Exercise 1

(1) Before we perform any calculation, let us note that the proposed function $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = x - x^3$ is odd, since:

$$\forall x \in (-1, 1), \ f(-x) = (-x) - (-x)^3 = -(x - x^3) = -f(x).$$

Hence, all the coefficients a_n, $n = 0, \dots$ appearing in the full Fourier expansion of f vanish, which leaves us with the sole calculation of the b_n coefficients, for $n = 1, \dots$.
We now use the formula from the lectures:

$$
\begin{aligned}
b_n &= \int_{-1}^{1} f(x) \sin(n\pi x) \, dx \\
&= 2 \int_{0}^{1} (x - x^3) \sin(n\pi x) \, dx \\
&= 2 \left(\left[-(x - x^3) \frac{\cos(n\pi x)}{n\pi} \right]_0^1 + \frac{1}{n\pi} \int_0^1 (1 - 3x^2) \cos(n\pi x) \, dx \right) \\
&= \frac{2}{n\pi} \left(\left[(1 - 3x^2) \frac{\sin(n\pi x)}{n\pi} \right]_0^1 + \frac{6}{n\pi} \int_0^1 x \sin(n\pi x) \, dx \right) \\
&= \frac{12}{n^2 \pi^2} \left(\left[-x \frac{\cos(n\pi x)}{n\pi} \right]_0^1 + \frac{1}{n\pi} \int_0^1 \cos(n\pi x) \, dx \right) \\
&= \frac{-12 \cos(n\pi)}{n^3 \pi^3} \\
&= \frac{12(-1)^{n+1}}{n^3 \pi^3}
\end{aligned}
$$

where the second line is obtained from the first one by using the fact that f is odd.

(2) Let us apply the theorem of the lectures about uniform convergence of Fourier series. We know that:

- f, f' and f'' exist and are continuous over the interval $(-1, 1)$,
- since we consider the full Fourier expansion of f, that is is expansion along the eigenfunctions of the operator $X \mapsto -X''$ over functions $X : (-1, 1) \to \mathbb{R}$ enjoying *periodic boundary conditions*, i.e.

$$X(-1) = X(1), \ X'(-1) = X'(1),$$

we have to check that f itself complies with these boundary conditions. We have actually:

$$f(-1) = f(1) = 0, \ \text{and} \ f'(-1) = f'(1) = 1 - 2.(-1)^2 = -1,$$

which is the desired result.

By the theory of uniform convergence for Fourier series, it follows that the expansion derived in Question (1) converges uniformly to f on $(-1, 1)$. In particular, for any $x \in [-1, 1]$ (that is, including the endpoints), one has:

$$\sum_{n=1}^{N} b_n \sin(n\pi x) \overset{N \to \infty}{\longrightarrow} f(x).$$

(3) It is easily checked that:

- if n is even, say $n = 2p$ for some $p = 1, \dots$, then $\sin(\frac{n\pi}{2}) = \sin(p\pi) = 0$,
- if n is odd, say $n = 2p + 1$, then $\sin(\frac{n\pi}{2}) = \sin\left(\left(p + \frac{1}{2}\right)\pi\right) = (-1)^p$.

(4) As suggested by the Result of the previous question, let us evaluate the series at $x = \frac{1}{2}$ (which is possible since we know that it converges pointwise at any point). This supplies:

$$
\begin{aligned}
f\left(\tfrac{1}{2}\right) = \tfrac{1}{2} - \tfrac{1}{8} = \tfrac{3}{8} \quad &= \quad \sum_{n=1}^{\infty} b_n \sin\left(\tfrac{n\pi}{2}\right) \\
&= \quad \sum_{p=0}^{\infty} b_{2p+1}(-1)^p \\
&= \quad \sum_{p=0}^{\infty} \tfrac{12(-1)^{2p+2}}{(2p+1)^3 \pi^3}(-1)^p \\
&= \quad \tfrac{12}{\pi^3} \sum_{p=0}^{\infty} \tfrac{(-1)^p}{(2p+1)^3}
\end{aligned}
$$

which readily yields that:

$$
\sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)^3} = \frac{\pi^3}{32}.
$$

