An introduction to optimal design

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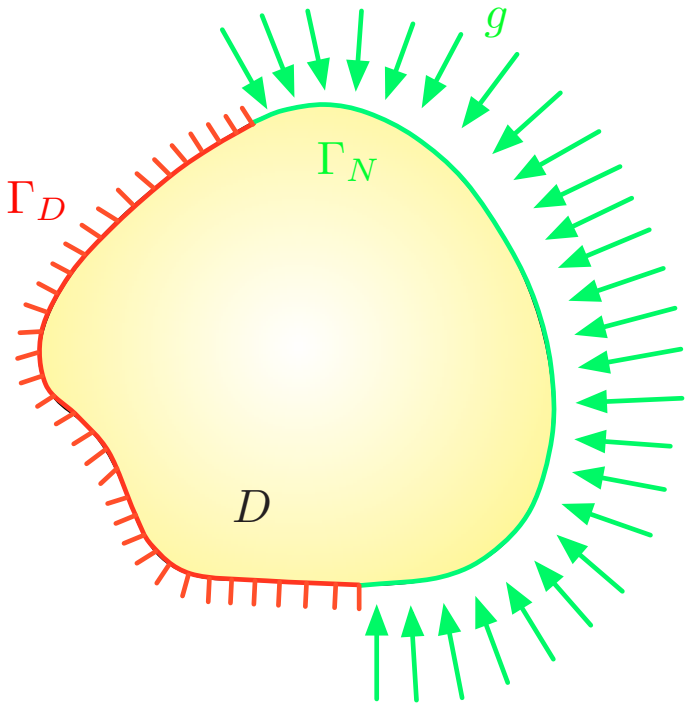
Summary of the previous lectures (I)

- We consider an optimization problem of the conductivity $h : D \to \mathbb{R}$ in a domain $D \subset \mathbb{R}^d$.
- The temperature u_h is the solution in $H^1(D)$ to the "state", conductivity equation:

$$\left\{ \begin{array}{rcll} -\mathrm{div}(h\nabla u_h) & = & f & \text{in } D, \\ u_h & = & 0 & \text{on } \Gamma_D, \\ h\frac{\partial u_h}{\partial n} & = & g & \text{on } \Gamma_N, \end{array} \right.$$

where $f \in L^2(D)$ and $g \in L^2(\Gamma_N)$ are sources.



- The set $\mathcal{U}_{\mathrm{ad}}$ of design variables is that of admissible conductivity functions:

$$\mathcal{U}_{\mathrm{ad}} = \left\{ h \in L^\infty(D),\ \alpha \leq h(x) \leq \beta \text{ a.e. } x \in D \right\} \subset L^\infty(D),$$

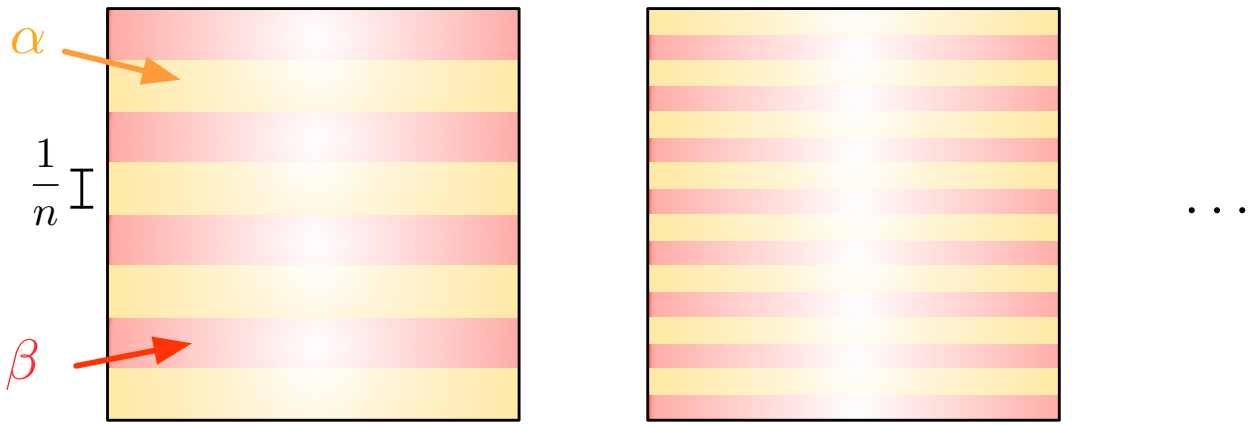
where $0 < \alpha < \beta$ are fixed bounds.

- We wish to minimize the mean temperature within D under a volume constraint:

$$J(h) = \frac{1}{|D|} \int_D u_h \, \mathrm{d}x \text{ and } G(h) = \int_D h \, \mathrm{d}x - V_T.$$

Summary of the previous lectures (II)

- This problem may not have a global solution, due to the homogenization phenomenon.



To get better and better, designs tend to develop features at lower and lower scales.

- We shall instead search for a local minimizer, by means of a gradient algorithm.
- In this perspective, we have calculated he derivative of $J(h)$, in the form:

$$J(h)(\widehat{h}) = \int_D v_h \widehat{h} \, \mathrm{d}x, \text{ where } v_h = \gamma \nabla u_h \cdot \nabla p_h,$$

and the adjoint state p_h is the solution to:

$$\begin{cases} -\mathrm{div}(h \nabla p_h) = -j'(u_h) & \text{in } D, \\ p_h = 0 & \text{on } \partial D. \end{cases}$$

Part III

Numerical study of parametric optimization

Numerical algorithms (I)

We solve the optimization problem:

$$\min_{h \in \mathcal{U}_{\text{ad}}} J(h), \text{ where } J(h) = \int_D j(u_h)\,\mathrm{d}x + \ell \int_D h\,\mathrm{d}x;$$

in there:

- The set $\mathcal{U}_{\text{ad}}$ is: $\mathcal{U}_{\text{ad}} = \{h \in L^\infty(D),\ \alpha < h(x) < \beta \text{ a.e. } x \in D\}$;
- A constraint on the high values of h is added by a fixed penalization.

A projected gradient algorithm for this problem reads:

Initialization: Start from an initial design h^0,

For $n = 0, \ldots$ **convergence:**

1. Calculate the state u_{h^n} and the adjoint p_{h^n} at $h = h^n$;
2. Calculate the descent direction $\widehat{h}^n = -\nabla u_{h^n} \cdot \nabla p_{h^n} - \ell$.
3. Select an appropriate time step $\tau^n > 0$;
4. Update the design as: $h^{n+1} = \min(\beta, \max(\alpha, h^n + \tau^n \widehat{h}^n))$.

Numerical algorithms (II)

In practice,

- The domain D is equipped with a fixed mesh $\mathcal{T}$, composed e.g. of triangles.
- The optimized conductivity h is discretized on this mesh, e.g. as a $\mathbb{P}_0$ or $\mathbb{P}_1$ finite element function.
- For a given value of h, the solutions u_h and p_h to the state and adjoint equations are calculated by the Finite Element Method on the mesh $\mathcal{T}$.

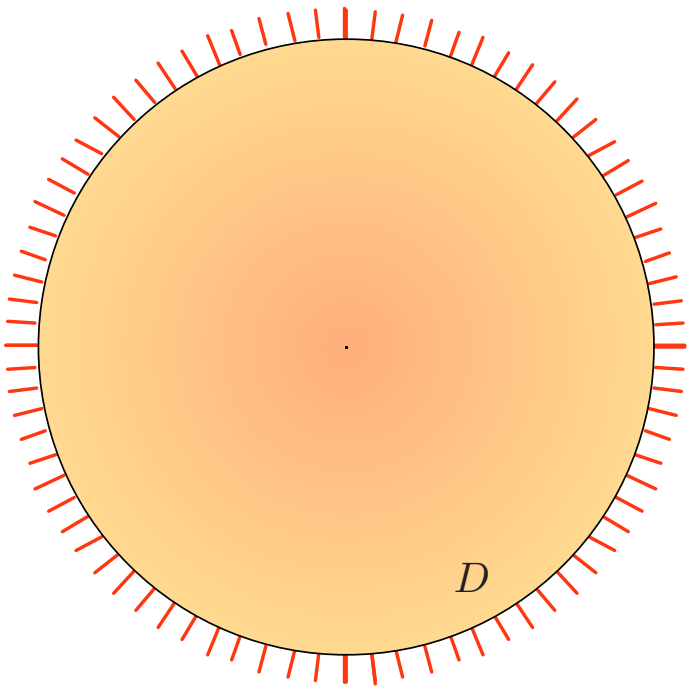
Example: the optimal radiator (I)

We consider the problem:

$$\min_{h \in \mathcal{U}_{\mathrm{ad}}} J(h), \text{ where } J(h) = \int_D u_h \, \mathrm{d}x + \ell \int_D h \, \mathrm{d}x,$$

and the temperature $u_h \in H_0^1(D)$ is the solution to:

