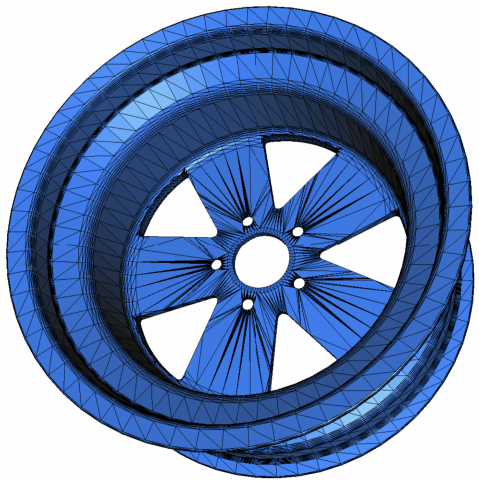
The Level Set Method for structural topology optimization: Part III

Grégoire Allaire[1], Charles Dapogny[2]

[1] Centre de Mathématiques Appliquées, École Polytechnique, Palaiseau, France
[2] Laboratoire Jacques-Louis Lions, Sorbonne Université, Paris, France

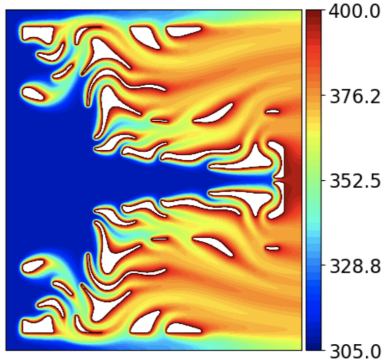
28th May, 2026

- A taste of meshing and remeshing.

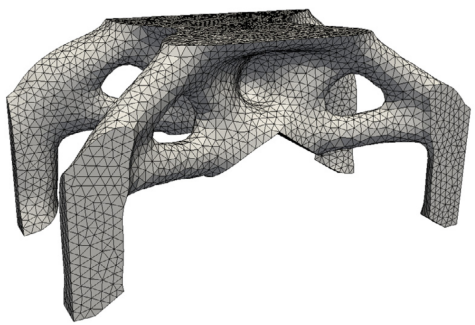




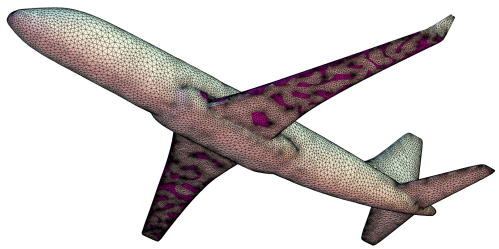
- Presentation of the body-fitted Level Set Method with remeshing.
- Multi-physics applications and beyond.



Optimized 2d section of an air-oil heat exchanger.



Optimized wedge withstanding fracture

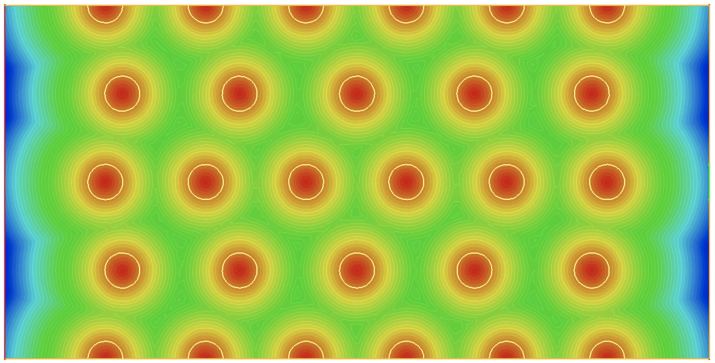


Optimized placement of a sound-soft material on a plane.

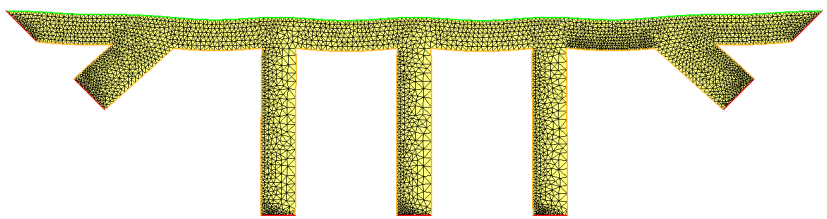
Reminder: The "classical" Level Set Method

- The Level Set Method describes arbitrary deformations of an optimized shape Ω.