Exercise 2

(1) The coefficients a_n of the cosine Fourier series of f write:

$$
\begin{aligned}
a_0 &= \quad \tfrac{2}{\pi} \int_0^\pi (\pi - x)\, dx \\
&= \quad \tfrac{2}{\pi} \left[-\tfrac{(\pi-x)^2}{2} \right]_0^\pi \\
&= \quad \pi
\end{aligned}
$$

and, for $n \geq 1$:

$$
\begin{aligned}
a_n &= \quad \frac{2}{\pi} \int_0^\pi (\pi - x)\cos(nx)\, dx \\
&= \quad \frac{2}{\pi} \left(\left[(\pi - x)\frac{\sin(nx)}{n} \right]_0^\pi + \frac{1}{n} \int_0^\pi \sin(nx) \right) \\
&= \quad \frac{2}{n\pi} \int_0^\pi \sin(nx) \\
&= \quad \frac{2}{n^2\pi} \left[-\cos(nx) \right]_0^\pi \\
&= \quad \frac{2}{n^2\pi}(1 - (-1)^n) \\
&= \quad \begin{cases} 0 & \text{if n is even,} \\ \frac{4}{n^2\pi} & \text{if n is odd} \end{cases}
\end{aligned}
$$

The cosine Fourier expansion of f over $(0, \pi)$ then writes:

(1)
$$
f(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} \cos((2p+1)x).
$$

(2) In order to use the pointwise convergence theory for f, we must return to the context of full Fourier series. We know that the cosine Fourier expansion (1) of f over $(0, \pi)$ is also the full Fourier expansion of the even extension f_{even} of f to the symmetric interval $(-\pi, \pi)$; see Figure 1.

We notice that the periodic extension $f_{\text{even,per}}$ of the even extension f_{even} of f is continuous over $\mathbb{R}$. We then have, for any $x \in [-\pi, \pi]$:

$$
\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) \xrightarrow{N \to \infty} \frac{f_{\text{even,per}}(x^-) + f_{\text{even,per}}(x^+)}{2} = f_{\text{even,per}}(x).
$$

The pointwise limit of the series $\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx)$ is thus $\pi - x$ if $x \in [0, \pi]$ and $\pi + x$ if $x \in [-\pi, 0]$.
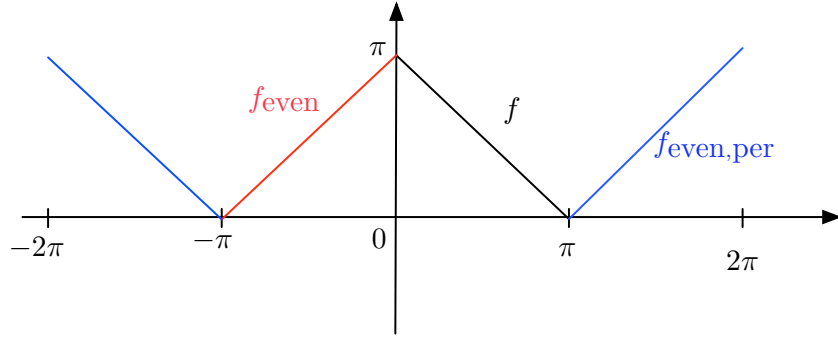
FIGURE 1. Even and periodic extensions of f.

(3) We evaluate the series (1) at $x = 0$, which is possible because of the pointwise convergence result of the previous question. It comes:

$$\pi = \frac{\pi}{2} + \frac{4}{\pi} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} \cos(0),$$

thus:

$$\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{\pi}{2} \cdot \frac{\pi}{4} = \frac{\pi^2}{8}.$$

(4) The only thing to check when it comes to the L^2 convergence of Fourier series is that: $||f||_{L^2} := \sqrt{\int_0^\pi f^2(x)\, dx} < \infty$. We calculate:

$$
\begin{aligned}
||f||_{L^2}^2 &= \int_0^\pi f^2(x)\, dx \\
&= \int_0^\pi (\pi - x)^2\, dx \\
&= \left[-\frac{(\pi - x)^3}{3} \right]_0^\pi \\
&= \frac{\pi^3}{3},
\end{aligned}
$$

which is obviously finite. Hence, the considered cosine Fourier series converges to f in the L^2 sense.

