Advanced Calculus I: Workshop 5

Exercise 1

Let $D \subset \mathbb{R}$, x_0 be an accumulation point of D, and consider a function $f : D \to \mathbb{R}$.

(1) Recall the meaning, in terms of quantifiers of the following sentence: 'f has a left-limit ℓ at x_0'.
(2) Recall the meaning, in terms of quantifiers of the following sentence: 'f has a right-limit ℓ at x_0'.
(3) Show that f has a limit ℓ at x_0 if and only if it has both a left-limit ℓ_1 and a right-limit ℓ_2 at x_0, and $\ell_1 = \ell_2$.

Exercise 2

Let $f : (0,1) \to \mathbb{R}$ be the function defined by:

$$\forall x \in (0,1), \;\; f(x) = \cos\left(\frac{1}{x}\right).$$

Find two sequences $\{x_n\}$ and $\{y_n\}$ of elements of $(0,1)$ which both converge to 0, such that the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ have different limits. Conclude that f does not have a limit at 0.