics, mechanical engineering and computer science.
- Shape optimization is a burning field of research!





Contents

- The present course is composed of
  - 12 lectures, covering the main theoretical aspects;
  - A set of appendices, at the end of the slides, where basic notions are recalled, and topics related to those of the course are broached.
  - A set of codes, dedicated to the numerical implementation of basic shape and topology optimization algorithms, written in `FreeFem++`.
- All the material for the course (slides of the lectures and commented, demonstration programs) is available on the webpage of the course:

  `https://ljk.imag.fr/membres/Charles.Dapogny/coursoptim.html`

- For any comment, suggestion or question, do not hesitate to contact either of the instructors:

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Part I

Introduction, history and generalities about shape optimization

Dido's problem (I)

- Dido's problem is reported in the myth of the foundation of Carthage by Phœnician princess Dido, in 814 B.C. (cf. Virgil's *Aeneid*, $\approx$ 100 B.C.).

- Dido fled from Tyr (actual Lebanon) after her husband got murdered by her brother Pygmalion.

- Accompanied by her fellows, she reached the Tunisian shore, where she required a land from local king Jarbas...

- *... They came to this spot, where to-day you can behold the mighty*
  Battlements and the rising citadel of New Carthage,
  And purchased a site, which was named 'Bull's Hide' after the bargain
  By which they should get as much land as they could enclose with a bull's hide...

[Virgil, Aeneid]

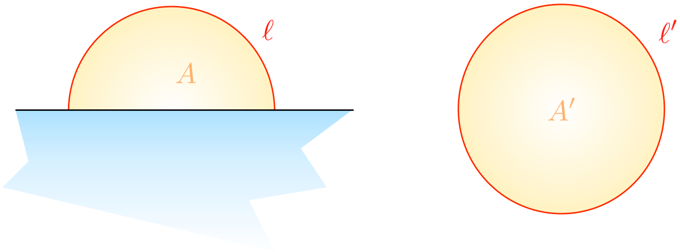
Dido's problem (II)

W. Turner: "Dido Building Carthage" or "The Rise of the Carthaginian Empire" (1815).

Dido's problem (III)

Using modern terminology:

How to surround the largest possible area A with a given contour length ℓ?



(Left) The solution to Dido's problem in the case where the surrounded domain is limited by the sea; (right) an "unconstrained" version of Dido's problem.

The isoperimetric inequality (I)

- Without knowing it, Queen Dido had just discovered the isoperimetric inequality:

Let $\Omega \subset \mathbb{R}^2$ be a domain with "smooth enough" boundary $\partial\Omega$. Let A be the area covered by Ω, and ℓ be the length of $\partial\Omega$. Then,

$$4\pi A \leq \ell^2,$$

where equality holds if and only if Ω is a disk.

- Equivalently,

  - Among all domains $\Omega \subset \mathbb{R}^2$ with prescribed perimeter, that with maximum area is the disk.
  - Among all domains $\Omega \subset \mathbb{R}^2$ with prescribed area, that with minimum perimeter is the disk.

- Multiple variants of this problem exist.

  Example: One may impose that the boundary of Ω should contain a non optimizable region (a segment).

The isoperimetric inequality (II)

- However intuitive, the first proof of this fact was obtained in 1838 by J. Steiner, ... but the proof was false! Actually, J. Steiner proved that, assuming that an optimal shape exist... it should then be a disk.

- However, many shape optimization problems do not have a solution, for deep mathematical and physical reasons.

- Only in 1860 did K. Weierstrass complete the proof of the isoperimetric inequality in two dimensions.

- The isoperimetric inequality holds in more general contexts, for instance in three space dimensions (H. Schwarz, 1884):

> Among all domains $\Omega \subset \mathbb{R}^3$ with prescribed volume, that with minimum surface is the ball.

Another occurrence of the isoperimetric inequality

Medieval cities often have a circular shape so as to minimize the perimeter of the necessary fortifications around a given population (i.e. their area).

Map of Paris during the Dark Ages.

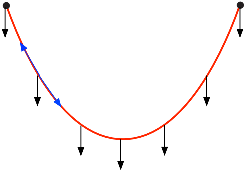
Part I

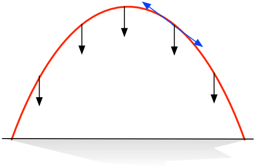
Introduction, history and generalities about shape optimization

The quest of architects for optimal design (I)

- Structural optimization has long been a central concern in architectural design.
- One crucial step towards modern design: the Hooke's theorem (1675)

"As hangs the flexible chain, so but inverted will stand the rigid arch."





(Left) A chain hanging in equilibrium under the action of gravity and tension forces; (right) an arch standing in equilibrium under gravity and compression forces.

The quest of architects for optimal design (II)

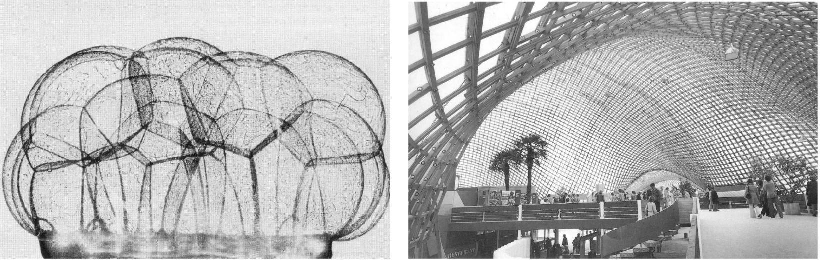
- A. Gaudi sketched the plans of the church of the Colònia Güell (1889-1914) by relying on a funicular model so as to determine a stable assembly of columns and vaults.