(5) We apply Parseval's formula:

$$||f||_{L^2}^2 = \left(\frac{a_0}{2} \right)^2 ||1||^2 + \sum_{n=1}^{\infty} b_n^2 ||\cos(nx)||^2,$$

and from the results of Question (1) and (4), and the fact that:

$$
\begin{aligned}
||\cos(nx)||^2 &= \int_0^\pi \cos^2(n\pi x)\, dx \\
&= \int_0^\pi \frac{1 + \cos(2n\pi x)}{2} \quad, \\
&= \frac{\pi}{2}
\end{aligned}
$$

and $||1||^2 = \pi$, we obtain:

$$
\begin{aligned}
\frac{\pi^3}{3} &= \pi \left(\frac{\pi}{2} \right)^2 + \sum_{p=0}^{\infty} \frac{16}{\pi^2 (2p+1)^4} \frac{\pi}{2} \\
&= \frac{\pi^3}{4} + \frac{8}{\pi} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^4}
\end{aligned}
\quad,
$$

3

thus:
$$\sum_{p=0}^{\infty} \frac{1}{(2p+1)^4} = \frac{\pi^4}{96}.$$

(5) As suggested, we observe that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{p=1}^{\infty} \frac{1}{(2p)^2} + \sum_{p=1}^{\infty} \frac{1}{(2p+1)^2}.$$

Hence:
$$\left(1 - \frac{1}{4}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{p=1}^{\infty} \frac{1}{(2p+1)^2},$$

that is, using the result of Question (3):
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{p=1}^{\infty} \frac{1}{(2p+1)^2} = \frac{\pi^2}{6}.$$

Exercise 3

We consider the PDE:
$$(2) \qquad \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0.$$

(1) The considered PDE is a first-order linear PDE. It can then be solved by means of the method of characteristics. The characteristic curves $s \mapsto (t(s), x(s))$ are solution to:
$$\begin{cases} t'(s) &= 1 \\ x'(s) &= 1 \end{cases} \Rightarrow \begin{cases} t(s) &= s \\ x(s) &= s + c \end{cases},$$

up to a change in the parametrization of the curves, for some constant c. Now, fixing a particular characteristic curve, the value function $z(s) := u(t(s), x(s))$ along this characteristic curves is constant, since $z'(s) = 0$. There exists then a function f such that:
$$u(t(s), x(s)) = f(c),$$

and returning to the original variables (t, x), the solutions to the considered PDE are of the form:
$$u(t, x) = f(x - t),$$

for an arbitrary differentiable function f.

(2) Let $u(t, x) = T(t)X(x)$ be a separated solution to (2). Then one has:
$$\frac{\partial u}{\partial t}(t, x) = T'(t)X(x), \quad \frac{\partial u}{\partial x}(t, x) = T(t)X'(x).$$

Hence, using these expression in (2), we have:
$$\forall t > 0, x \in \mathbb{R}, \ T'(t)X(x) + T(t)X'(x) = 0,$$

thus:
$$-\frac{T'(t)}{T(t)} = \frac{X'(x)}{X(x)}.$$

The last equality is an equality between a function which only depends on the t variable (on the left-hand side) and an function which only depends on the x variable (on the right-hand side). Since it holds for any values $t > 0$ and $x \in \mathbb{R}$, we conclude that both are equal to a constant $\lambda \in \mathbb{R}$, *which is the same for both*. In other terms:
$$-\frac{T'(t)}{T(t)} = \frac{X'(x)}{X(x)} = \lambda,$$

which readily leads to the desired conclusion.

(3) By using the system

$$\begin{cases} X'(x) = \lambda X(x) \\ T'(t) = -\lambda T(t) \end{cases},$$

it comes that there exist constants $c, d \in \mathbb{R}$ such that:

$$\begin{cases} \forall x \in \mathbb{R}, \ X(x) = ce^{\lambda x} \\ \forall t > 0, \ T(t) = de^{-\lambda t} \end{cases}.$$

The solution $u(t,x) = T(t)X(x)$ then reads:

(3) $$u(t,x) = ce^{\lambda(x-t)},$$

for some constants $c, \lambda \in \mathbb{R}$. Conversely, any function of the form (3) for some constants c, λ is obviously a separated to the PDE (2), which means that these functions are exactly all the separated solutions to (2).

