Advanced Calculus I: Workshop 11

Exercise 1

Let $a < b$ be two real numbers, and $f : (a, b) \to \mathbb{R}$ be a differentiable function whose derivative f' is bounded, i.e. there exists a real number $M > 0$ with:

$$\forall x \in (a, b),\ \ |f'(x)| \leq M.$$

(1) Is it possible to directly apply Heine's theorem to conclude that f is uniformly continuous on (a, b)?
(2) Prove that:
$$\forall x, y \in (a, b),\ \ |f(x) - f(y| \leq M|x - y|,$$
(3) Infer from (2) that f is uniformly continuous on (a, b).
(4) Give an example of a function which is uniformly continuous on some interval (a, b) of your choice, which is differentiable on (a, b), but whose derivative is unbounded on (a, b).

Exercise 2

Let $a < b$ be two real numbers, and $x_0 \in (a, b)$. Let $f : (a, b) \to \mathbb{R}$ be a function such that:

- f is continuous on (a, b),
- f is differentiable at every point of (a, b) except at x_0,
- the limit $\ell = \lim_{x \to x_0} f'(x)$ exists and is finite.

Prove that f is also differentiable at x_0, with $f'(x_0) = \ell$, and that the function f' is continuous at x_0.