(Left) Gaudi's experimental device, (right) model of the Colònia Güell ***(Photo credits: http://www.gaudidesigner.com)***.

The quest of architects for optimal design (III)

Since then, optimal design concepts have attracted the attention of world-renowned architects: Heinz Isler, Gustave Eiffel, Frei Otto, etc.

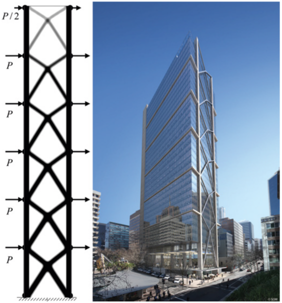
- They allow to model complex geometric criteria, related to the æstethics, the constructibility, and the mechanical performance of structures.
- Optimized shapes with respect to mechanical considerations have often "elegant" outlines: their organic nature is very appreciated by architects.



(Left) A soap-film structure, coined by Frei Otto, (right) interior view of the Manheim Garden festival.

The quest of architects for optimal design (IV)

- Nowadays, modern structural optimization techniques are currently employed for the design of large-scale buildings.



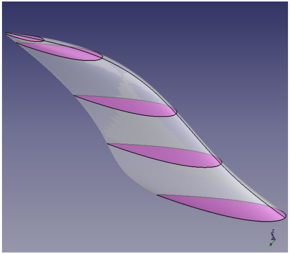
(Left) Entrance of the Qatar National Convention Center, in Doha [Sasaki et al]. (Right) Sketch of a 288m high skyscraper in Australia by Skidmore, Owings & Merrill.

Part I

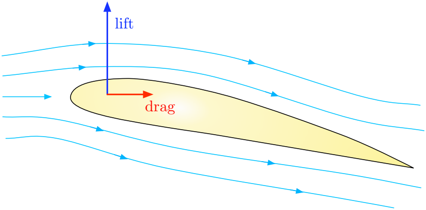
Introduction, history and generalities about shape optimization

Towards "modern" shape and topology optimization (I)

- More advanced shape optimization methods have emerged since the 1960's, mainly due to
  - The development of efficient numerical tools for simulating complex physical phenomena (notably the finite element method);
  - The increase in computational power.
- One of the first fields involved is aeronautics, where engineers were motivated to optimize airfoils so as to
  - Minimize the drag of aircrafts;
  - Increase their lift.



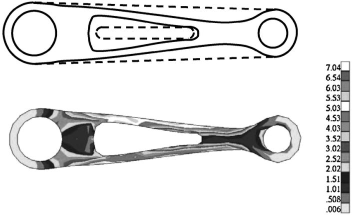
Sketch of the wing of an aircraft



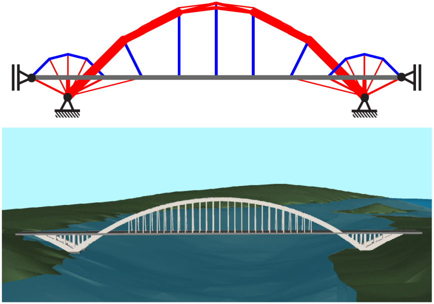
An airfoil subjected to the reaction of air

Towards "modern" shape and topology optimization (II)

Concurrently, such computer-aided methods have aroused a great enthusiasm in civil and mechanical engineering.



Optimization of a torque arm (from [KiWan])

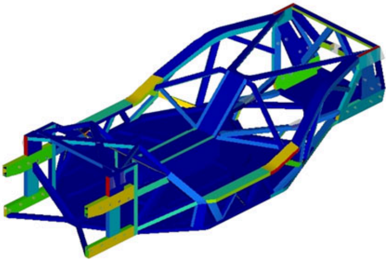


Optimization of an arch bridge (from [ZhaMa])

Towards “modern” shape and topology optimization (III)

- Since then, much headway has been made in the mathematical and algorithmic practice of shape and topology optimization.
- Nowadays, shape and topology optimization techniques are consistently used in industry in a wide variety of situations.
- Several industrial softwares are available: `OptiStruct`, `Ansys`, `Tosca`, etc.

Optimization of a hip prosthesis ***(Photo credits:*** *[Al]****)***



Optimization of an automotive chassis

(from *[CaBa]****)***

Disclaimer

Disclaimer

- This course is very introductory, and by no means exhaustive, as well for theoretical as for numerical purposes.
- See the (non exhaustive) References section to go further.