Exercise 4

(1) The graph of f is reported on Figure 2.

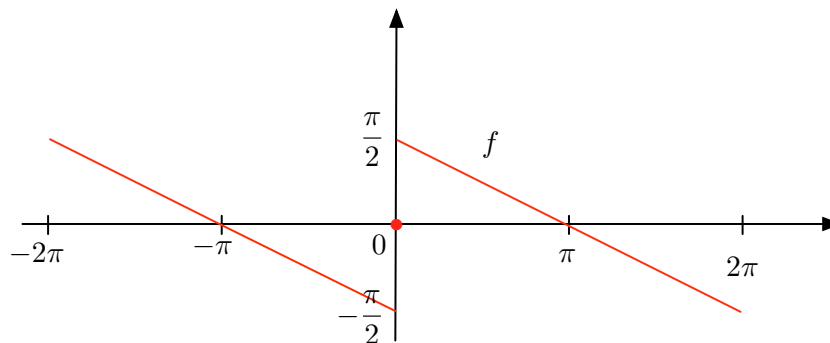

FIGURE 2. Graph of f in Exercise 4.

(2) It is impossible that the full Fourier series of f converge uniformly to f on $(-\pi, \pi)$, for if it were the case, it would converge pointwise to f, which is not the case since f presents jump discontinuities.

(3) We apply the pointwise convergence theory to the full Fourier series of f on $[-\pi, \pi]$, which is legitimate since f is piecewise continuous on $(-\pi, \pi)$, as well as its derivative f'. We notice that the only jump discontinuity of f (when seen as a function defined over the whole $\mathbb{R}$, extended from $(-\pi, \pi)$ by 2π-periodicity) is at $x = 0$ (notice that this periodic extension is continuous at $x = -\pi$ and $x = \pi$). Thus, the full Fourier series $\frac{a_0}{2} + \sum a_n \cos(nx) + b_n \sin(nx)$ of f has the pointwise limit:

$$\forall x \in [-\pi, \pi] \setminus \{0\}, \quad \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx) \overset{N \to \infty}{\longrightarrow} f(x),$$

and, for $x = 0$, the pointwise limit is $\frac{f(0^-)+f(0^+)}{2} = \frac{-\frac{\pi}{2}+\frac{\pi}{2}}{2} = 0$.

(4) Let us now calculate the coefficients of the full Fourier expansion of f over $(-\pi, \pi)$. First, since f

is odd, we have $a_n = 0$, $n = 0, ...$ Then, for $n \geq 1$:

$$
\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \\
&= \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx) \, dx \\
&= \frac{2}{\pi} \int_{0}^{\pi} \frac{\pi - x}{2} \sin(nx) \, dx \\
&= \frac{2}{\pi} \left(\left[-\frac{\pi - x}{2} \frac{\cos(nx)}{n} \right]_{0}^{\pi} - \frac{1}{2n} \int_{0}^{\pi} \cos(nx) \, dx \right) \\
&= \frac{1}{n},
\end{aligned}
$$

where we used the fact that f is odd to pass from the first line to the second.

(5) We use the pointwise convergence result of Question (3) at $x = 1$:

$$
f(1) = \frac{\pi - 1}{2} = \sum_{n=1}^{\infty} \frac{\sin(n)}{n}.
$$

(6) We compute the squared L^2 norm:

$$
\begin{aligned}
||f||_{L^2}^2 &= \int_{-\pi}^{\pi} f^2(x) \, dx \\
&= 2 \int_{0}^{\pi} f^2(x) \, dx \\
&= \frac{1}{2} \int_{0}^{\pi} (\pi - x)^2 \, dx \\
&= \frac{1}{6} \left[-(\pi - x)^3 \right]_{0}^{\pi} \\
&= \frac{\pi^3}{6}
\end{aligned}
$$

where we used the fact that f^2 is even (for f is odd) to pass from the first line to the second. Thus $||f||_{L^2} = \sqrt{\frac{\pi^3}{6}}$ is finite, and the full Fourier expansion of f converges to f in the L^2 sense.