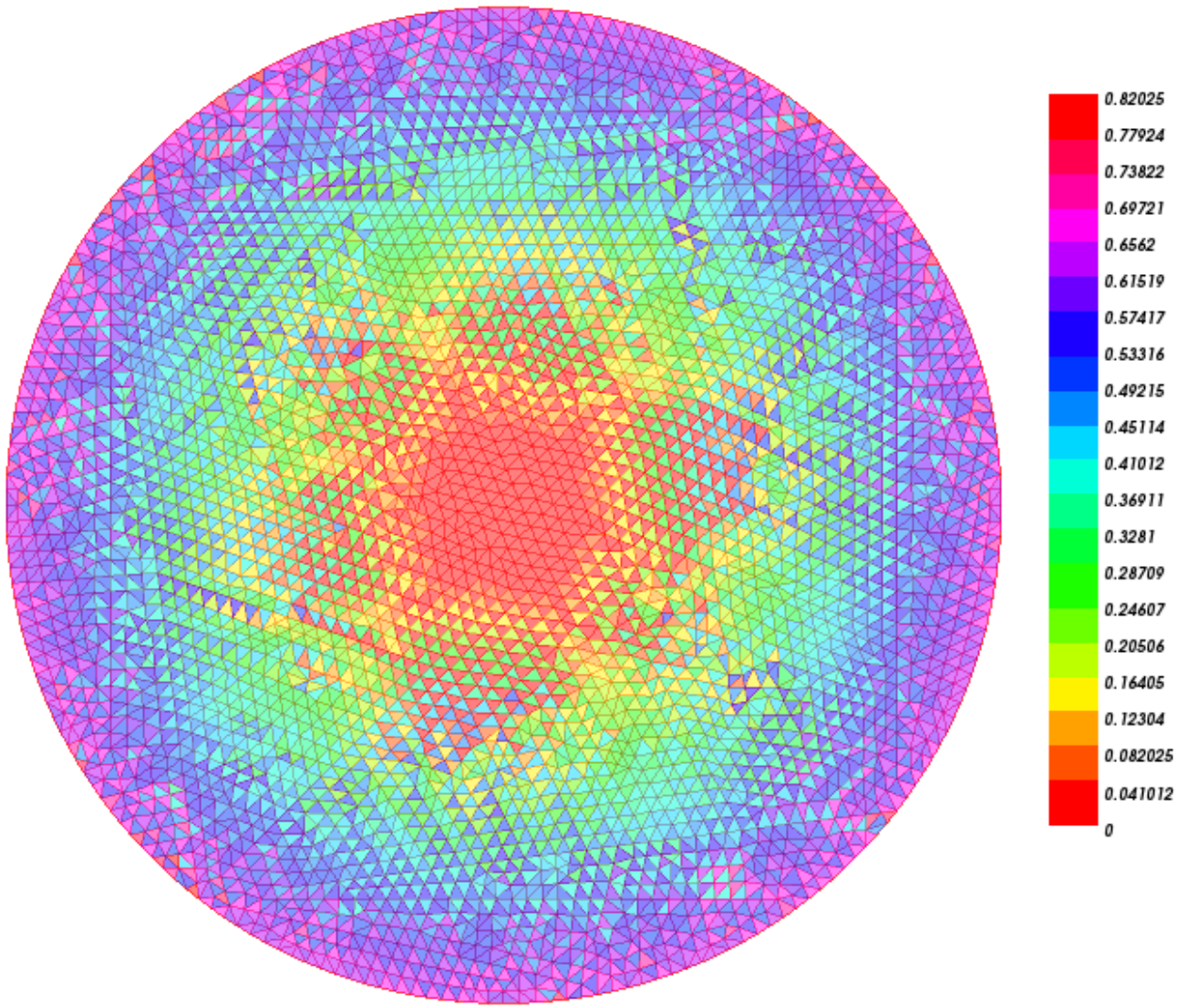
$$\begin{cases} -\mathrm{div}(h \nabla u_h) = 1 & \text{in } D, \\ u_h = 0 & \text{on } \partial D. \end{cases}$$



In other terms,

- The mean temperature inside D is minimized;
- High values of the conductivity are penalized in the objective function.

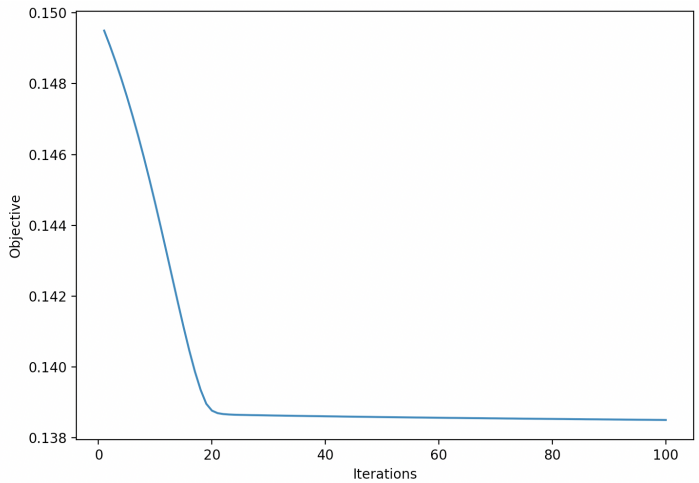
Example: the optimal radiator (II)



Optimized density in the thermal radiator problem.

Example: the optimal radiator (III)

- The "optimized" conductivity distribution h^* presents highly oscillating patterns: low and high values alternate at the level of the mesh size.
- This is not an error: the objective function is dutifully minimized.



Convergence history in the optimal radiator test case.

- The algorithm tries to reproduce the "homogenized" behavior of solutions.
- However natural, this behavior is highly undesirable in practice.
- **One remedy** Act on the descent direction, by changing inner products.

Part III

Numerical study of parametric optimization

Changing inner products (I)

- By definition of the Fréchet derivative, the following expansion holds:

$$J(h+\tau\widehat{h}) = J(h) + \tau J'(h)(\widehat{h}) + \mathrm{o}(\tau),$$

  and a descent direction for J from h is any $\widehat{h} \in L^\infty(D)$ such that $J'(h)(\widehat{h}) < 0$.

- Judging from the derivative formula:

$$J'(h)(\widehat{h}) = \int_D \widehat{h} \nabla u_h \cdot \nabla p_h \,\mathrm{d}x$$

  it is tempting to chose the descent direction:

$$\widehat{h} = -\nabla u_h \cdot \nabla p_h.$$

- Doing so, mathematically, $\widehat{h}$ is the (negative) $L^2(D)$ gradient g_0 of $J'(h)$:

$$\forall v \in L^2(D), \quad \underbrace{\int_D g_0 v \,\mathrm{d}x}_{\langle g_0, v\rangle_{L^2(D)}} = J'(h)(v).$$

Changing inner products (II)

- This choice is actually awkward:
  - Mathematically, it is ill-posed, since $\nabla u_h \cdot \nabla p_h$ may not belong to $L^2(D)$.
  - Even if ∇u_h and ∇p_h are more regular than $H^1(D)$, g_0 may be very irregular!

- **Main idea:** Select a descent direction as the negative gradients of $J'(h)$, obtained with another inner product than that of $L^2(D)$.

- This practice is classical and crucial in the treatment of gradient flows [Neu].

Changing inner products (III)

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$.

Solve the following identification problem: Search for $V \in H$ such that:

$$\boxed{\forall w \in H, \ \langle V, w \rangle_H = J'(h)(w) = \int_D w \nabla u_h \cdot \nabla p_h \, \mathrm{d}x.}$$

Then $-V$ is also a descent direction for $J(h)$, since for $\tau > 0$ small enough:

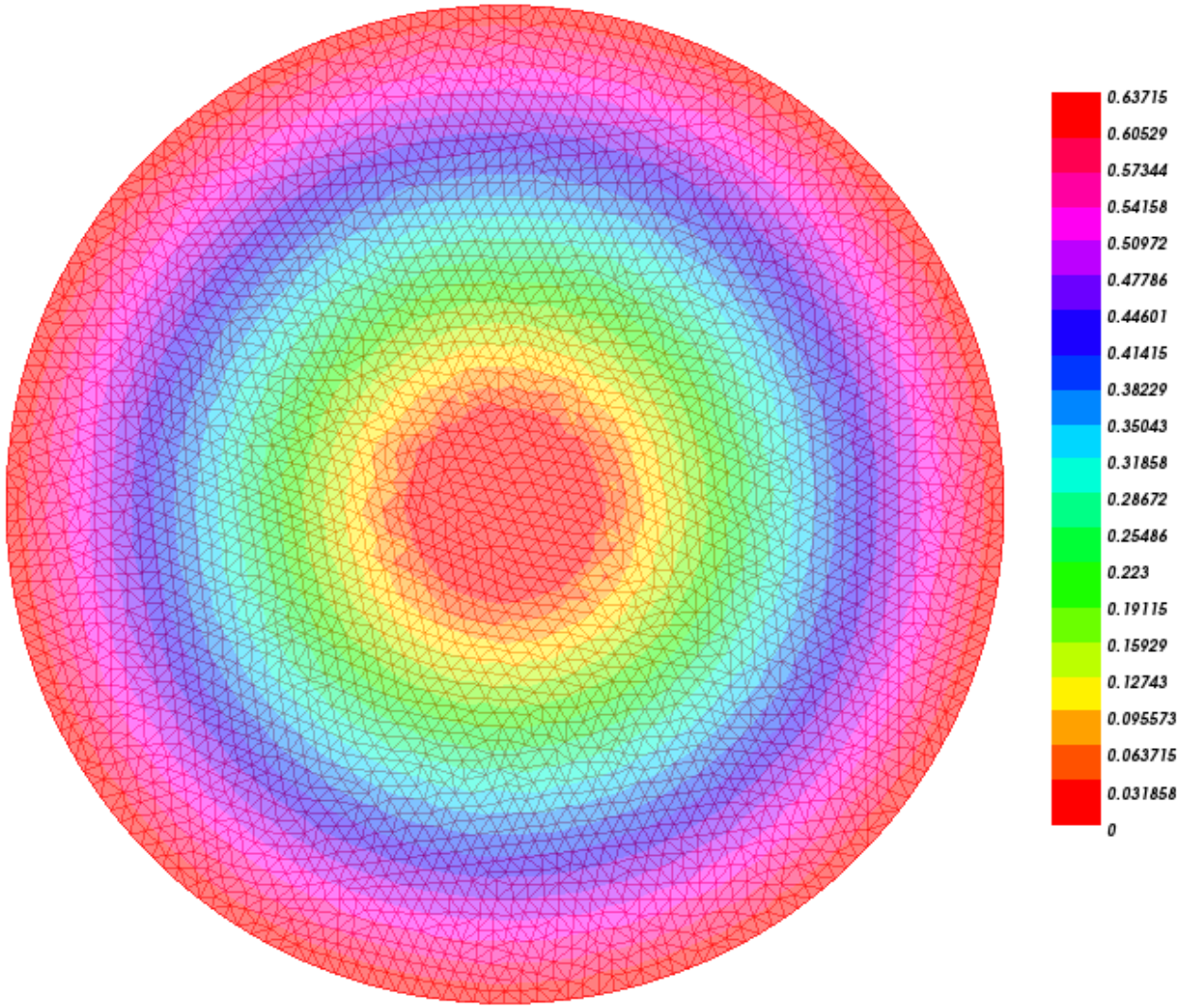
$$\begin{array}{rcl} J(h - \tau V) & = & J(h) - \tau J'(h)(V) + o(\tau) \\ & = & J(h) - \tau \langle V, V \rangle_H + o(\tau) \\ & < & J(h). \end{array}$$

Example: A more regular descent direction g_α than g_0 is obtained with the choice:

$$H = H^1(D), \text{ and } \langle u, v \rangle_H = \int_D (\alpha^2 \nabla u \cdot \nabla v + uv) \, \mathrm{d}x,$$

for α "small" (of the order of the mesh size).

The optimal radiator revisited



Optimized conductivity for the thermal radiator problem using the "change of inner product" trick.