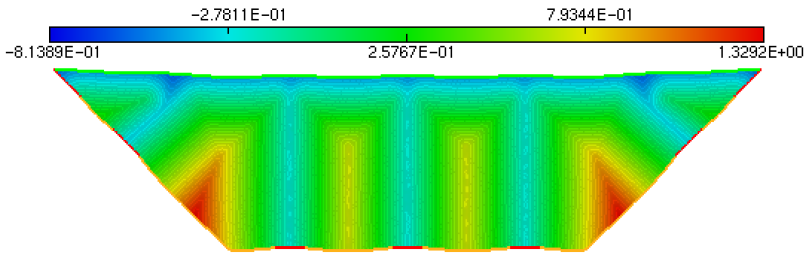
- A fixed mesh $\mathcal{T}$ of the domain D is used, and Ω is known via a Level Set function ϕ on $\mathcal{T}$.
- Mechanical computations use a fictitious domain approximation of the PDE on Ω by a PDE on D.



Example: The ersatz material approximation in linear elasticity.



Elastic displacement of a bridge $\Omega \subset D$.



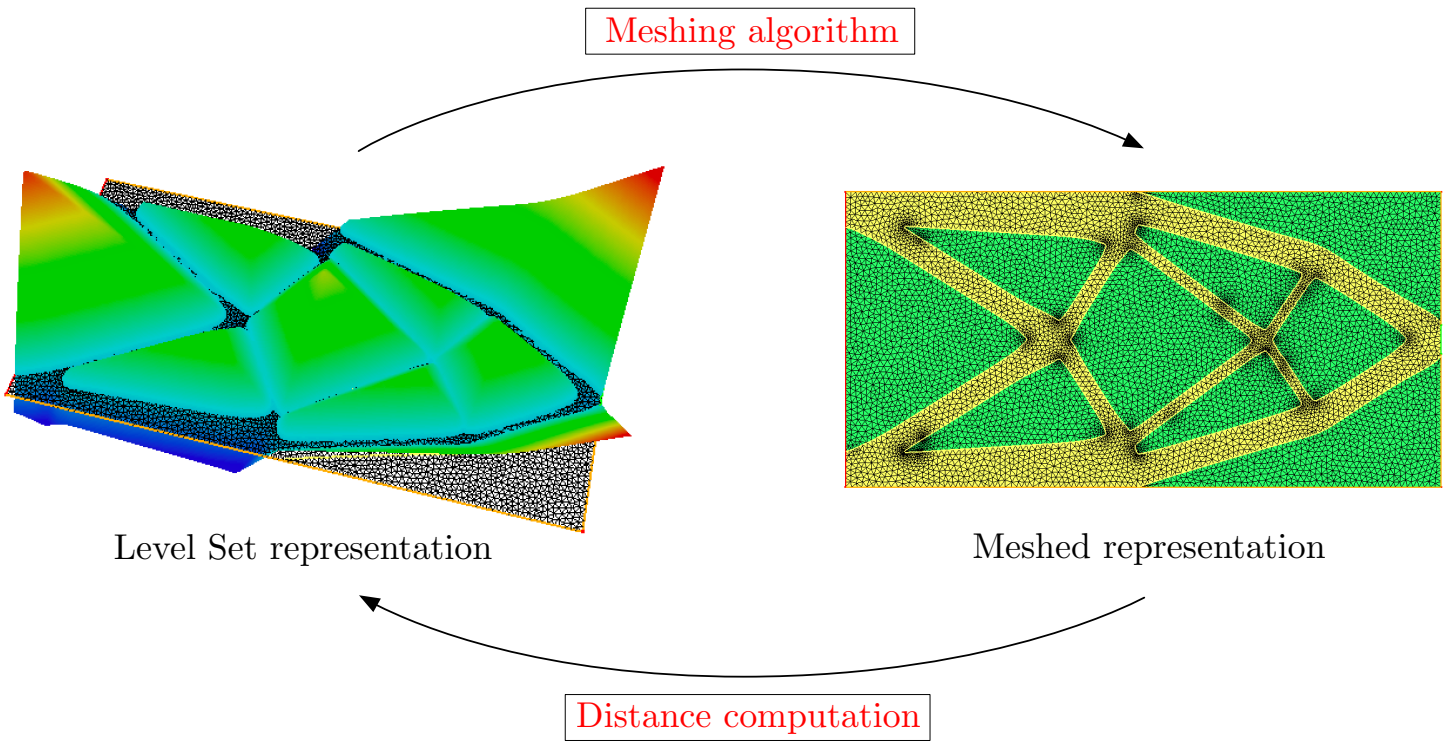
Approximate, ersatz material displacement of D when Ω is described by a Level Set function.