(7) Let us eventually apply Parseval's equality. We have already calculated $||f||_{L^2}^2$. We have:

$$
||f||_{L^2}^2 = \sum_{n=1}^{\infty} b_n^2 ||\sin(nx)||^2,
$$

where

$$
||\sin(nx)||^2 = \int_{-\pi}^{\pi} \sin^2(nx) \, dx = \int_{-\pi}^{\pi} \frac{1 - \cos(2nx)}{2} \, dx = \pi.
$$

Thus,

$$
\frac{\pi^3}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \pi,
$$

and we have proved that:

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
$$

Exercise 5

We consider the Laplace equation of a function $u(x, y)$ of two variables:

(4)
$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,
$$

when (x, y) belongs to the rectangles $R = \{0 \leq x \leq \ell, \ 0 \leq y \leq h\}$. Equation (4) is complemented by the following *non homogeneous* Dirichlet boundary conditions:

(5) $$\forall y \in [0, h], \ u(0, y) = g(y),$$

(6) $$\forall y \in [0, h], \ u(\ell, y) = 0,$$

(7) $$\forall x \in [0, \ell], \ u(x, 0) = 0,$$

(8) $$\forall y \in [0, \ell], \ u(x, h) = 0.$$

(1) Let $u(x, y) = X(x)Y(y)$ be a separated solution of (4) (6) (7) (8). We have:

$$\frac{\partial^2 u}{\partial x^2}(x, y) = X''(x)Y(y), \ \frac{\partial^2 u}{\partial y^2}(x, y) = X(x)Y''(y),$$

so that (4) rewrites:

$$X''(x)Y(y) + X(x)Y''(y) = 0,$$

or:

$$-\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)}.$$

This last equality is an equality between of function of the sole variable x and a function of the sole variable y. It holds for for any $x \in (0, \ell)$ and $y \in (0, h)$, so that both functions should be constant. Hence, there exists $\lambda \in \mathbb{R}$ such that:

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ Y''(y) + \lambda Y(y) = 0 \end{cases}.$$

What's more, using (6) yields that

$$y \in [0, h], \ X(\ell)Y(y) = 0,$$

and for Y (thus u) to be non identically 0, one should have $X(\ell) = 0$. The boundary conditions $Y(0) = Y(h) = 0$ for Y are deduced from (7) and (8) in the same way.

(2) This is exactly the same thing as we did during the lectures! Consider a function Y such that $Y''(y) - \beta^2 Y(y) = 0$ for $y \in (0, h)$. Then, Y is of the form:

$$Y(h) = a \cosh(\beta x) + b \sinh(\beta x),$$

for some constants a, b to be determined. But $Y(0) = 0$ implies that $a = 0$. Now, $Y(h) = 0$ implies that $b \sinh(\beta h) = 0$, and since $\beta h \neq 0$, $\sinh(\beta h) \neq 0$, so that necessarily $b = 0$, and Y is the null function.

Consequently, $\lambda = -\beta^2$ is not an eigenvalue of the problem.

(3) This is again as in the lectures! If Y is such that $Y''(y) - 0.Y(y) = Y''(y) = 0$ for $y \in (0, h)$, then $Y(y)$ is of the form $Y(y) = ay + b$, for a, b to be determined. But $Y(0) = 0$ implies that $b = 0$, and then $Y(h) = 0$ implies that $a = 0$. Again, Y is the null function and 0 is not an eigenvalue of the problem.

(4) This is (again) as in the lectures! Let Y be a function such that $Y''(y) + \beta^2 Y(y) = 0$ for $y \in (0, h)$. Then, Y is of the form:

$$Y(h) = a \cos(\beta x) + b \sin(\beta x),$$

for a, b to be determined. The condition $Y(0) = 0$ now implies that $a = 0$, and the condition $Y(h) = 0$ gives:

$$b \sin(\beta h) = 0.$$

Now, for Y to be non identically 0, one should have $b \neq 0$, thus $\sin(\beta h) = 0$. Then β is of the form $\beta_n = \frac{n\pi}{h}$, for $n = 1, ...$ In this case, $Y \equiv Y_n$ takes the form

$$Y_n(y) = b \sin\left(\frac{n\pi y}{h}\right),$$

for some constant b.