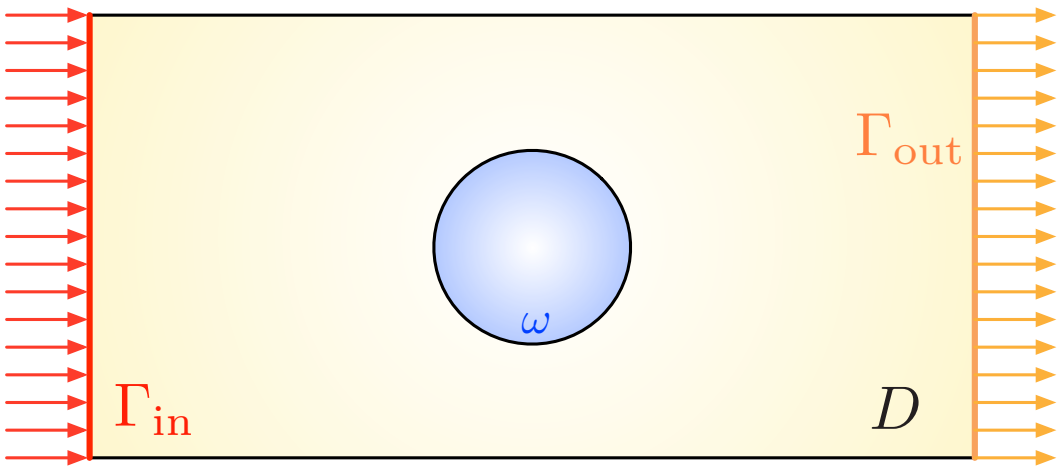
Another example: design of a "heat lens" (I)

As proposed in [Che], the problem

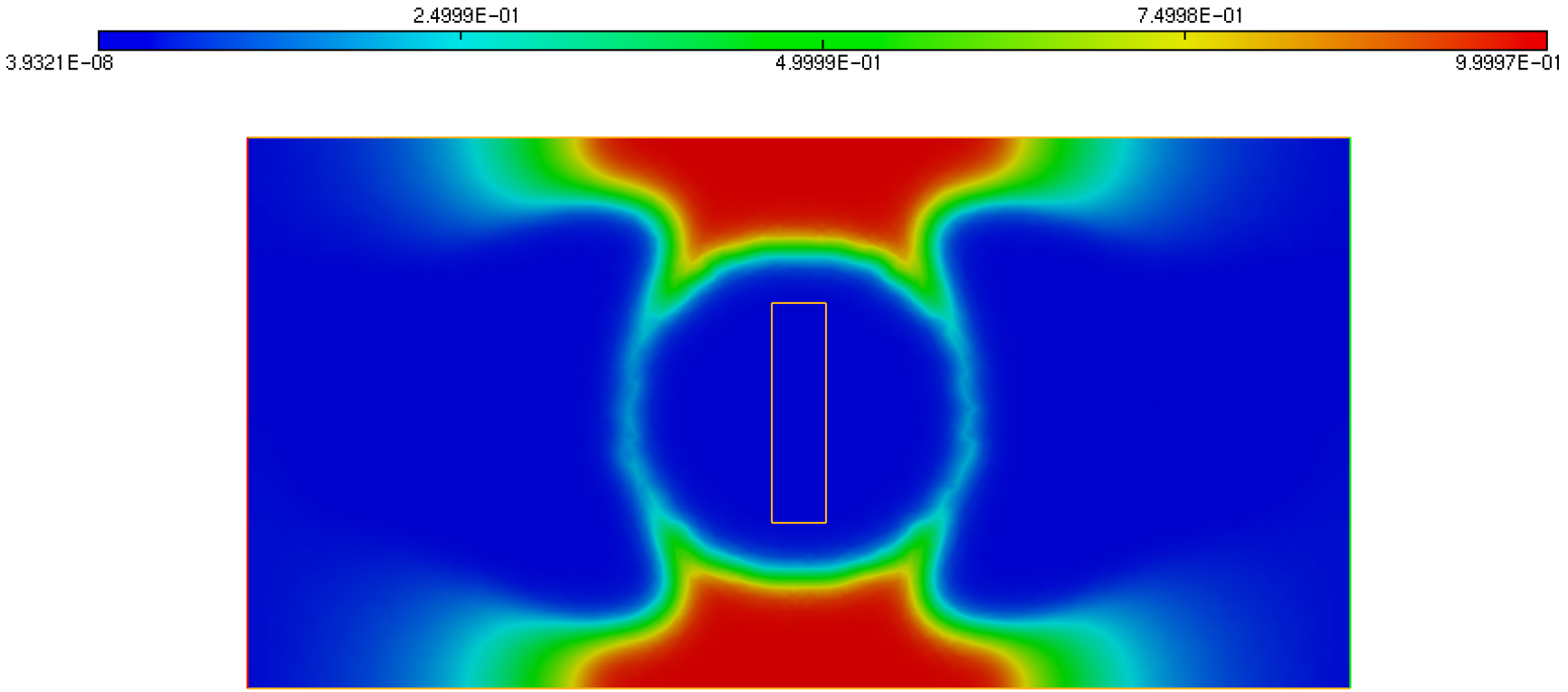
$$\min J(h) \text{ where } J(h) = \int_\omega \left| \alpha \frac{\partial u_h}{\partial x_1} \right|^2 \mathrm{d}x + \ell \int_D h \, \mathrm{d}x$$

is considered:

- The horizontal heat flux through a non optimizable region ω is minimized;
- High values of the conductivity h are penalized.



Another example: design of a "heat lens" (II)



Optimized heat lens under a penalization of high values of the conductivity.

Final remarks

- Other algorithms may be used, to impose constraints on the amount of high conductivity material, such as the Augmented Lagrangian algorithm.

- With a little more work, even more efficient algorithms could be used, such as:
  - Quasi-Newton methods, such as the Gauss-Newton or the BFGS algorithms;
  - "True" second-order algorithms, based on the Hessian of the mapping $h \mapsto J(h)$.

Part III

Numerical study of parametric optimization

Density-based topology optimization (I)

- The treatment of parametric optimal design problems is simplified by the fact that the design variable belongs to a fixed vector space.
- However, this convenient setting suffers from two major drawbacks:
  - An a priori parametrization of designs is required, which is often unrealistic.
  - The freedom about the sought optimal design is severely limited.
- The density-based topology optimization framework overcomes this issue.
- From the technical viewpoint, it essentially boils down to parametric optimal design problems.

Optimized 2d beam with the SIMP method (from [LaSi]).

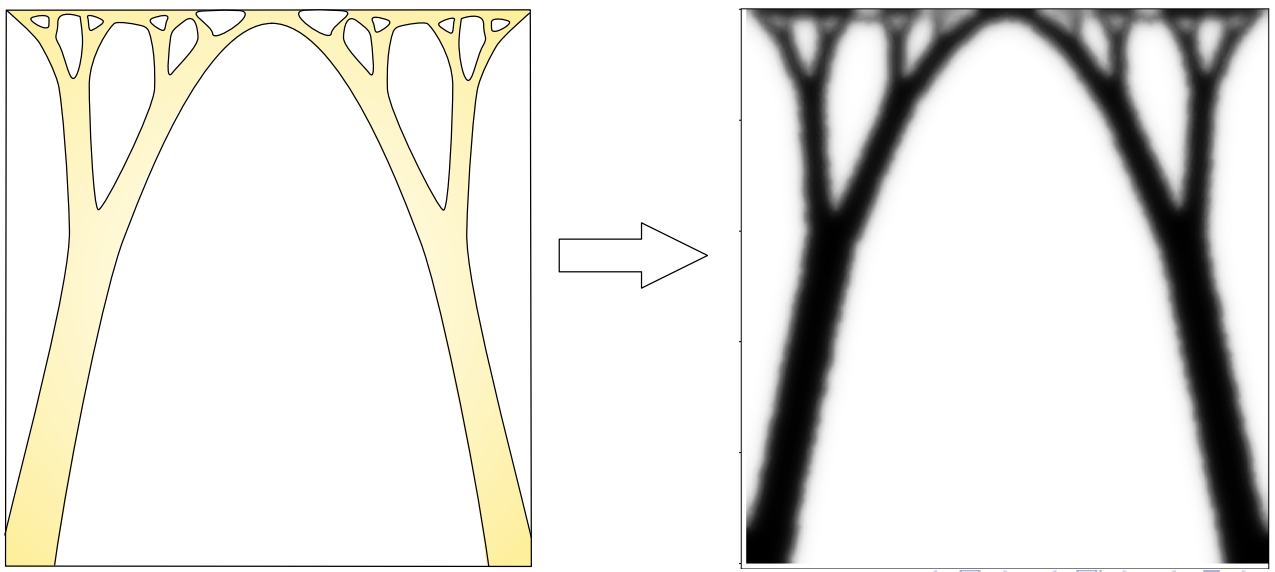
Optimized shape of a wing (from [DTU]).

Density-based topology optimization (II)

- Density-based methods – a.k.a. topology optimization in engineering – are (often formal) relaxations of "true" shape optimization problems, of the form:

$$\min_{\Omega \subset \mathbb{R}^d} J(\Omega) \text{ s.t. } G(\Omega) = 0,$$

- They give a meaning to such problems for shapes Ω with "microscopic patterns".
- **Main idea:** (Heuristic inspiration from the homogenization theory) The optimized shape Ω is replaced by a density function $h : D \to [0, 1]$.

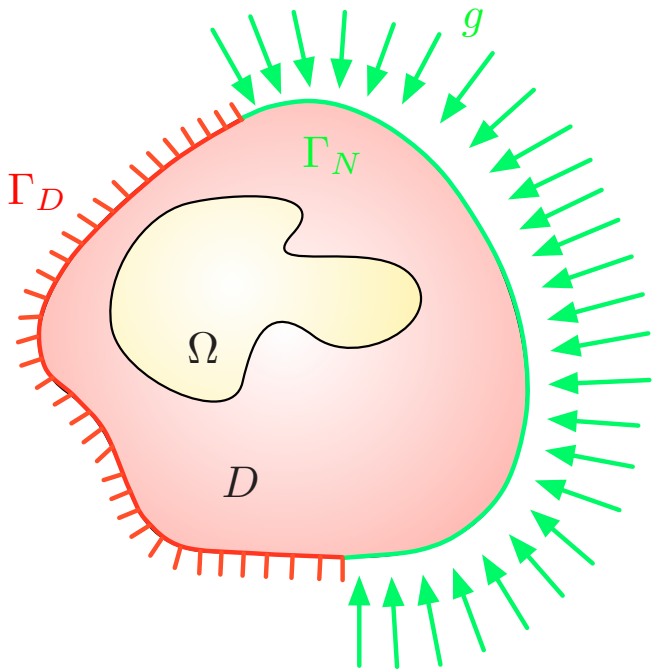


Density-based topology optimization (I)

- Let us consider the two-phase shape optimization counterpart of our toy problem:

$$\min_{\Omega \subset D} J(\Omega), \text{ where } J(\Omega) := \int_D j(u_\Omega)\,\mathrm{d}x, \qquad \text{(SO)}$$

and $j : \mathbb{R} \to \mathbb{R}$ is a given, smooth function.