- This approximation may be unacceptable in a variety of important situations.

Example: In multi-physics optimization, where $\partial\Omega$ bears complex physical effects.

The body-fitted Level Set Method

- This `webinar` deals with a recent variant of the Level Set Method.
- The latter enjoys:
  - Accurate and non intrusive computations on an exact mesh of the shape.
  - A robust tracking of arbitrary shape evolution by the Level Set Method.
- The method combines two shape descriptions:



Part I

A look at meshing and remeshing

1. A look at meshing and remeshing
   - A few definitions and key concepts
   - Isosurface discretization
2. The body-fitted variant of the Level Set Method
3. A few applications and extensions

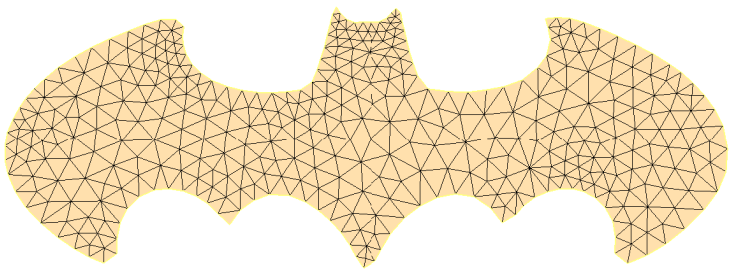
A few definitions about meshes (I)

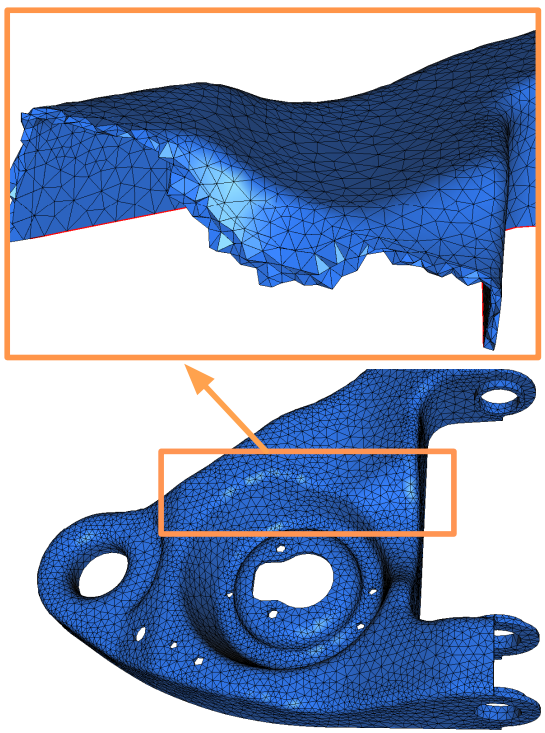
Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a polyhedral domain.

Definition (mesh).

A simplicial mesh $\mathcal{T}$ of Ω is a collection $\{T_i\}_{i=1,\dots,N_T}$ of simplices (triangles in 2d, tetrahedra in 3d) such that