Part I

Introduction, history and generalities about shape optimization

1. Some selected milestones in shape optimization

2. Generalities about shape optimization problems and examples
   - What is a shape optimization problem?
   - Examples of model problems

What is a shape optimization problem? (I)

- A typical shape optimization problem arises under the form:

$$\boxed{\min_{\Omega \in \mathcal{U}_{\mathbf{ad}}} J(\Omega), \text{ s.t. } C(\Omega) \leq 0,}$$

  where

  - Ω is the shape, or the design variable;
  - $J(\Omega)$ is an objective function to be minimized;
  - $C(\Omega)$ is a constraint function;
  - $\mathcal{U}_{\text{ad}}$ is a set of admissible shapes;

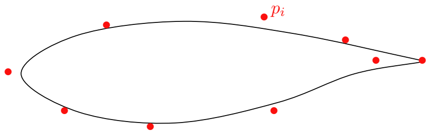
- In this course, the considered problems are motivated by mechanical or physical applications; $J(\Omega)$ and $C(\Omega)$ often depend on Ω via a state u_Ω, solution to a PDE posed on Ω (e.g. the linear elasticity system, or the Stokes equations).

What is a shape optimization problem? (II)

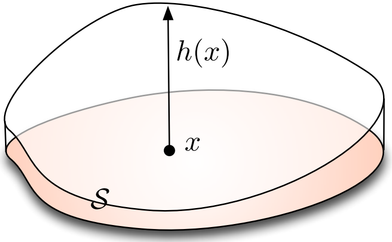
- A shape optimization process is the combination of:
  - A physical model, often based on PDE (e.g. the linear elasticity equations, the Stokes system, etc...) describing the mechanical behavior of shapes,
  - A mathematical representation of shapes and their variations (e.g. as sets of parameters, density functions, etc...),
  - A numerical description of shapes (e.g. by a mesh, a spline model, etc...)
- These choices are strongly inter-dependent and they are often guided by the particular application.
- Roughly speaking, shape and topology optimization problems fall intro three main categories: parametric, geometric and topology optimization.
- This classification is quite arbitrary; it mainly reflects a point of view about what is important in the problem. The associated mathematical and numerical methods share a lot of common features.

I. Parametric optimization

The considered shapes are described by means of a set of physical parameters $\{p_i\}_{i=1,...,N}$, typically thicknesses, curvature radii, etc...



Description of a wing by NURBS; the parameters of the representation are the control points p_i.



A plate with fixed cross-section $\mathcal{S}$ is parametrized by its thickness function $h : \mathcal{S} \to \mathbb{R}$.

Various settings for shape optimization (II)

- The parameters describing shapes are the only optimization variables, and the shape optimization problem rewrites:

$$\min_{\{p_i\} \in \mathcal{P}_{\mathbf{ad}}} J(p_1, ..., p_N),$$

where $\mathcal{P}_{\text{ad}}$ is a set of admissible parameters.

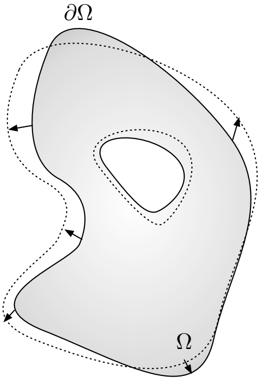
- Parametric shape optimization is eased by the fact that it is straightforward to account for variations of a shape $\{p_i\}_{i=1,\ldots,N}$:

$$\{p_i\}_{i=1,\ldots,N} \to \{p_i + \delta p_i\}_{i=1,\ldots,N}.$$

- However, the variety of possible designs is severely restricted, and the use of such methods relies on an a priori knowledge about the sought optimal design.

II. Geometric shape optimization

- The topology of shapes (i.e. their number of holes in 2d) is fixed.
- The whole boundary $\partial\Omega$ of shapes Ω is the optimization variable.
- Geometric optimization allows for more freedom than parametric optimization, since no a priori knowledge of the relevant regions of shapes to act on is required.



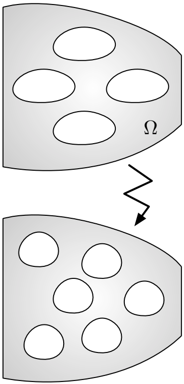
Optimization of Ω via "free" perturbations of the boundary $\partial\Omega$.

III. Topology optimization

- In many applications, the suitable topology of shapes is unknown, and it should also be subject to optimization.

- In this context, it is often preferred not to represent the boundaries of shapes, but to employ different descriptions which allow for a more natural account of topological changes.

  Example Describing shapes Ω as density functions $h : D \to [0, 1]$.



Optimizing a shape by acting on its topology.

Part I

Introduction, history and generalities about shape optimization

A simplified, academic example (I)

A cavity $D \subset \mathbb{R}^d$ is filled with a material with thermal conductivity $h : D \to \mathbb{R}$.

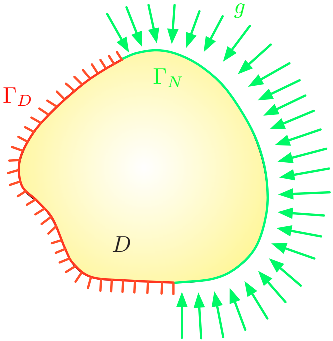
- A region $\Gamma_D \subset \partial D$ is kept at temperature 0.
- A heat flux g is applied on $\Gamma_N := \partial D \setminus \overline{\Gamma_D}$.
- A heat source or sink $f : D \to \mathbb{R}$ is acting inside D.