- Here, the temperature $u_\Omega \in H_0^1(D)$ is the solution to:

$$\begin{cases} -\mathrm{div}(\gamma_\Omega \nabla u_\Omega) &= f & \text{in } D, \\ u_\Omega &= 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial u_\Omega}{\partial n} &= g & \text{on } \Gamma_N, \end{cases}$$

where the conductivity γ_Ω is of the form:

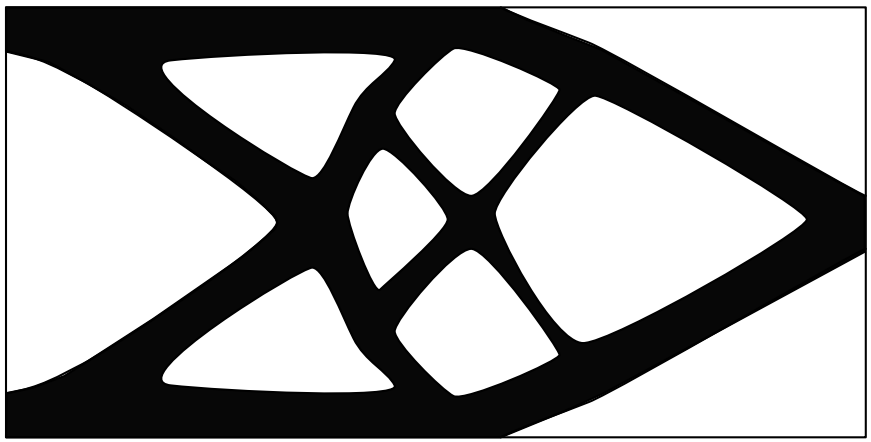
$$\gamma_\Omega(x) = \alpha + \chi_\Omega(x)(\beta - \alpha), \quad x \in D.$$

- This problem does not have a solution in general: minimizing sequences tend to create "microstructures".

Density-based topology optimization (II)

- The (sought) "black-and-white" characteristic function $\chi_\Omega : D \to \{0, 1\}$ of the shape Ω, is replaced by a "grayscale" density function $h : D \to [0, 1]$.
- The properties (diffusion) of a region with intermediate density $h(x) \in (0, 1)$ are described via an empirical isotropic interpolation law $\zeta(h)$ between α and β:

$$\zeta(0) = \alpha, \text{ and } \zeta(1) = \beta.$$



"Black-and-white" shape Ω

Density function $h : D \to [0, 1]$.

Density-based topology optimization (III)

- The approximate density optimization problem rewrites:

$$\min_{h \in \mathcal{U}_{\mathrm{ad}}} J(h),$$

where:

- $\mathcal{U}_{\mathrm{ad}} = L^\infty(D, [0,1])$ is the set of density functions;
- The objective function is $J(h) = \int_D j(u_h)\,\mathrm{d}x$;
- The state $u_h \in H^1(D)$ is the solution to:

$$\begin{cases} -\mathrm{div}(\zeta(h)\nabla u_h) = f & \text{in } D, \\ u_h = 0 & \text{on } \Gamma_D, \\ \zeta(h)\frac{\partial u_h}{\partial n} = g & \text{on } \Gamma_N. \end{cases}$$

- See the appendix for a heuristic derivation of this model from homogenization theory.

Density-based topology optimization (IV)

This problem falls into the setting of parametric optimization!

Theorem.

The objective function

$$J(h) = \int_D j(u_h)\,\mathrm{d}x$$

is Fréchet differentiable at any $h \in \mathcal{U}_{\mathrm{ad}}$*, and its derivative reads*

$$\forall \widehat{h} \in L^\infty(D),\ \ J'(h)(\widehat{h}) = \int_D \zeta'(h)(\nabla u_h \cdot \nabla p_h)\widehat{h}\,\mathrm{d}x,$$

where the adjoint state $p_h \in H^1(D)$ *is the unique solution to the system:*

$$\begin{cases} -\mathrm{div}(\zeta(h)\nabla p_h) = -j'(u_h) & \text{in } D, \\ p_h = 0 & \text{on } \Gamma_D, \\ \zeta(h)\frac{\partial p_h}{\partial n} = 0 & \text{on } \Gamma_N. \end{cases}$$

The interpolation profile

- The interpolation profile $\zeta(h)$ prescribes material properties (diffusion, etc.) to regions with (fictitious) intermediate densities.
- In the practice of the Solid Isotropic Method with Penalization (SIMP), a power law of the form
$$\zeta(h) = \alpha + h^p(\beta - \alpha)$$
is used (often, $p = 3$).
- This has the effect to penalize the presence of "grayscale" intermediate regions, and to steer the optimized density towards a "black-and white" function.
- This interpolation law is empirical: there is no guarantee that a material with such properties does exist!
- In the article [Am2], other, more "physical" choices for $\zeta(h)$ are discussed.

Part V

Topology optimization

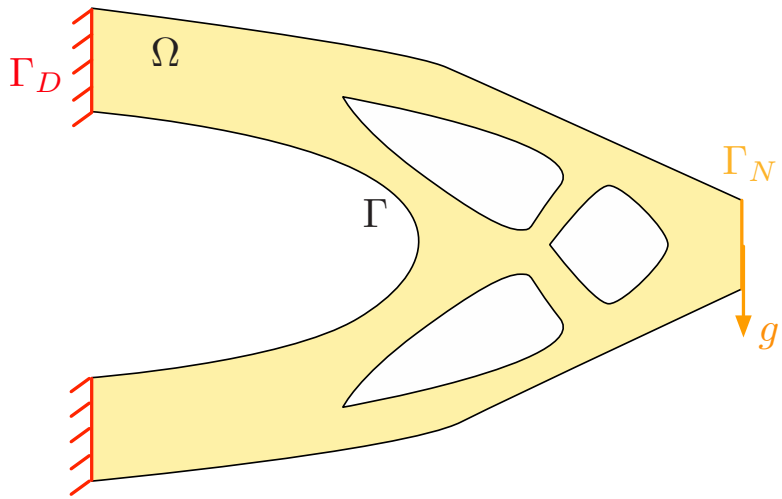
The linearized elasticity setting (I)

- We consider the structural optimization problem:

$$\min_{\Omega} J(\Omega),$$

where

  - The shape Ω is a mechanical structure;
  - $J(\Omega) = \int_\Omega j(u_\Omega)\,\mathrm{d}x$,
  - $j : \mathbb{R}^d \to \mathbb{R}$ is a smooth function.



- The displacement u_Ω is the unique solution to the linearized elasticity system:

$$\begin{cases} -\mathrm{div}(Ae(u_\Omega)) = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ Ae(u_\Omega)n = g & \text{on } \Gamma_N, \\ Ae(u_\Omega)n = 0 & \text{on } \Gamma, \end{cases} \tag{E}$$

where the optimized boundary Γ is traction-free.

The linearized elasticity setting (II)

The "one-phase and void" problem for u_Ω is replaced by the two-phase Ersatz material approximation:

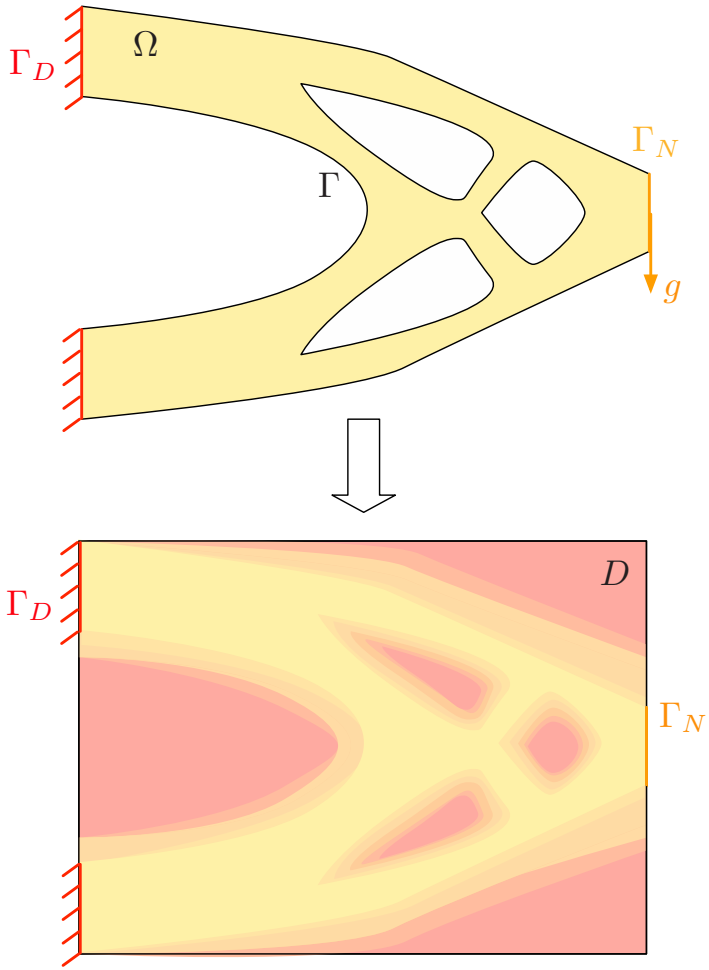
- A large computational box D is considered.
- The void $D \setminus \overline{\Omega}$ is filled with a very soft material, with Hooke's tensor εA, $\varepsilon \ll 1$.
- The solution u_Ω to (E) is replaced by that u_ε to:

$$\begin{cases} -\mathrm{div}(A_\Omega e(u_\varepsilon)) = 0 & \text{in } D, \\ u_\varepsilon = 0 & \text{on } \Gamma_D, \\ A_\Omega e(u_\varepsilon) n = g & \text{on } \Gamma_N, \\ A_\Omega e(u_\Omega) n = 0 & \text{on } \partial D \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N}), \end{cases}$$

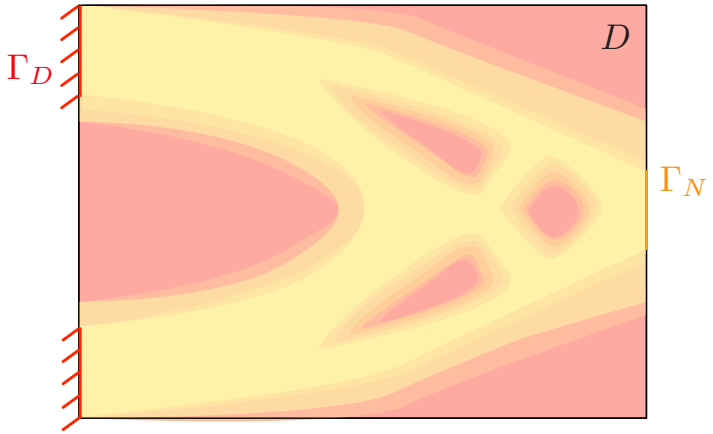
with:

$$A_\Omega = \varepsilon A + \chi_\Omega (1-\varepsilon) A.$$

- This is possible because the boundary Γ in contact with void is traction-free.