$$\overline{\Omega} = \bigcup_{i=1}^{N_T} \overline{T_i}.$$

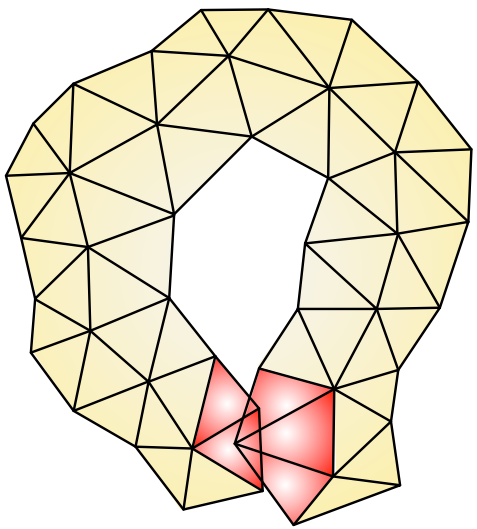




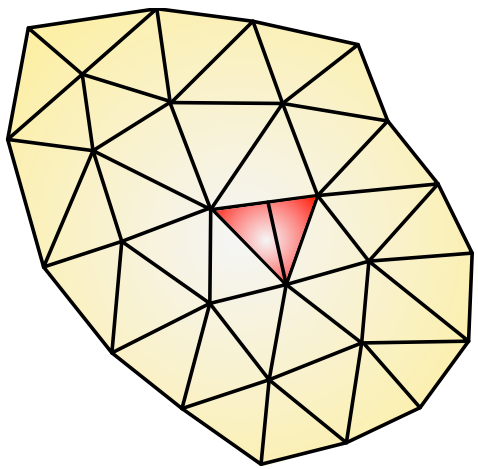
A few definitions about meshes (II)

Most often, the mesh $\mathcal{T}$ is required to be

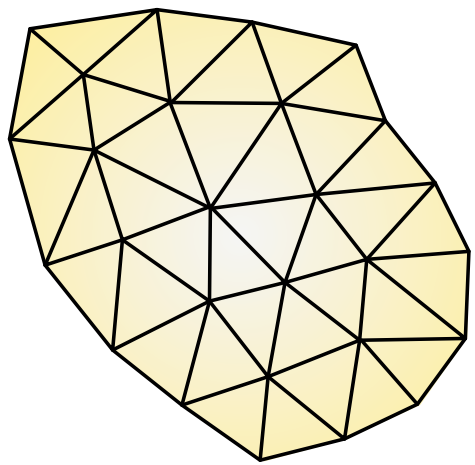
- Valid: the simplices T_i are mutually disjoint: $T_i \cap T_j = \emptyset$ when $i \neq j$;
- Conforming: each intersection $\overline{T_i} \cap \overline{T_j}$, $i \neq j$ is reduced to either a vertex, an edge, or a face of the mesh.



Overlapping elements in an invalid mesh



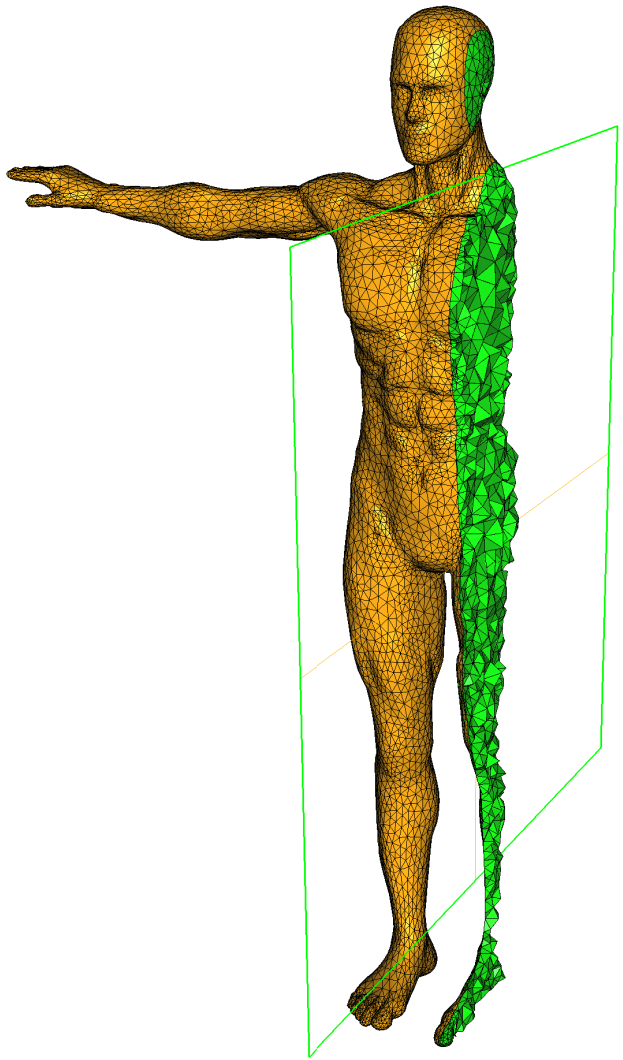
A non conforming mesh



A valid, conforming mesh

A few definitions about meshes (III)

- The mesh $\mathcal{T}$ naturally comprises a surface mesh $\mathcal{S}_\mathcal{T}$ for the boundary $\partial\Omega$:
  - In 2d, $\mathcal{S}_\mathcal{T}$ is a collection of segments;
  - In 3d, $\mathcal{S}_\mathcal{T}$ is a surface triangulation.
- In practice, the meshed domain Ω is *not* polyhedral and $\mathcal{S}_\mathcal{T}$ is meant to be an approximation of $\partial\Omega$.



