Elementary partial differential equations: 2^{nd} midterm

This midterm is purposely too long for you to complete within the allotted time, and of course, it is not assumed that you cover all the exercises to get the maximal grade. The indicated grading policy is only provisional, and aimed at illustrating the relative weights of the different exercises.

Exercise 1 (5 points)

Let $f : (-1, 1) \to \mathbb{R}$ be the function defined by $f(x) = x - x^3$.

(1) Calculate the coefficients a_n, $n = 0, 1, ...$ and b_n, $n = 1, ...$ appearing in the full Fourier series of f:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\pi x) + b_n \sin(n\pi x)). \tag{1}$$

[Hint: a preliminary study of the parity of f can save a lot of effort.]

(2) Show that this series converges uniformly to f on the interval $(-1, 1)$, and infer the pointwise limit of the series of Question (1), for any point $x \in \mathbb{R}$.

(3) For $n = 1, 2, ...$, compute the value of $\sin(\frac{n\pi}{2})$ depending on the parity of n.

(4) By evaluating the series (1) at a particular point $x \in \mathbb{R}$, show that:

$$\sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)^3} = \frac{\pi^3}{32}.$$

Exercise 2 (8 points)

Let $f : (0, \pi) \to \mathbb{R}$ be the function defined by $f(x) = \pi - x$.

(1) Compute the cosine Fourier coefficients of f and write down the corresponding expansion on $(0, \pi)$.

(2) By using the pointwise convergence theory for Fourier series, compute the limit of the expansion of Question (1) for any point $x \in [-\pi, \pi]$ (*including the endpoints*).

[Warning: Remember that the study of the pointwise convergence of Fourier series only applies to full Fourier series.]

(3) By evaluating this series at a particular point $x \in [-\pi, \pi]$, calculate the value of the infinite sum $\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2}$.

(4) Does the cosine Fourier series of Question (1) converge to f in the L^2 sense?

(5) By using Parseval's formula, calculate the value of the infinite sum $\sum_{p=0}^{\infty} \frac{1}{(2p+1)^4}$.

[Warning: Remember to be careful about how to deal with the term for $n = 0$ with Parseval's formula].

(6) Calculate the value of the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

[Hint: notice that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{p=1}^{\infty} \frac{1}{(2p)^2} + \sum_{p=1}^{\infty} \frac{1}{(2p+1)^2}$ and use the result of Question (3).]

Exercise 3 (4 points)

Let us consider the following PDE of a function $u(t, x)$ of two real variables:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{2}$$

(1) What can you say about this PDE (order, linearity, ...)? By using the method of characteristics, find the general form of its solutions.

(2) We are now only interested in the *separated* solutions of (2), that is, those solutions of the form $u(t,x) = T(t)X(x)$, for some functions $T(t)$ and $X(x)$. Show that, if $u(t,x)$ is such a solution, there exists a real λ such that the following holds:

$$\begin{cases} X'(x) = \lambda X(x) \\ T'(t) = -\lambda T(t) \end{cases}.$$

(3) Find all the separated solutions of (2).

Exercise 4 (8 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the following construction:

- $\forall x \in (0,\pi),\ \ f(x) = \frac{\pi - x}{2}$,
- f is odd,
- f is 2π-periodic.

(1) Draw the graph of f over several periods.
(2) We are interested in the full Fourier expansion of f (over $(-\pi,\pi)$). Without computing its coefficients, does this series converge uniformly to f?
(3) Still without computing its coefficients, apply the pointwise convergence theory to this series, and find its pointwise limit at any point $x \in [-\pi,\pi]$ (that is, including the endpoints).
(4) Compute the coefficients of the full Fourier expansion of f.
(5) By using the result of Question (3) at a particular point x, calculate the value of the infinite sum $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$.
(6) Calculate the L^2-norm

$$||f|| = \sqrt{\int_{-\pi}^{\pi} f^2(x)\, dx}$$

of f over $(-\pi,\pi)$. Does the considered series converge in the L^2 sense to f?
(7) By using Parseval's equality, compute the value of the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Exercise 5 (10 points)

The purpose of this exercise is to study the Laplace equation of a function $u(x,y)$ of two variables:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{3}$$

when (x,y) belongs to the rectangles $R = \{0 \le x \le \ell,\ 0 \le y \le h\}$. Equation (3) is complemented by the following *non homogeneous* Dirichlet boundary conditions (see Figure 1):

$$\forall y \in [0,h],\ \ u(0,y) = g(y), \tag{4}$$

$$\forall y \in [0,h],\ \ u(\ell,y) = 0, \tag{5}$$

$$\forall x \in [0,\ell],\ \ u(x,0) = 0, \tag{6}$$

$$\forall y \in [0,\ell],\ \ u(x,h) = 0. \tag{7}$$

In this setting where there is no time-dependency in the considered PDE, we shall deal with the non homogeneous boundary condition (4) as we did for the (non homogeneous) initial conditions in the lectures, in the context of time-dependent PDE. In other words, we are first going to search for the general form of solutions of (3) which fulfill the three boundary conditions (5) (6) and (7) (that is, 'forgetting' the boundary condition (4) for a moment), and then see how we can choose one of these solutions to match with (4).