The linearized elasticity setting (III)



- The approximate density-based problem is:

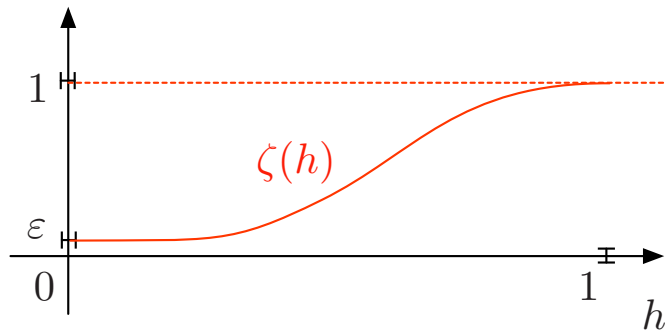
$$\min_{h \in L^\infty(D,[0,1])} J(h), \text{ where } J(h) = \int_D j(u_h)\,\mathrm{d}x.$$

- The approximate displacement u_h is the solution to:

$$\begin{cases} -\mathrm{div}(\zeta(h)Ae(u_h)) = 0 & \text{in } D, \\ u_h = 0 & \text{on } \Gamma_D, \\ \zeta(h)Ae(u_h)n = g & \text{on } \Gamma_N, \\ \zeta(h)Ae(u_h)n = 0 & \text{on } \partial D \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N}). \end{cases}$$

- The transition between the soft and bulk materials is realized by a smooth interpolation profile:

$$\zeta : [0,1] \to [0,1], \quad \zeta(0) = \varepsilon, \text{ and } \zeta(1) = 1.$$



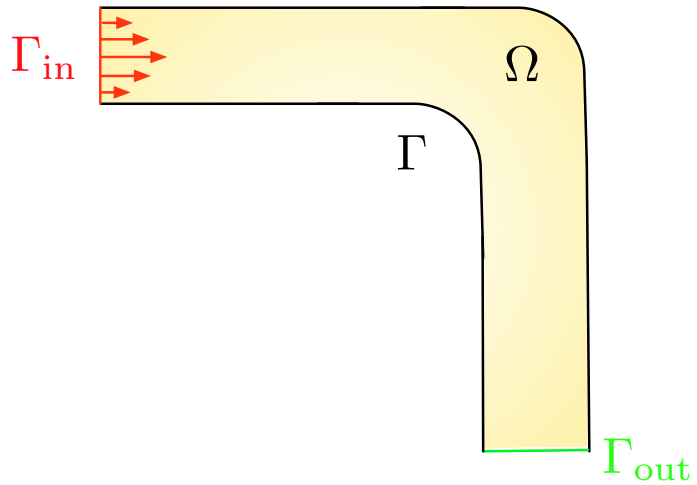
For instance:

$$\zeta(h) = \varepsilon + h^3(1-\varepsilon).$$

Treatment of fluid mechanics by the artifical porosity method (I)

- We consider the shape optimization problem:

$$\min_{\Omega} J(\Omega),$$



where

  - The shape Ω is a duct conveying a fluid;
  - $J(\Omega) = \int_\Omega j(u_\Omega)\,\mathrm{d}x$,
  - $j : \mathbb{R}^d \to \mathbb{R}$ is a smooth function.

- The velocity and pressure $u_\Omega : \Omega \to \mathbb{R}^d$, $p_\Omega : \Omega \to \mathbb{R}$ are solutions to the incompressible Stokes equations:

$$\begin{cases} -\nu\Delta u_\Omega + \nabla p_\Omega = 0 & \text{in } \Omega, \\ \operatorname{div}(u_\Omega) = 0 & \text{in } \Omega, \\ u_\Omega = u_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ \nu\frac{\partial u_\Omega}{\partial n} - pn = -p_{\text{out}}\, n & \text{on } \Gamma_{\text{out}}, \\ u_\Omega = 0 & \text{on } \Gamma, \end{cases} \tag{S}$$

where no-slip boundary conditions are imposed on the optimized boundary Γ.

Treatment of fluid mechanics by the artifical porosity method (II)

The state equation on a shape Ω is approximated by the artificial porosity method:

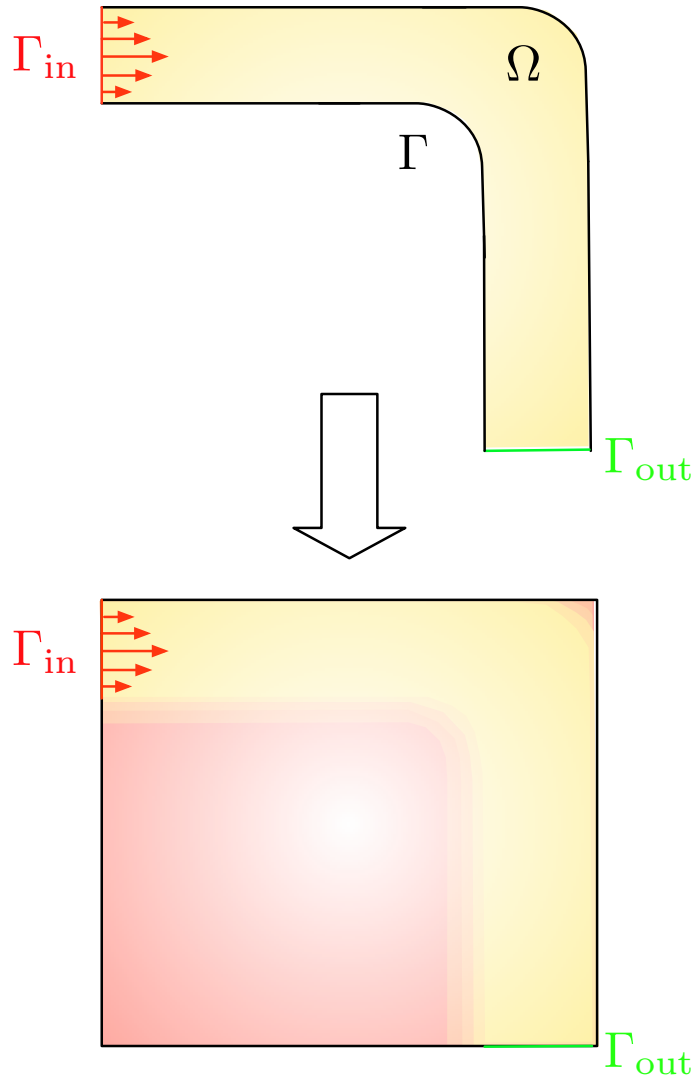
- A large computational box D is considered.
- The solid region $D \setminus \overline{\Omega}$ is filled with a solid material with very small porosity $\varepsilon \ll 1$.
- The velocity u_Ω and pressure p_Ω are replaced by the solutions u_ε, p_ε to the Brinkman equation:

$$\begin{cases} -\nu\Delta u_\varepsilon + \nabla p_\varepsilon + \alpha_\Omega u_\varepsilon = 0 & \text{in } D, \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } \Omega, \\ u_\varepsilon = u_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ \nu \frac{\partial u_\varepsilon}{\partial n} - p_\varepsilon n = -p_{\text{out}} n & \text{on } \Gamma_{\text{out}}, \\ u_\varepsilon = 0 \quad \text{on } \partial D \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N}), \end{cases}$$

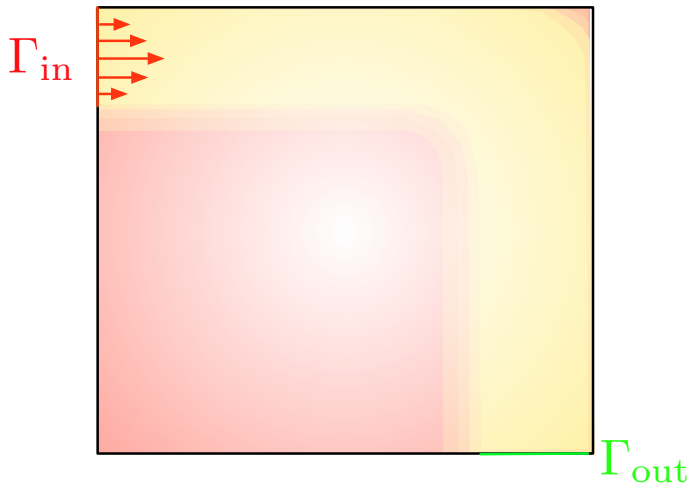
where α_Ω takes very large values in the "solid":

$$\alpha_\Omega := \frac{1}{\varepsilon}(1 - \chi_\Omega).$$

- This approximation is possible because the boundary Γ bears no slip boundary conditions.



Treatment of fluid mechanics by the artifical porosity method (III)



- The density-based version of this problem reads:

$$\min_{h \in L^\infty(D,[0,1])} J(h), \text{ where } J(h) = \int_D j(u_h)\,\mathrm{d}x.$$

- The velocity u_h and pressure p_h are the solutions to the Brinkman equations:

$$\begin{cases} -\nu\Delta u_h + \nabla p_h + \alpha(h)u_h = 0 & \text{in } D, \\ \operatorname{div}(u_h) = 0 & \text{in } D, \\ u_h = u_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ \nu\frac{\partial u_h}{\partial n} - p_h n = -p_{\text{out}} n & \text{on } \Gamma_{\text{out}}, \\ u_h = 0 & \text{on } \partial D \setminus (\overline{\Gamma_{\text{in}}} \cup \overline{\Gamma_{\text{out}}}). \end{cases}$$

- The artificial porosity coefficient $\alpha(h)$ is of the form:

$$\alpha(h) = \alpha_{\max} + (\alpha_{\min} - \alpha_{\max})h\frac{1+q}{h+q},$$

where q is a parameter (e.g. $q = 3$), $\alpha_{\max}$ is typically 10^6 and $\alpha_{\min} \approx 0$.

Part V

Topology optimization

Generalities about density-based relaxation (I)

- The density-based formulation of our model problem reads:

$$\min_{h \in L^\infty(D,[0,1])} J(h), \text{ where } J(h) := \int_D j(u_h)\,\mathrm{d}x$$

and $u_h \in H^1(D)$ is the solution to:

$$\begin{cases} -\mathrm{div}(\zeta(h)\nabla u_h) = f & \text{in } D, \\ u_h = 0 & \text{on } \Gamma_D, \\ \zeta(h)\frac{\partial u_h}{\partial n} = g & \text{on } \Gamma_N. \end{cases}$$

This is a parametric optimization problem, similar to those considered hitherto.