The temperature $u_h : D \to \mathbb{R}$ within the cavity is the solution to the conductivity equation:

$$\left\{ \begin{array}{rcll} -\text{div}(h\nabla u_h) & = & f & \text{in } D, \\ u_h & = & 0 & \text{on } \Gamma_D, \\ h\frac{\partial u_h}{\partial n} & = & g & \text{on } \Gamma_N. \end{array} \right. .$$

Parametric optimization problem: the design variable is the conductivity distribution $h \in \mathcal{U}_{\text{ad}}$, where

$$\mathcal{U}_{\text{ad}} = \{h \in L^\infty(D), \ \alpha \le h(x) \le \beta, \ x \in D\}.$$



The considered cavity

A simplified, academic example (II)

Examples of objective functions:

- The compliance $C(h)$ of the cavity D:

$$C(h) = \int_D h|\nabla u_h|^2 \mathrm{d}x = \int_\Omega f u_h \mathrm{d}x + \int_{\Gamma_N} g u_h \, \mathrm{d}s,$$

  measuring the heat power inside D, or the work of the heat sources on D.

- A least-square error between u_h and a target temperature u_0:

$$D(h) = \left(\int_D k(x) |u_h - u_0|^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}},$$

  where α is a fixed parameter, and $k(x)$ is a weight factor.

- The opposite of the first eigenvalue of the cavity:

$$-\lambda_1(h), \text{ where } \lambda_1(h) = \min_{\substack{u \in H^1(D) \\ u = 0 \text{ on } \Gamma_D}} \frac{\int_D h|\nabla u|^2 \, \mathrm{d}x}{\int_\Omega u^2 \, \mathrm{d}x},$$

  which characterizes the decay rate of the heat inside D in the transient version of the conductivity equation.

A simplified, academic example (III)

This problem has a geometric optimization variant, where the conductivity inside D takes

- A high value β inside a region $\Omega \subset D$;
- A low value α inside $D \setminus \overline{\Omega}$;

that is:

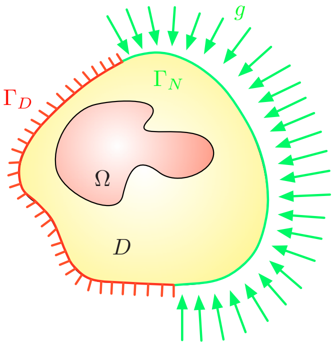
$$h_\Omega = \alpha + \chi_\Omega(\beta - \alpha),$$

where χ_Ω is the characteristic function of Ω.

The temperature $u_\Omega : D \to \mathbb{R}$ is the solution to the conductivity equation:

$$\left\{ \begin{array}{ccll} -\mathrm{div}(h_\Omega \nabla u_\Omega) & = & f & \text{in } D, \\ u_\Omega & = & 0 & \text{on } \Gamma_D, \\ h_\Omega \frac{\partial u_\Omega}{\partial n} & = & g & \text{on } \Gamma_N. \end{array} \right. .$$

Geometric optimization problem: the design variable is the geometry Ω of the good conducting phase.

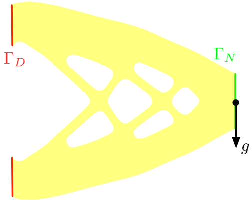


The two-phase conductivity setting

Shape optimization in structural mechanics (I)

We consider a structure $\Omega \subset \mathbb{R}^d$, that is, a bounded domain which is

- Fixed on a part $\Gamma_D \subset \partial\Omega$ of its boundary,
- Submitted to surface loads g, applied on $\Gamma_N \subset \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$.



A "Cantilever" beam

The displacement vector field $u_\Omega : \Omega \to \mathbb{R}^d$ is governed by the linear elasticity system:

$$\begin{cases} -\mathrm{div}(Ae(u_\Omega)) = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ Ae(u_\Omega)n = g & \text{on } \Gamma_N, \\ Ae(u_\Omega)n = 0 & \text{on } \Gamma := \partial\Omega \setminus (\Gamma_D \cup \Gamma_N), \end{cases}$$

where $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the strain tensor field, and A is the Hooke's law of the material.

The deformed cantilever

Shape optimization in structure mechanics (II)

Examples of objective functions:

- The work of the external loads g or compliance $C(\Omega)$ of domain Ω:

$$C(\Omega) = \int_{\Omega} Ae(u_\Omega) : e(u_\Omega)\, \mathrm{d}x = \int_{\Gamma_N} g.u_\Omega\, \mathrm{d}s$$

- A least-square discrepancy between the displacement u_Ω and a target displacement u_0 (useful when designing micro-mechanisms):

$$D(\Omega) = \left(\int_{\Omega} k(x) |u_\Omega - u_0|^{\alpha} \mathrm{d}x \right)^{\frac{1}{\alpha}},$$

where α is a fixed parameter, and $k(x)$ is a weight factor.

Shape optimization in structure mechanics (III)

Examples of constraints:

- A constraint on the volume $\text{Vol}(\Omega)$, or the perimeter $\text{Per}(\Omega)$ of shapes.

$$\text{Vol}(\Omega) = \int_\Omega \mathrm{d}x, \;\; \text{Per}(\Omega) = \int_{\partial\Omega} \mathrm{d}s.$$

- A constraint on the total stress developped in shapes:

$$S(\Omega) = \int_\Omega ||\sigma(u_\Omega)||^2 \, \mathrm{d}x,$$

where $\sigma(u) = Ae(u)$ is the stress tensor.