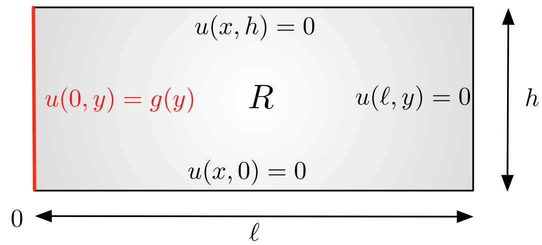

FIGURE 1. Setting for Exercise 5.

(1) We first search for *separated* solutions $u(x,y)$ of (3)-(5)-(6)-(7), i.e. solutions of the form $u(x,y) = X(x)Y(y)$, where $X(x)$ and $Y(y)$ are two functions of one variable to be determined. Show that if $u(x,y) = X(x)Y(y)$ is such a *non zero* separated solution, then there exists a constant $\lambda \in \mathbb{R}$ such that:

$$\left\{ \begin{array}{c} X''(x) - \lambda X(x) = 0 \\ Y''(y) + \lambda Y(y) = 0 \end{array} \right. , \left\{ \begin{array}{c} X(\ell) = 0 \\ Y(0) = Y(h) = 0 \end{array} \right. . \tag{8}$$

(2) We first search for the *eigenvalues* λ such that a *non zero* function Y satisfying the associated conditions in (8) exists. Show that there is no such negative eigenvalue $\lambda = -\beta^2$, $\beta > 0$.

(3) Is $\lambda = 0$ an eigenvalue of this problem?

(4) Find all the positive eigenvalues $\lambda_n = \beta_n^2$, $\beta_n > 0$ of this system, as well as corresponding eigenfunctions $Y_n(y)$.

(5) We now turn to the X equation in (8). Show that a corresponding eigenfunction $X_n(x)$ to the eigenvalue $\lambda_n = \beta_n^2$ is:

$$\forall x \in [0,\ell], \;\; X_n(x) = \sinh(\beta_n(x-\ell)).$$

[Hint: recall the following formula: for any real numbers a, b: $\sinh(a-b) = \sinh(a)\cosh(b) - \sinh(b)\cosh(a)$.]

(6) Show that the general solution to (3)-(5)-(6)-(7) can be written as an infinite sum:

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{h}\right) \sinh\left(\frac{n\pi}{h}(x-\ell)\right), \tag{9}$$

where the coefficients b_n are constants yet to be determined.

(7) Eventually, we turn to the non homogeneous boundary condition (4). Give an explicit formula of the coefficients b_n such that $u(x,y)$, defined as an infinite sum of the form (9) (which you may suppose to converge pointwise) satisfies (4) in addition to (3)-(5)-(6)-(7).

Formula sheet

I. Formulae around Fourier series

	sine Fourier series	cosine Fourier series	full Fourier series
Typical interval for functions	$(0,\ell)$	$(0,\ell)$	$(-\ell,\ell)$
Boundary conditions	$\varphi(0)=0$ $\varphi(\ell)=0$	$\varphi'(0)=0$ $\varphi'(\ell)=0$	$\varphi(0)=\varphi(\ell)$ $\varphi'(0)=\varphi'(\ell)$
Eigenvalues	$\left(\frac{n\pi}{\ell}\right)^2$, $n=1,2,...$	$\left(\frac{n\pi}{\ell}\right)^2$, $n=0,1,2,...$	$\left(\frac{n\pi}{\ell}\right)^2$ $n=0,1,2,...$
Eigenfunctions	$\sin\left(\frac{n\pi x}{\ell}\right)$, $n=1,2,...$	$\cos\left(\frac{n\pi x}{\ell}\right)$, $n=0,1,2,...$	$\sin\left(\frac{n\pi x}{\ell}\right)$, and $\cos\left(\frac{n\pi x}{\ell}\right)$ $n=0,1,2,...$
Expansion	$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$	$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$	$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)+b_n \sin\left(\frac{n\pi x}{\ell}\right)$
Coefficients	$b_n=\frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx$	$a_0=\frac{2}{\ell}\int_0^\ell f(x)\,dx$ $a_n=\frac{2}{\ell}\int_0^\ell f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx$	$a_0=\frac{1}{\ell}\int_{-\ell}^\ell f(x)\,dx$ $a_n=\frac{1}{\ell}\int_{-\ell}^\ell f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx$ $b_n=\frac{1}{\ell}\int_{-\ell}^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx$

II. Trigonometric identities

For $a,b\in\mathbb{R}$,

$$\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b),\ \ \cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b),$$
$$\sin(a+b)=\sin(a)\cos(b)+\sin(b)\cos(a),\ \ \sin(a-b)=\sin(a)\cos(b)-\sin(b)\cos(a).$$

For $a\in\mathbb{R}$,

$$\cos^2(a)=\frac{1+\cos(2a)}{2},\ \ \sin^2(a)=\frac{1-\cos(2a)}{2}.$$