- As we have seen, the simple parametric setting allows for the use of basic gradient algorithms, but also more advanced optimization algorithms, such as:
  - The Method of Moving Asymptotes (MMA);
  - The quasi-Newton BFGS method;
  - Conjugate gradient algorithms.

Generalities about density-based relaxation (II)

- As the result of a density-based topology optimization process, a density function h is obtained, which may present "grayscale" values.
- However, in general, a real "black-and-white" design $\Omega \subset D$ is expected.
- Hence there is the need to threshold the density h, i.e. for an adequate value $c \in (0, 1)$, Ω is obtained as:

$$\Omega = \{x \in D, \ \ h(x) > c\}, \text{ that is,}$$

  - Regions where $0 \leq h(x) \leq c$ are considered to be "void";
  - Regions where $c < h(x) \leq 1$ are considered to be "full of material".

- One simple idea is to use dichotomy to select c so that Ω satisfies a volume target.

Part V

Topology optimization

Density filters (I)

- Often, desired properties of the density h (regularity, etc.) are enforced by filtering: h appears in the state (and adjoint) equations via the quantity Lh, where

$$L : L^\infty(D, [0,1]) \to L^\infty(D, [0,1])$$

is a linear operator called filter.

- The problem rewrites:

$$\min_{h \in \mathcal{U}_{\mathrm{ad}}} J(h), \text{ where } J(h) = \int_D j(u_h)\, dx,$$

and u_h is the solution to:

$$\begin{cases} -\mathrm{div}(\zeta(Lh)\nabla u_h) = f & \text{in } D, \\ u_h = 0 & \text{on } \Gamma_D, \\ (\zeta(Lh)\nabla u_h)n = g & \text{on } \Gamma_N. \end{cases}$$

- The calculation of the derivative of $J(h)$ now reads:

$$\begin{aligned} J'(h)(\widehat{h}) &= \int_D \zeta'(Lh)(\nabla u_h \cdot \nabla p_h)(L\widehat{h})\, \mathrm{d}x, \\ &= \int_D L^T \left(\zeta'(Lh)(\nabla u_h \cdot \nabla p_h)\right) \widehat{h}\, \mathrm{d}x. \end{aligned}$$

Density filters (II)

Here are some examples of regularization filters:

- *Convolution-based filter:* For ε "small" ($\varepsilon \approx$ mesh size), one defines:

$$L_\varepsilon h = h * \eta_\varepsilon,$$

where η_ε is a mollifying kernel; i.e. $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\eta(\frac{x}{\varepsilon})$,

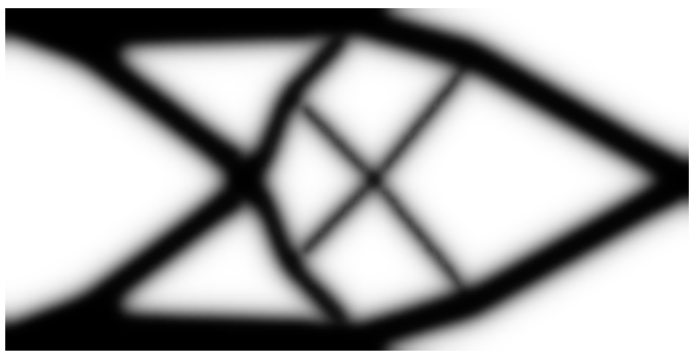
$$\eta \in \mathcal{C}_c^\infty(\mathbb{R}^d),\ \operatorname{supp}(\eta) \subset B(0,1),\ \text{and}\ \int_{\mathbb{R}^d} \eta \, \mathrm{d}x = 1.$$

- *PDE-based filter:* For small ε,

$$L_\varepsilon h = q,$$

where q is the unique solution in $H^1(D)$ to the problem:

$$\begin{cases} -\varepsilon^2 \Delta q + q = h & \text{in } D, \\ \frac{\partial q}{\partial n} = 0 & \text{on } \partial D. \end{cases}$$



Original density function

Filtered density function

Density filters (III)

Morphological filters are often useful to express the idea of geometric robustness:

- The *dilation filter*

$$D_\delta h(x) = \frac{1}{\beta} \log \left(\frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} e^{\beta h(z)} \, \mathrm{d}z \right)$$

interpolates between the mean and the max. of h on $B(x,\delta)$:

$$D_\delta h(x) \xrightarrow{\beta \to 0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} h(z) \, \mathrm{d}z, \text{ and } D_\delta h(x) \xrightarrow{\beta \to \infty} \max_{z \in B(x,\delta)} h(z).$$

- The *erosion filter*

$$E_\delta h(x) = 1 - \frac{1}{\beta} \log \left(\frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} e^{\beta(1-h(z))} \, \mathrm{d}z \right)$$

interpolates between the mean and the min. of h on $B(x,\delta)$:

$$E_\delta h(x) \xrightarrow{\beta \to 0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} h(z) \, \mathrm{d}z, \text{ and } D_\delta h(x) \xrightarrow{\beta \to \infty} \min_{z \in B(x,\delta)} h(z).$$

Density filters (III)

The *Heaviside filter* allows to steer the optimized density towards values 0 and 1 during the optimization:

$$\widetilde{H_{\beta,\eta}h} = \frac{\tanh(\beta\eta) + \tanh(\beta(h-\eta))}{\tanh(\beta\eta) + \tanh(\beta(1-\eta))},$$

where β and η are user-defined parameters which may be updated in the course of the process.

- All these filters can be combined (i.e. composed).
- See for instance [WanSig] for other examples of filters.

Sensitivity filters

- As in the parametric optimization context, the expression of the derivative

$$\forall \widehat{h} \in L^\infty(D),\ \ J'(h)(\widehat{h}) = \int_D \zeta'(h)(\nabla u_h \cdot \nabla p_h)\widehat{h}\,\mathrm{d}x$$

  lends itself to a straightforward choice of a descent direction:

$$\widehat{h} = -\zeta'(h)(\nabla u_h \cdot \nabla p_h),$$

  that is, $\widehat{h}$ is the (negative) $L^2(D)$ gradient of $J'(h)$.

- Other choices are possible (and often more adequate) by changing inner products:

$$\widehat{h} = -V,$$

  where V solves:

$$\forall w \in H, \quad \langle V, w\rangle_H = J'(h)(w),$$

  for an adapted choice of Hilbert space and inner product H and $\langle\cdot,\cdot\rangle_H$.

- This operation is often called sensitivity filtering in density-based methods.

Part V

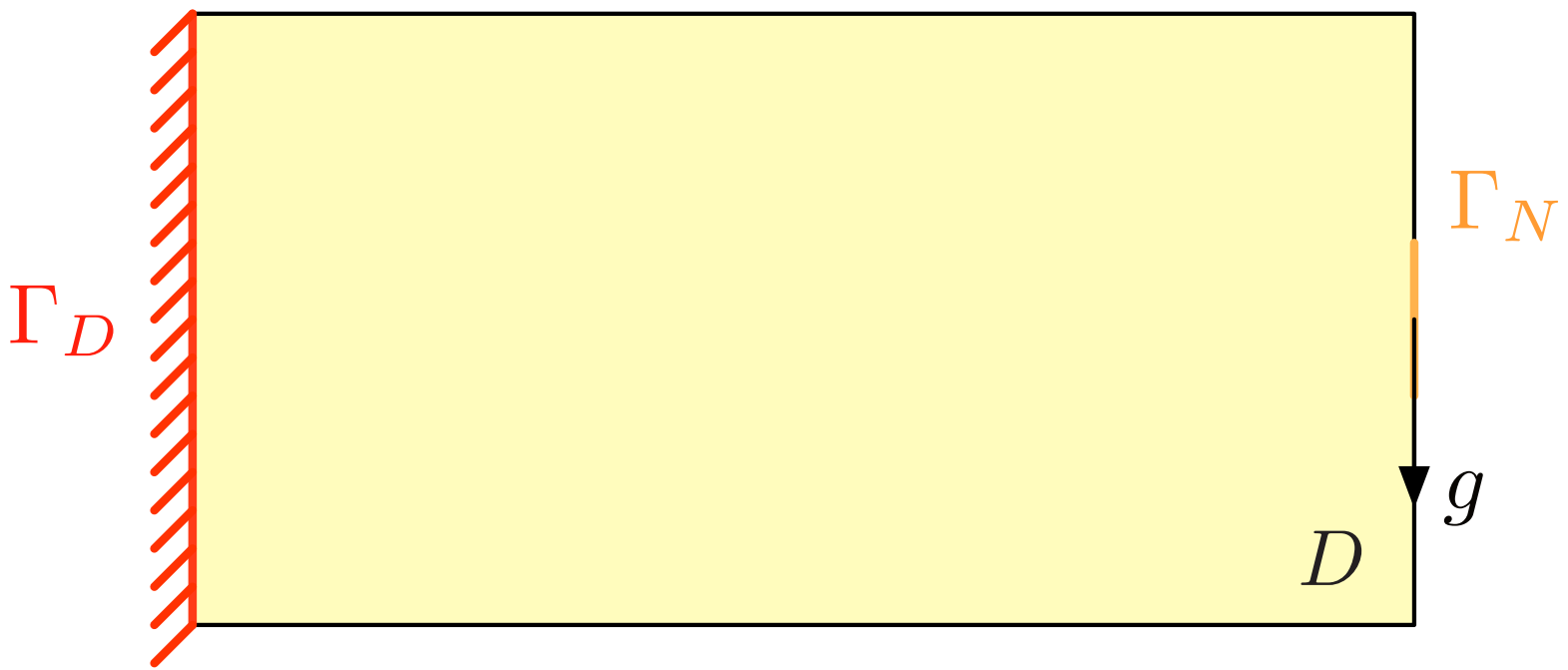
Topology optimization

Example: the cantilever benchmark

- In the context of linearized elasticity, the compliance of a cantilever beam is minimized:

$$C(h) = \int_D \zeta(h) A e(u_h) : e(u_h) \, \mathrm{d}x.$$

- A constraint on the volume $\mathrm{Vol}(h) = \int_D h \, dx$ of the structure is added.