- Geometric constraints, e.g.
  - Constraints on the minimal and maximum thickness of shapes;
  - Constraints on their curvature radii;

such constraints are often imposed by the manufacturing process.

Shape optimization in fluid mechanics (I)

An incompressible fluid with kinematic viscosity ν occupies a domain $\Omega \subset \mathbb{R}^d$.

- The flow u_{in} through the input boundary Γ_{in} is known.
- A pressure profile p_{out} is imposed on the exit boundary Γ_{out}.
- No slip boundary conditions are considered on the free boundary $\partial\Omega \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}})$.

The velocity $u_\Omega : \Omega \to \mathbb{R}^d$ and pressure $p_\Omega : \Omega \to \mathbb{R}$ of the fluid satisfy the Stokes equations:

$$\left\{ \begin{array}{ll} -2\nu \text{div}(D(u)) + \nabla p = f & \text{in } \Omega \\ \text{div}(u) = 0 & \text{in } \Omega \\ u = u_{\text{in}} & \text{on } \Gamma_{\text{in}} \\ u = 0 & \text{on } \Gamma \\ \sigma(u)n = -p_{\text{out}} & \text{on } \Gamma_{\text{out}} \end{array} \right. ,$$

where $D(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the rate of strain tensor.

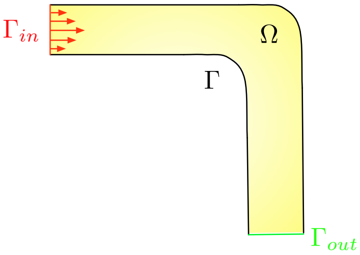
Shape optimization in fluid mechanics (II)

Model problem I: *Optimization of the shape of a pipe.*

- The shape Ω is a pipe, connecting the (fixed) input area Γ_{in} and output area Γ_{out}.
- One is interested in minimizing the total work of the viscous forces inside Ω:

$$J(\Omega) = 2\nu \int_\Omega D(u_\Omega) : D(u_\Omega)\, \mathrm{d}x.$$

- A constraint on the volume $\mathrm{Vol}(\Omega)$ of the pipe is enforced.

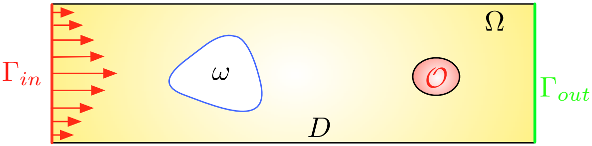


Shape optimization in fluid mechanics (III)

Model problem II: *Reconstruction of the shape of an obstacle.*

- An obstacle of unknown shape ω is immersed in a fixed domain D filled by the considered fluid.
- Given a mesure u_{meas} of the velocity u_Ω of the fluid inside a small observation area $\mathcal{O}$, one aims at reconstructing the shape of ω.
- The optimized domain is $\Omega := D \setminus \overline{\omega}$, and only the part $\partial\omega$ of $\partial\Omega$ is optimized. One then minimizes the least-square criterion:

$$J(\Omega) = \int_{\mathcal{O}} |u_\Omega - u_{\text{meas}}|^2 \, \mathrm{d}x.$$



And yet more examples

- *Optimization of the shape of an airfoil:* reducing the drag acting on airplanes (even by a few percents) has been a tremendous challenge in the aerodynamic industry for decades.
- *Optimization of the microstructure of composite materials:* in linear elasticity, one is interested in the design of negative Poisson ratio materials, etc...
- *Optimization of the shape of wave guides* (e.g. optical fibers), in order to minimize the power loss of conducted electromagnetic waves.
- etc...

Why are shape optimization problems difficult?

- *From the modeling viewpoint:* difficulty to describe the physical problem at stake by a model which is relevant (thus complicated enough), yet tractable (i.e. simple enough).

- *From the theoretical viewpoint:* often, optimal shapes do not exist, and shape optimization problems enjoy at best local optima.

- *From both theoretical and numerical viewpoints:* the optimization variable is the domain! Hence the need for of a means to differentiate functions depending on the domain, and before that, to parametrize shapes and their variations.

- *On the numerical side:* difficulty to represent shapes and their evolutions.

- *On the numerical side:* shape optimization problems may be very sensitive and can be completely dominated by discretization errors.

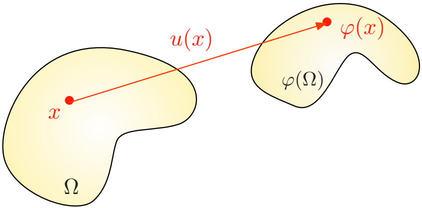
Appendix: physical models

A primer in linear elasticity

Elasticity is the ability of a structure Ω to resist an input stress, and to return exactly to its original state when the stress is relieved ($\neq$ plasticity or fracture).

The motion of an elastic shape Ω is described by:

- The deformation $\varphi : \Omega \to \varphi(\Omega)$;
- The displacement $u(x) = \varphi(x) - x$.