Tetrahedral mesh $\mathcal{T}$ (in green) and associated surface triangulation $\mathcal{S}_\mathcal{T}$ (in yellow).

Quality of elements

- The accuracy of most numerical methods using $\mathcal{T}$ (e.g. Finite Element computations) crucially depends on the quality of the simplices $T \in \mathcal{T}$.

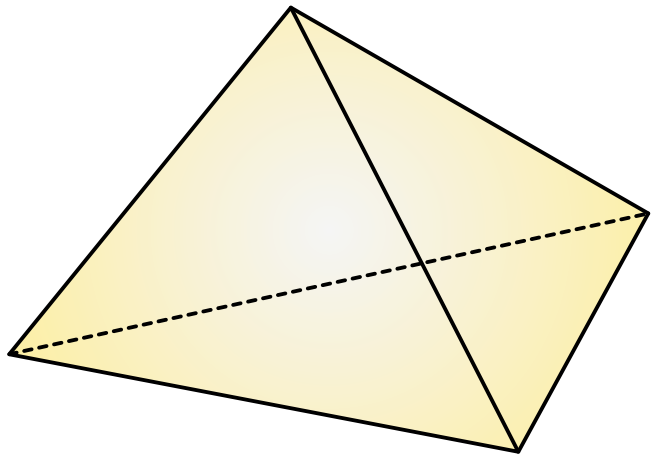
- The latter is measured by a quality factor $\mathcal{Q}(T)$:
  - $\mathcal{Q}(T) \approx 1$ when T is close to regular;
  - $\mathcal{Q}(T) \approx 0$ when T is nearly flat.

- In practice, $\mathcal{Q}(T)$ should "smoothly" discriminate "good", "bad" and "not so good" simplices T.

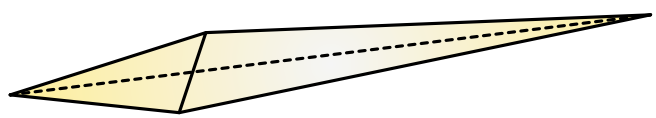
- **Example:** A popular quality factor is:

$$\mathcal{Q}(T) = \alpha \frac{\mathrm{Vol}(T)}{\left(\sum_{i=1}^{d(d+1)/2} |e_i|^2 \right)^{\frac{d}{2}}},$$

where the e_i are the edges of T.



A regular tetrahedron ($\mathcal{Q}(T) \approx 1$).



A nearly degenerate tetrahedron ($\mathcal{Q}(T) \approx 0$).

Quality of the geometric approximation (I)

The surface mesh $\mathcal{S}_\mathcal{T}$ should be a close approximation of $\partial\Omega$, e.g.:

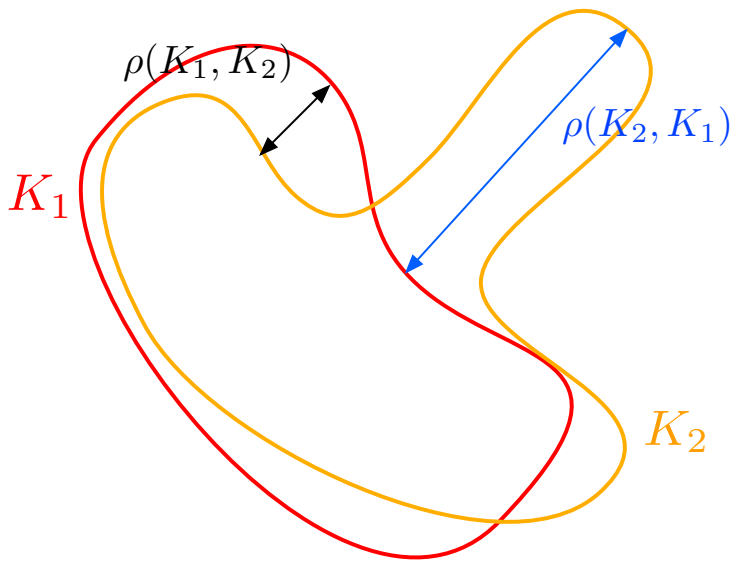
$$d^H(\mathcal{S}_\mathcal{T}, \partial\Omega) \leq \varepsilon,$$

where ε is a user-defined threshold and $d^H(\cdot,\cdot)$ stands for the Hausdorff distance.