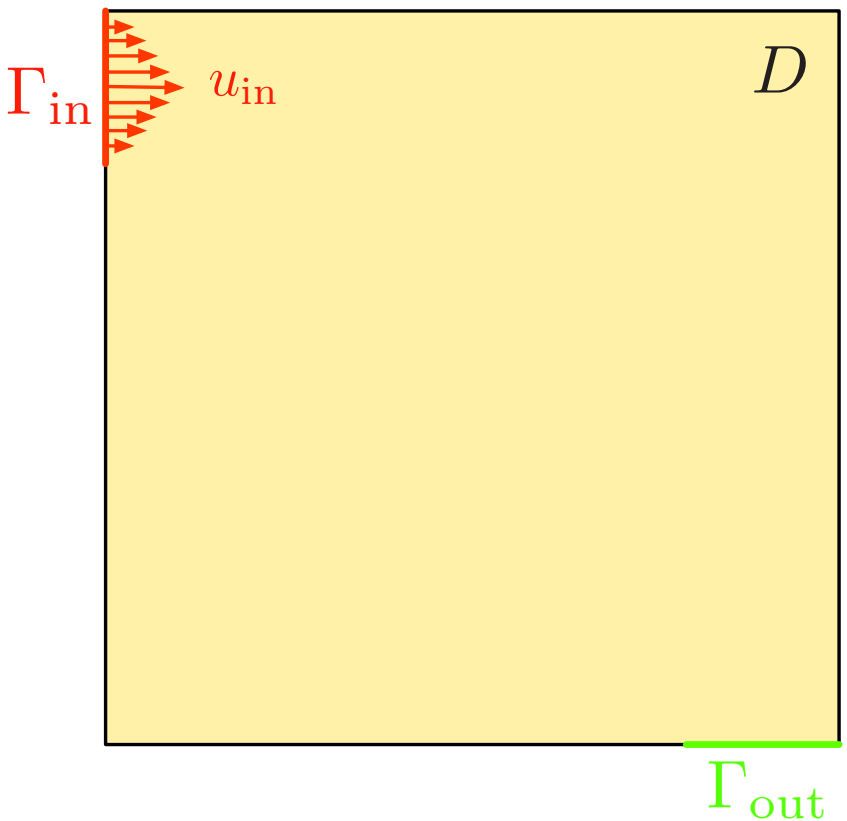


Example: the pipe benchmark

- In the context of fluid mechanics, the viscous dissipation within a pipe is minimized:

$$C(h) = 2\mu \int_D \zeta(h) D(u_h) : D(u_h) \, \mathrm{d}x.$$

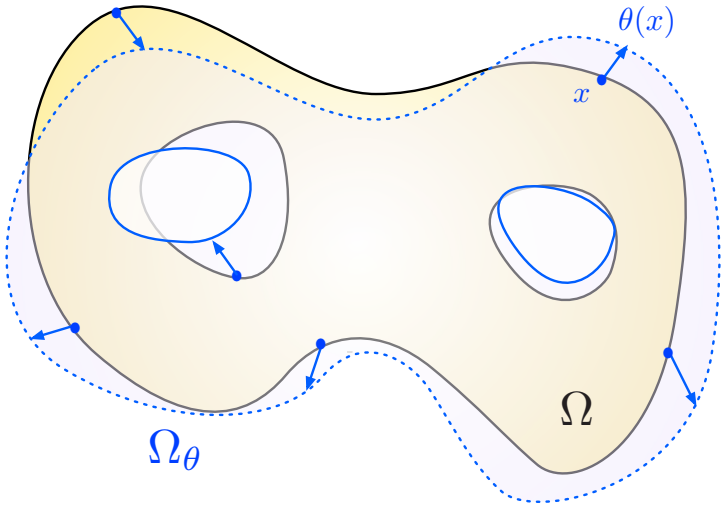
- A constraint on the volume $\mathrm{Vol}(h) = \int_D h \, dx$ of the pipe is added.



Overview of the next courses

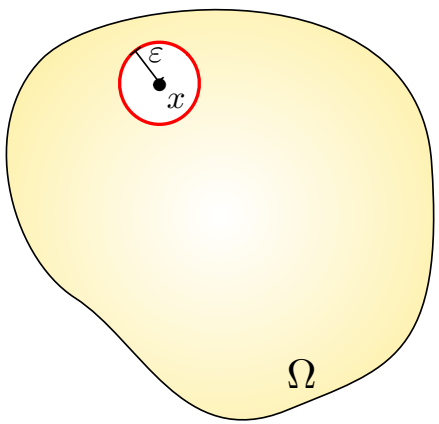
- `Course Diaraf` Topology optimization.

  - Topological derivatives;
  - A few worked examples.



- `Course Ilias` Geometric shape optimization.

  - Existence and non existence of optimal shapes;
  - Shape derivatives;
  - (If time permits) Use of neural network techniques.



Thank you for your attention!

Technical appendix

Fréchet and Gateaux derivatives

Several notions of derivative are available for a function $F : U \to V$ between two normed vector spaces $(U, ||\cdot||_U)$ and $(V, ||\cdot||_V)$.

Definition (Fréchet differentiability).

- *A function $F : U \to V$ is called Fréchet differentiable at some point $x \in U$ if there exists a linear, continuous mapping $L_x : U \to V$ such that:*

$$F(x+v) = F(x) + L_x(v) + \mathrm{o}(||v||_U), \text{ where } \frac{||\mathrm{o}(||v||_U)||_V}{||v||_U} \xrightarrow{v \to 0} 0.$$

- *The mapping $v \mapsto L_x(v)$ is denoted by $v \mapsto F'(x)(v)$, or $\mathrm{d}_x F(v)$ and is called the Fréchet derivative of F at x.*

- *The function $F : U \to V$ is called Gateaux differentiable at $x \in U$ if for any direction $v \in U$, the following limit exists:*

$$\lim_{\substack{t \to 0 \\ t > 0}} \frac{F(x+tv) - F(x)}{t}.$$

Remark: The notion of Fréchet differentiability is stronger than that of Gateaux differentiability, which is a generalization of directional differentiability.

Fréchet derivatives: the "chain rule"

The chain rule is a *fundamental result*, which supplies the Fréchet derivative of the composite $G \circ F$ of two functions

$$F : U \to V \text{ and } G : V \to W$$

between three normed vector spaces $(U, ||\cdot||_U)$, $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$.

Theorem (Chain rule).

Let $x \in U$ be a point such that:

- *F is Fréchet differentiable at x;*
- *G is Fréchet differentiable at $F(x) \in V$.*

Then, the composite function $G \circ F : U \to W$ is Fréchet differentiable at x, and its Fréchet derivative $v \mapsto (G \circ F)'(x)(v)$ is the linear mapping defined by:

$$\forall v \in U, \ (G \circ F)'(x)(v) = G'(F(x))(F'(x)(v)).$$

The homogenization method in a nutshell (I)

- We consider the following wo-phase shape optimization problem:

$$\min_{\Omega \subset D} J(\Omega), \text{ where } J(\Omega) := \int_D j(u_\Omega)\,\mathrm{d}x, \qquad (\mathrm{SO})$$

and $j : \mathbb{R} \to \mathbb{R}$ is a given, smooth function.

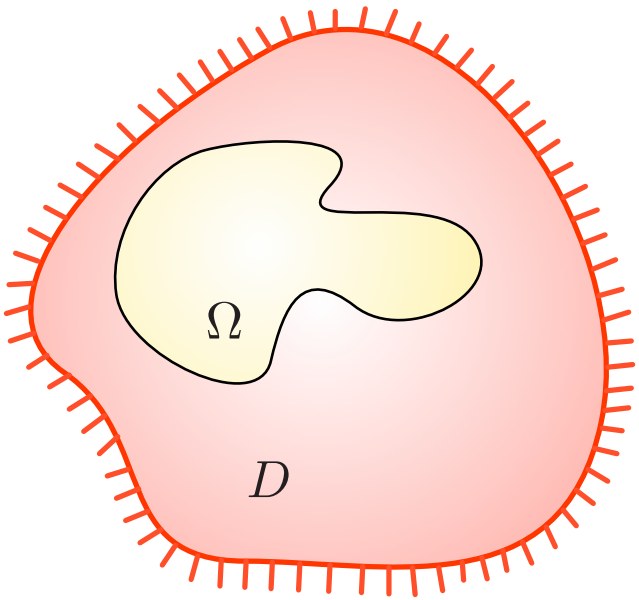
- Here, the temperature $u_\Omega \in H_0^1(D)$ is the solution to:

$$\begin{cases} -\mathrm{div}(\gamma_\Omega \nabla u_\Omega) &= f & \text{in } D, \\ u_\Omega &= 0 & \text{on } \Gamma_D, \end{cases}$$

where the conductivity γ_Ω is of the form:

$$\gamma_\Omega(x) = \alpha + \chi_\Omega(x)(\beta - \alpha), \quad x \in D.$$

- This problem does not have a solution in general: minimizing sequences tend to create "microstructures".



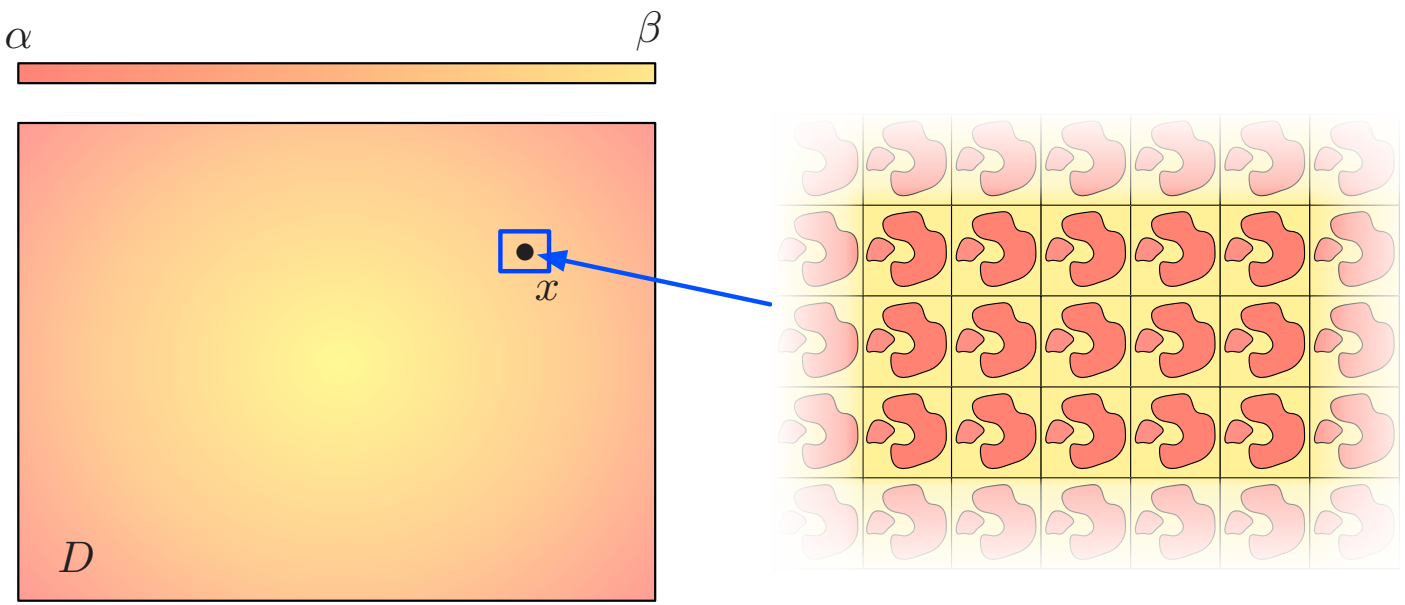
The homogenization method in a nutshell (II)

The homogenized formulation of the problem extends the set of designs to composite structures:

$$(\theta, A) \in \mathcal{CD} := \left\{ \theta \in L^\infty(D, [0,1]),\ \ A(x) \in G_{\theta(x)} \text{ a.e. } x \in D \right\},$$

where:

- $\theta(x) \in [0,1]$ is the local density of material β around $x \in D$;
- $A(x) \in \mathbb{R}^{d \times d}$ is the microstructure tensor, in the set $G_{\theta(x)}$ of all the tensors obtained by homogenizing materials α, β in proportions $(1 - \theta(x))$ and $\theta(x)$.