Lagrangian point of view: the considered quantity is the motion $u(x)$ of the constituent particles x of the structure at rest Ω, which serves as reference.

The strain tensor (I)

The Cauchy-Green strain tensor $C(x)$ measures how φ distorts lengths.

A curve $(0,1) \ni t \mapsto \gamma(t) \in \Omega$ with length

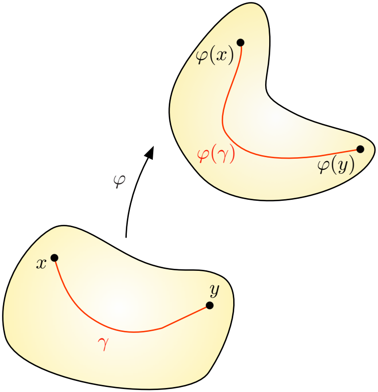
$$\ell(\gamma) := \int_0^1 |\gamma'(t)| \mathrm{d}t;$$

is transformed into $t \mapsto \varphi(\gamma(t))$, with length:

$$\ell(\varphi(\gamma)) = \int_0^1 \sqrt{(C(\gamma(t))\gamma'(t)) \cdot \gamma'(t)}\, \mathrm{d}t,$$

where

$$\begin{aligned} C(x) &= (\nabla\varphi(x))^T(\nabla\varphi(x)) \\ &= (\mathrm{I} + \nabla u(x))^T(\mathrm{I} + \nabla u(x)). \end{aligned}$$



The strain tensor (II)

Geometric linearity Assuming that the displacement u is "small", one approximates:

$$C(x) \approx \mathrm{I} + \nabla u(x) + \nabla u^T(x).$$

- The linearized strain tensor

$$e(u)(x) := \frac{1}{2}(\nabla u(x) + \nabla u^T(x))$$

then satisfies:

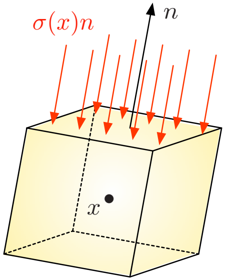
$$\ell(\varphi(\gamma)) \approx \ell(\gamma) + \int_0^1 \left(e(u)(\gamma(t)) \frac{\gamma'(t)}{|\gamma'(t)|} \right) \cdot \frac{\gamma'(t)}{|\gamma'(t)|} |\gamma'(t)| \,\mathrm{d}t.$$

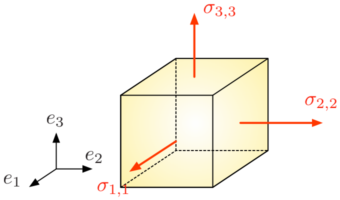
- In the usual Cartesian coordinates, $e(u)$ is defined by:

$$e(u)_{i,j} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i,j = 1, \dots, d.$$

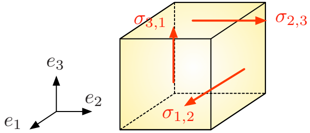
The stress tensor

- The stress tensor σ encodes the internal efforts within the body.
- For $x \in \Omega$, $n \in \mathbb{R}^d$ with $|n| = 1$, $\sigma(x)n \in \mathbb{R}^d$ is the force applied by the surrounding medium on the face oriented by n of a small cube around x.
- Cauchy's theorem: as a consequence of balance of momentum, $\sigma(x)$ is a $d \times d$ symmetric matrix.





The diagonal entries of σ account for traction and compression forces.



The off-diagonal entries of σ account for shear effects.

The equilibrium equations

- The equilibrium equations relate internal efforts with external stresses.
- If body forces $f : \Omega \to \mathbb{R}^d$ (e.g. gravity) occur, it holds, for any subset $\mathcal{V} \subset \Omega$,

$$\underbrace{\int_{\partial\mathcal{V}} \sigma \cdot n \, \mathrm{d}s}_{\text{Efforts applied on } \partial\mathcal{V}} + \underbrace{\int_{\mathcal{V}} f \, \mathrm{d}x}_{\text{Body efforts within } \mathcal{V}} = 0,$$

and so, using Green's formula:

$$\int_{\mathcal{V}} (\mathrm{div}(\sigma) + f) \, \mathrm{d}x = 0.$$

Since the latter relation holds for any subset $\mathcal{V} \subset \Omega$, it follows:

$$\boxed{-\mathrm{div}(\sigma) = f \text{ in } \Omega.}$$

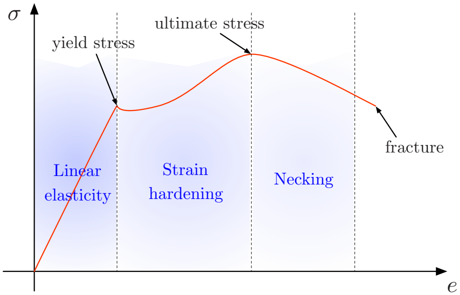
- Likewise, if external forces (e.g. traction loads) $g : \partial\Omega \to \mathbb{R}^d$ are applied, it holds:

$$\boxed{\sigma n = g \text{ on } \partial\Omega.}$$

The constitutive relation (I)