Definition.

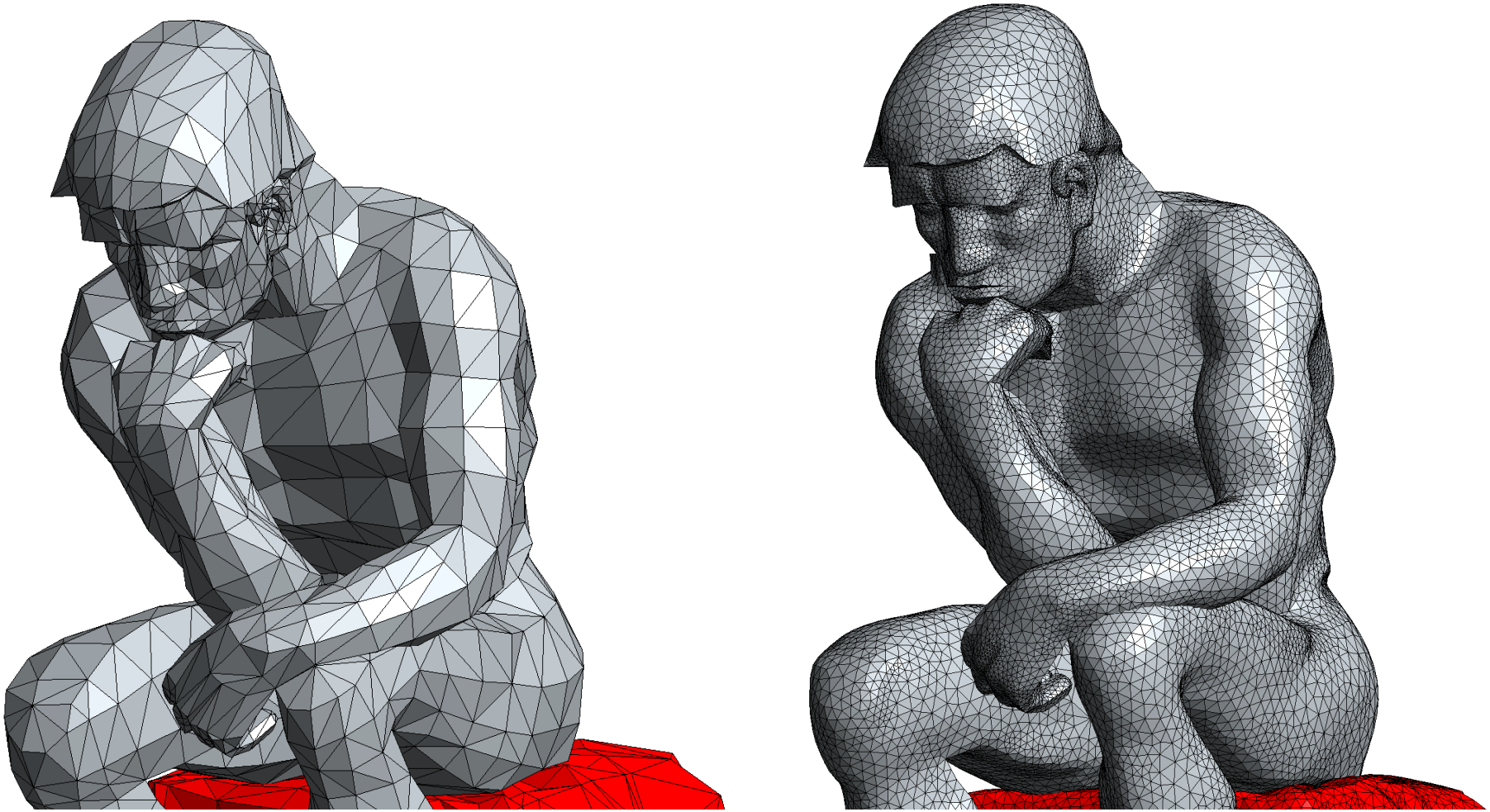
The Hausdorff distance $d^H(K_1, K_2)$ between two compact subsets $K_1, K_2 \subset \mathbb{R}^d$ is:

$$d^H(K_1, K_2) = \max(\rho(K_1, K_2), \rho(K_2, K_1)), \text{ where } \rho(K_1, K_2) := \max_{x \in K_1} d(x, K_2).$$



The Hausdorff distance $d^H(K_1, K_2)$ measures the "maximum gap" between K_1 and K_2.