The homogenization method in a nutshell (III)

The relaxed version of the original problem is:

$$\min_{(\theta,A)\in\mathcal{CD}} J(\theta,A) := \int_D j(u_{\theta,A})\,\mathrm{d}x, \tag{H}$$

where the temperature $u_{\theta,A}$ induced by the composite structure (θ, A) satisfies:

$$\begin{cases} -\mathrm{div}(A\nabla u_{\theta,A}) = f & \text{in } D, \\ u_{\theta,A} = 0 & \text{on } \partial D. \end{cases}$$

Under "suitable" hypotheses, the following facts hold true:

- The problem (H) has at least one global minimizer (θ^*, A^*).
- Every such global minimizer is the limit, in the sense of homogenization, of a sequence of "classical designs" $\Omega^n \subset D$.
- Every minimizing sequence Ω^n of the original problem (SO) converges in the sense of homogenization to a global minimizer of the relaxed problem (H).

Composite materials (I)

- A pattern $\omega \subset Y := (0,1)^d$ induces a periodic distribution of the two conductive phases α and β at any scale ε within D:

$$A_\omega^\varepsilon(x) = A_\omega\left(\frac{x}{\varepsilon}\right), \quad x \in D,$$

where for $y \in Y$,

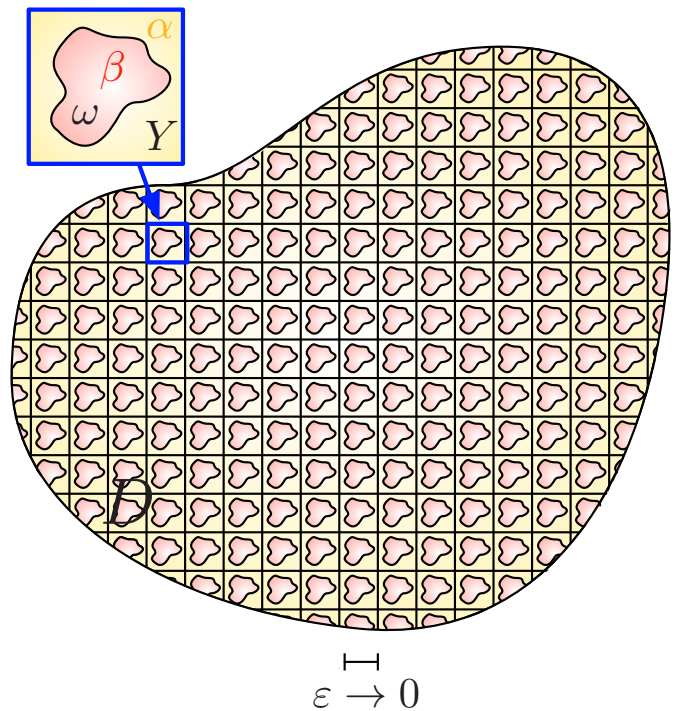
$$A_\omega(y) := \begin{cases} \beta & \text{if } x \in \omega, \\ \alpha & \text{if } x \notin \omega. \end{cases}$$

- The effective conductivity inside D as $\varepsilon \to 0$ is given by the matrix A_ω^* with entries:

$$(A_\omega^*)_{i,j} = \int_Y A_\omega(y)(e_i + \nabla w_i(y)) \cdot (e_j + \nabla w_j(y))\,\mathrm{d}y,$$

where the $w_i \in H^1_\#(Y)$ are the cell functions:

$$\begin{cases} -\mathrm{div}(A_\omega(y)(e_i + \nabla w_i)) = 0 & \text{in } Y, \\ y \mapsto w_i(y) & \text{is } Y-\text{periodic}. \end{cases}$$



Composite materials (II)

- The set G_θ is made of all the effective conductivities A^*_ω of periodic arrangements of the phases α and β induced by patterns $\omega \subset Y$ with volume fraction $|\omega| = \theta$.

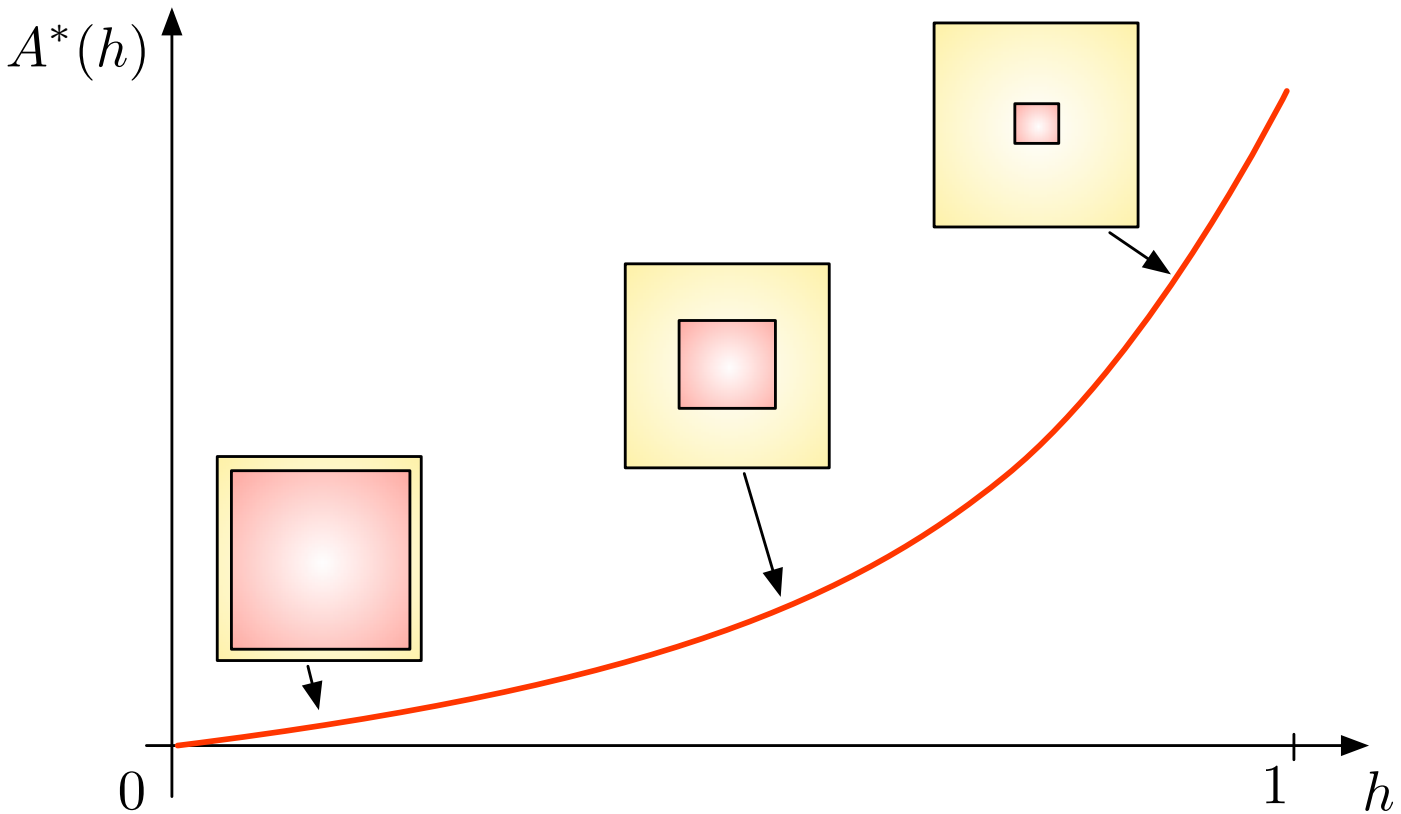
- This set can be characterized explicitly in the present two-phase conductivity setting, but this characterization is difficult to handle in practice.

- In general, only bounds are known about G_θ: the Hashin-Shtrikman bounds.

From homogenization to density-based methods (III)

One simplification proposition [BenKik]:

- Only one microstructure pattern (e.g. a square inclusion) is retained for a given value $h \in [0, 1]$ of the volume fraction, or density.
- The homogenization tensor $A^*(h)$ is then completely parametrized by h.



From homogenization to density-based methods

- In numerical practice, a continuous expression $h \mapsto A^*(h)$ is constructed by:
  - Calculating $A^*(h_i)$ from the homogenization formulas, using the Finite Element Method on the unit cell Y, for sample values $0 \leq h_1 < \ldots < h_N \leq 1$.
  - Interpolating the quantities $A^*(h_i)$ using e.g. polynomial functions (splines).
- The density-based version of the problem reads:

$$\min_{h \in L^\infty(D,[0,1])} J(h), \quad J(h) = \int_D j(u_h)\,\mathrm{d}x,$$

where $u_h \in H^1_0(D)$ is the solution to:

$$\begin{cases} -\mathrm{div}(A^*(h)\nabla u_h) = f & \text{in } D, \\ u_h = 0 & \text{on } \partial D. \end{cases}$$

The problem falls in the setting of parametric optimization.

- One step further in this direction consists in dropping completely the requirement that the material law $h \mapsto A^*(h)$ be "physical"...

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[WanSig] F. Wang, B. S. Lazarov, and O. Sigmund, *On projection methods, convergence and robust formulations in topology optimization*, Structural and Multidisciplinary Optimization, 43 (2011), pp. 767–784.

Online resources I

- [DTU] *Web page of the Topopt group at DTU*, `http://www.topopt.dtu.dk`.
- [FreeFem++] O. Pironneau, F. Hecht, A. Le Hyaric, *FreeFem++ version 2.15-1*, `http://www.freefem.org/ff++/`.
- [TopWebinar] O. Sigmund, *Density-based topology optimization*, webinar on the SIMP method, `https://www.youtube.com/watch?v=DWo2v7QLH2M&t=265s`.