- The system of equations is completed with a constitutive relation between the stress tensor σ and the strain tensor $e(u)$, which describes, equivalently,
  - The deformation of a piece of material caused by a given stress;
  - The internal stress induced by an imposed deformation.
- Material linearity: σ depends linearly on $e(u)$, via the Hooke's law:

$$\sigma = Ae(u).$$



The constitutive relation (II)

The Hooke's law A of an isotropic material reads:

$$\forall e \in \mathcal{S}_d(\mathbb{R}), \;\; Ae = 2\mu e + \lambda \mathrm{tr}(e) I.$$

where the Lamé parameters λ, μ are related to the more physical quantities E and ν:

$$\mu = \frac{E}{2(1+\nu)}, \text{ and } \lambda = \frac{E\nu}{(1+\nu)(1+\nu(1-d))}.$$

- The Young's modulus

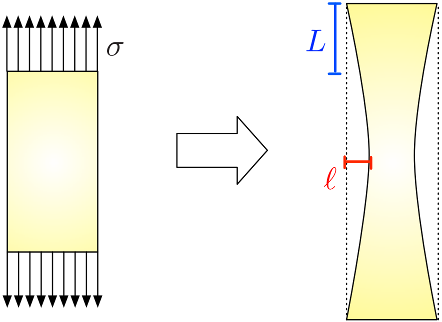
  $$E = \sigma / L$$

  measures the resistance to deformation under traction;

- The Poisson's ratio

  $$\nu = -\ell / L$$

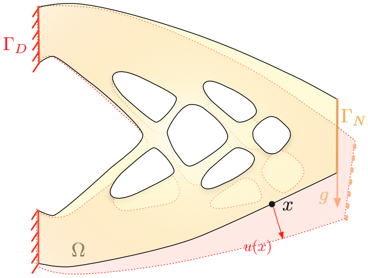
  accounts for the relative transverse displacement for a given longitudinal deformation.



Linear elasticity in a nutshell

In most concrete situations,

- The elastic shape Ω is attached on a region Γ_D of its boundary;
- Body forces $f : \Omega \to \mathbb{R}^d$ are at play;
- Surface loads $g : \Gamma_N \to \mathbb{R}^d$ are applied on a region $\Gamma_N \subset \partial\Omega$;
- The remaining region $\Gamma := \partial\Omega \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N})$ of $\partial\Omega$ is traction-free.



The displacement $u : \Omega \to \mathbb{R}^d$ of the shape in this context is the unique solution (in $H^1(\Omega)^d$) to:

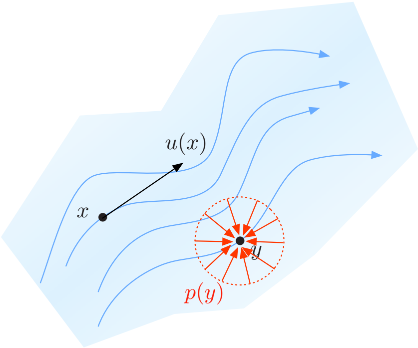
$$\begin{cases} -\mathrm{div}(Ae(u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ Ae(u)n = g & \text{on } \Gamma_N, \\ Ae(u)n = 0 & \text{on } \Gamma. \end{cases}$$

A glimpse of incompressible fluid mechanics

- A fluid medium is characterized by its inability to resist a permanent shear stress.
- Eulerian description: one looks at the properties inside the fluid domain Ω at all positions x, independently of the attached particle (the latter may change).
- We focus on the steady-state Stokes equations; see [ChoMar] for more advanced models: time-dependent, Navier-Stokes equations, turbulence phenomena, etc.

The state of the fluid is described in terms of

- The velocity $u : \Omega \to \mathbb{R}^d$;
- The pressure $p : \Omega \to \mathbb{R}$ inside the fluid.



Rate of strain and vorticity

Two important quantities related to the velocity of the fluid are:

- The rate of strain tensor:

$$D(u) = \frac{1}{2}(\nabla u + \nabla u^T).$$

- The vorticity ω is a scalar field if $d = 2$, and a vector field if $d = 3$:

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \text{ if } d = 2;$$

$$\omega = \nabla \times u = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \text{ if } d = 3.$$

Physical interpretation: A Taylor expansion around $x \in \Omega$ yields (for $d = 3$):

$$u(x+h) = u(x) + D(u)(x)h + \frac{1}{2}\omega(x) \times h + \mathcal{O}(|h|^2);$$

- The transformation $h \mapsto D(u)(x)h$ induces a stretching of lengths encoded by the eigenvalues of the symmetric matrix $D(u)$;
- The mapping $h \mapsto \omega(x) \times h$ is a rotation with axis $\frac{\omega(x)}{|\omega(x)|}$ and velocity $|\omega(x)|$.

The equilibrium equations (I)

- As in the case of elasticity, the balance of momentum at equilibrium implies that:

$$\boxed{-\mathrm{div}(\sigma) = f \text{ in } \Omega,}$$

where:

  - $\sigma : \Omega \to \mathbb{R}^{d\times d}$ is the stress tensor;
  - $f : \Omega \to \mathbb{R}^d$ represents volumic forces (e.g. gravity).