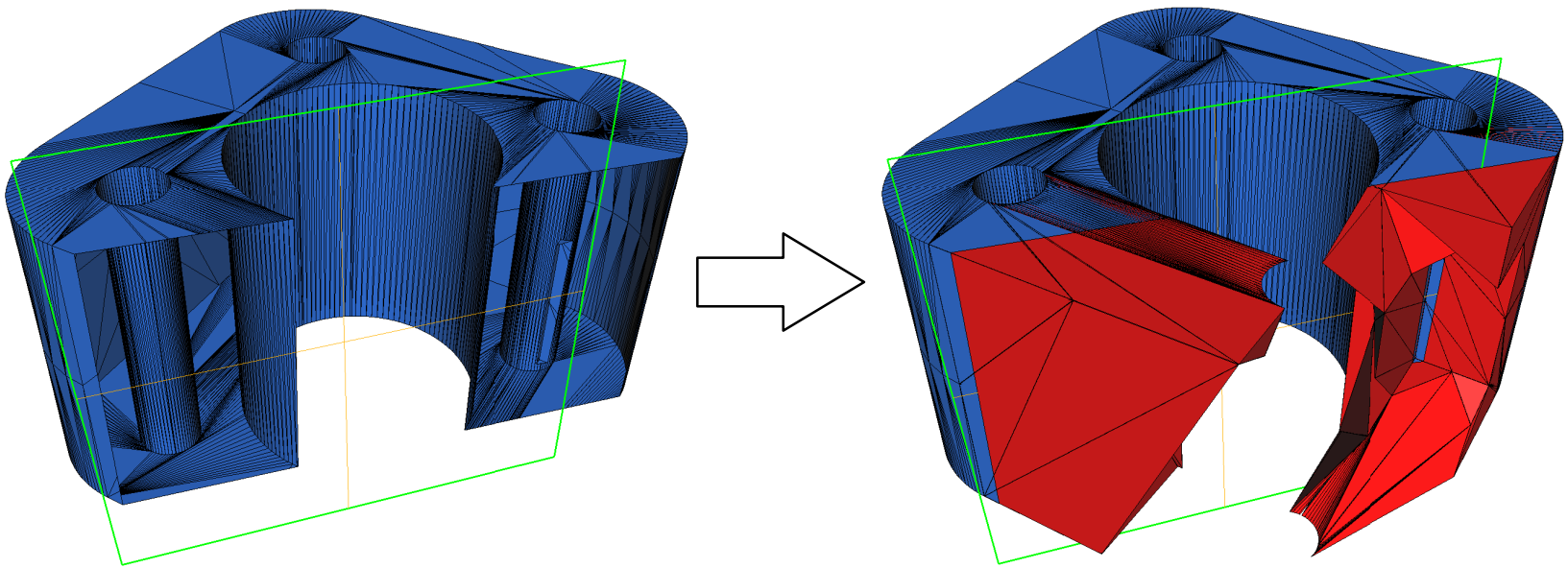
Quality of the geometric approximation (II)



(Left) Rough approximation of a domain $\Omega \subset \mathbb{R}^3$; (right) Fine geometric approximation of Ω.