Hence, the positive eigenvalues of the problem are the $\lambda_n := \left(\frac{n\pi}{h}\right)^2$, $n = 1, ...$, and a corresponding eigenfunction is given by $Y_n(y) = \sin\left(\frac{n\pi x}{h}\right)$.

(5) In the case $\lambda = \lambda_n$, the corresponding equation for $X(x) = X_n(x)$ is:

$$X_n''(x) - \beta_n^2 X_n(x) = 0,$$

supplemented with the condition $X_n(\ell) = 0$. Thus, X_n is of the form:

$$\forall x \in [0, \ell], \ \ X_n(x) = a\cosh(\beta_n x) + b\sinh(\beta_n x),$$

for some constants a, b to be determined. The condition $X_n(\ell) = 0$ now implies that:

$$a\cosh(\beta_n \ell) + b\sinh(\beta_n \ell) \Rightarrow a = -\frac{b\sinh(\beta_n \ell}{\cosh(\beta_n \ell)},$$

since $\cosh(\beta_n \ell) \neq 0$. Hence, X_n has the form:

$$\begin{aligned} X_n(x) &= b\frac{-\cosh(\beta_n x)\sinh(\beta_n \ell) + \cosh(\beta_n \ell) + \sinh(\beta_n x)}{\cosh(\beta_n \ell)} \\ &= b\frac{\sinh(\beta_n(x-\ell))}{\cosh(\beta_n \ell)} \end{aligned},$$

for some constant b, by using the identity $\sinh(a-b) = \sinh(a)\cosh(b) - \sinh(b)\cosh(a)$. Up to changing the constant b into another constant (still denoted as b), we can rewrite:

$$X_n(x) = b\sinh(\beta_n(x-\ell)).$$

(6) By invoking the superposition principle, any finite linear combination of solutions to the *homogeneous* PDE (4) with *homogeneous* boundary conditions (6) (7) (8) is again solution to the same system (in particular, we may use the separated solutions we have found in the previous questions). If we have at our disposal a convenient notion of convergence of infinite sums, it is then tempting to say that any infinite sum of such solutions is again solution, and that the general solution of (4) (6) (7) (8) can be written as an infinite sum of separated solutions:

$$(9) \qquad\qquad u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{h}\right) \sinh\left(\frac{n\pi}{h}(x-\ell)\right),$$

where the coefficients b_n are constants yet to be determined.

(7) Let now $u(t,x)$ be a solution to (4) (6) (7) (8), which also satisfies the non homogeneous boundary condition (5). In particular, $u(t,x)$ takes the general form (9), in which the coefficients b_n are to be found by using the fact that u also satisfies (5). Assuming that the infinite sum (9) converges pointwise, this reads:

$$\forall y \in [0, h], \ \ u(0,y) = \sum_{n=1}^{\infty} \left(-b_n \sinh\left(\frac{n\pi \ell}{h}\right) \right) \sin\left(\frac{n\pi y}{h}\right) = g(y),$$

which is then nothing but a sine Fourier decomposition for the boundary condition g! More precisely, this means that the coefficients $\left(-b_n \sinh\left(\frac{n\pi \ell}{h}\right) \right)$ should be the coefficients appearing in the sine Fourier decomposition of g over $(0, h)$. By using the explicit form of these coefficients, one has:

$$\left(-b_n \sinh\left(\frac{n\pi \ell}{h}\right) \right) = \frac{2}{h} \int_0^h g(y) \sin\left(\frac{n\pi y}{h}\right) \, dy,$$

whence:

$$b_n = -\frac{2}{h\sinh\left(\frac{n\pi \ell}{h}\right)} \int_0^h g(y) \sin\left(\frac{n\pi y}{h}\right) \, dy.$$