- The fluid is assumed to be incompressible: at equilibrium, the mass contained inside each subset $\mathcal{V} \subset \Omega$ is conserved:

$$\int_{\mathcal{V}} u \cdot n \, \mathrm{d}s = 0,$$

and so, by virtue of Green's formula:

$$\boxed{\mathrm{div}(u) = 0 \text{ in } \Omega.}$$

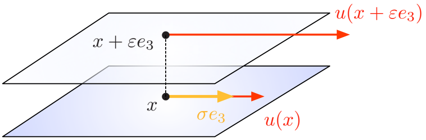
Newton's law

The stress tensor σ is related to the characteristics u, p of the fluid via Newton's law:

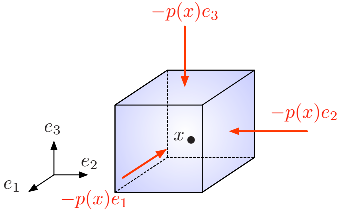
$$\sigma = 2\nu D(u) - p\mathrm{I}.$$

Here,

- The viscous forces $2\nu D(u)$ are frictional, slowering effects, proportional to the variations of the velocity within the fluid, via the viscosity coefficient ν.
- The pressure p induces a normal stress on every region of Ω.



The difference between the velocity at x and $x+\varepsilon e_3$ causes a friction force at x, proportional to the viscosity of the fluid.



Pressure forces act in a normal fashion.

Fluid mechanics in a nutshell

In most practical situations,

- The fluid is subjected to internal forces $f : \Omega \to \mathbb{R}^d$;
- The fluid enters the domain Ω via the region $\Gamma_{\text{in}} \subset \partial\Omega$, with a known velocity profile $u_{\text{in}} : \Gamma_{\text{in}} \to \mathbb{R}^d$;
- The fluid leaves Ω through the region $\Gamma_{\text{out}} \subset \partial\Omega$, with no applied stress;
- The fluid satisfies no slip boundary conditions on the remaining region $\Gamma := \partial\Omega \setminus (\overline{\Gamma_{\text{in}}} \cup \overline{\Gamma_{\text{out}}})$, i.e. the fluid "sticks" to the wall.

The Stokes equations read in this context:

$$\left\{ \begin{array}{cl} -2\nu \text{div}(D(u)) + \nabla p = f & \text{in } \Omega, \\ \text{div}(u) = 0 & \text{in } \Omega, \\ \sigma(u)n = 0 & \text{on } \Gamma_{\text{out}}, \\ u = u_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ u = 0 & \text{on } \Gamma. \end{array} \right.$$

Appendix: technical facts

Notation for differential calculus (I)

Let Ω be an open subset of $\mathbb{R}^d$;

- The gradient of a (differentiable) real-valued function $u : \Omega \to \mathbb{R}$ is the vector field:

$$\forall x \in \Omega, \ \ \nabla u(x) = \begin{pmatrix} \frac{\partial u}{\partial x_1}(x) \\ \vdots \\ \frac{\partial u}{\partial x_d}(x) \end{pmatrix}.$$

- The derivative of a vector-valued function $v : \Omega \to \mathbb{R}^d$ at $x \in \Omega$ is the matrix (tensor) field:

$$\forall x \in \Omega, \ \ \nabla v(x) = \begin{pmatrix} \frac{\partial v_1}{\partial x_1}(x) & \dots & \frac{\partial v_1}{\partial x_d}(x) \\ \vdots & \vdots & \vdots \\ \frac{\partial v_d}{\partial x_1}(x) & \dots & \frac{\partial v_d}{\partial x_d}(x) \end{pmatrix}$$

Notation for differential calculus (II)

- The divergence of a vector field $v : \Omega \to \mathbb{R}^d$ is the function:

$$\begin{aligned} \forall x \in \Omega,\ \operatorname{div}(v)(x) &= \operatorname{tr}(\nabla v(x)) \\ &= \frac{\partial v_1}{\partial x_1}(x) + \ldots + \frac{\partial v_d}{\partial x_d}(x) \end{aligned}$$

- The divergence of a tensor field $\sigma : \Omega \to \mathbb{R}^{d \times d}$ is the vector field whose entries are the divergences of the rows of σ:

$$\forall x \in \Omega,\ \operatorname{div}(\sigma)(x) = \begin{pmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \ldots + \frac{\partial \sigma_{1d}}{\partial x_d} \\ \vdots \\ \frac{\partial \sigma_{d1}}{\partial x_1} + \ldots + \frac{\partial \sigma_{dd}}{\partial x_d} \end{pmatrix}.$$

The Green's formula is a generalization of integration by parts to the case of multiple space dimensions.

Theorem 1 (Green's formula).

Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. Then, for any function $u \in W^{1,1}(\Omega)$,

$$\int_\Omega \frac{\partial u}{\partial x_i} \, \mathrm{d}x = \int_{\partial\Omega} u n_i \, \mathrm{d}s, \quad i = 1, \ldots, d,$$

where $n = (n_1, \ldots, n_d)$ is the unit normal vector to $\partial\Omega$, pointing outward Ω.

The Green's formula has a number of useful avatars, such as:

Corollary 2.

Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. Then, for any $u \in H^2(\Omega)$, $v \in H^1(\Omega)$:

$$\int_\Omega \Delta u \, v \, \mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, \mathrm{d}s - \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x.$$

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