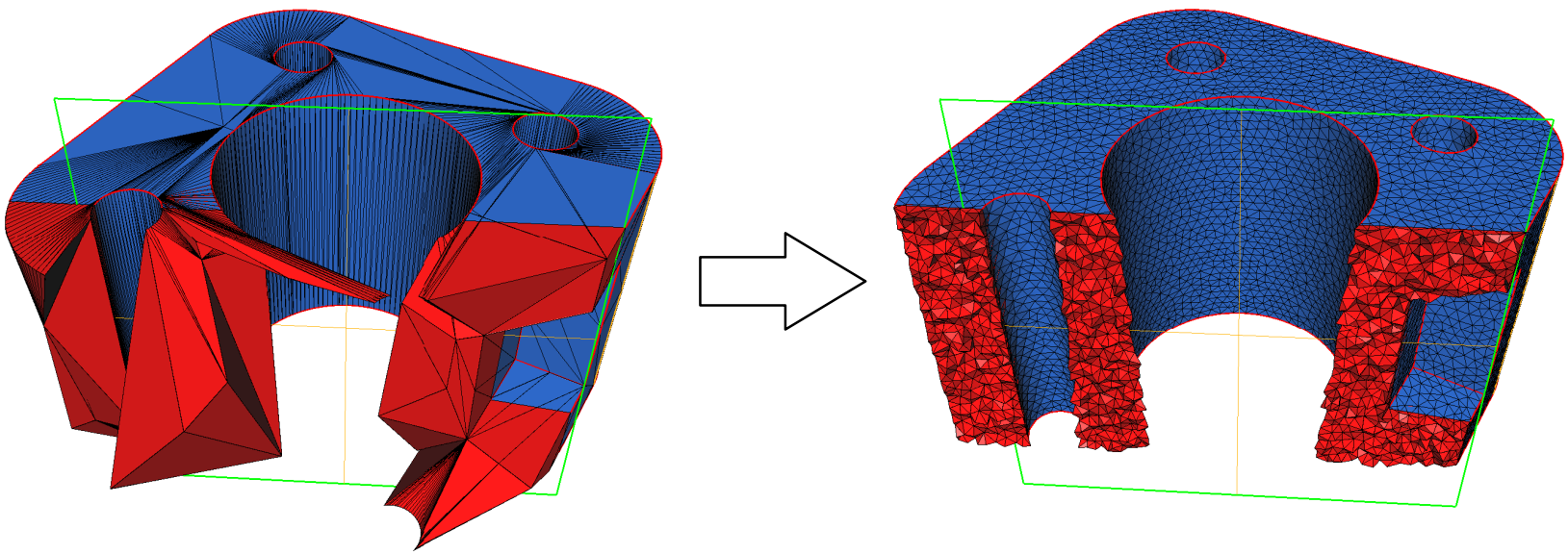
Meshing vs. remeshing (I)

- Meshing starts from the datum of a (line or surface) mesh $\mathcal{S}$ of the boundary $\partial\Omega$.
- It aims to fill the volume Ω with simplices, i.e. to construct a mesh $\mathcal{T}$ of Ω with surface part $\mathcal{S}_{\mathcal{T}} = \mathcal{S}$.



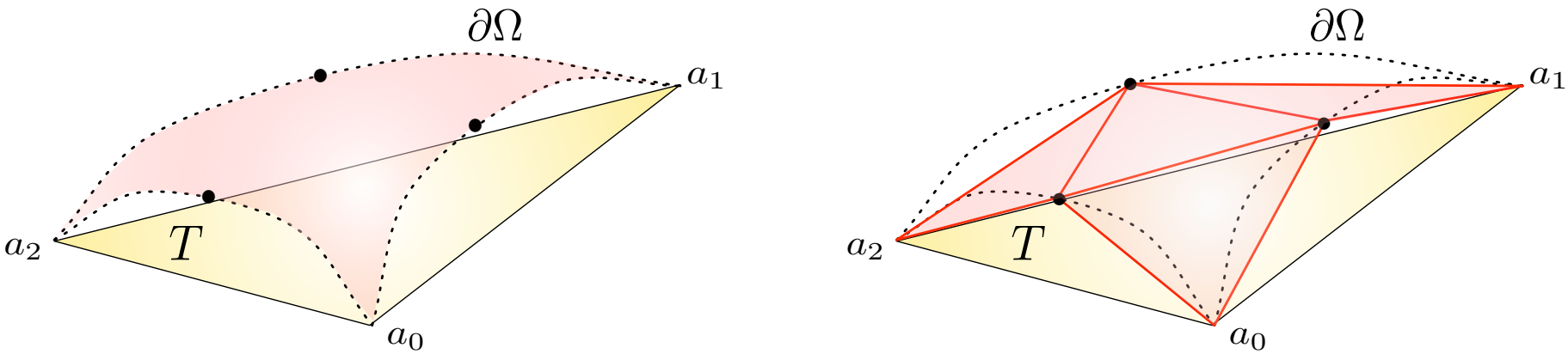
Meshing vs. remeshing (II)

- Remeshing assumes the input of a valid, conforming mesh $\mathcal{T}$ of Ω.
- It aims to modify $\mathcal{T}$ into a "better" mesh $\widetilde{\mathcal{T}}$ of Ω.

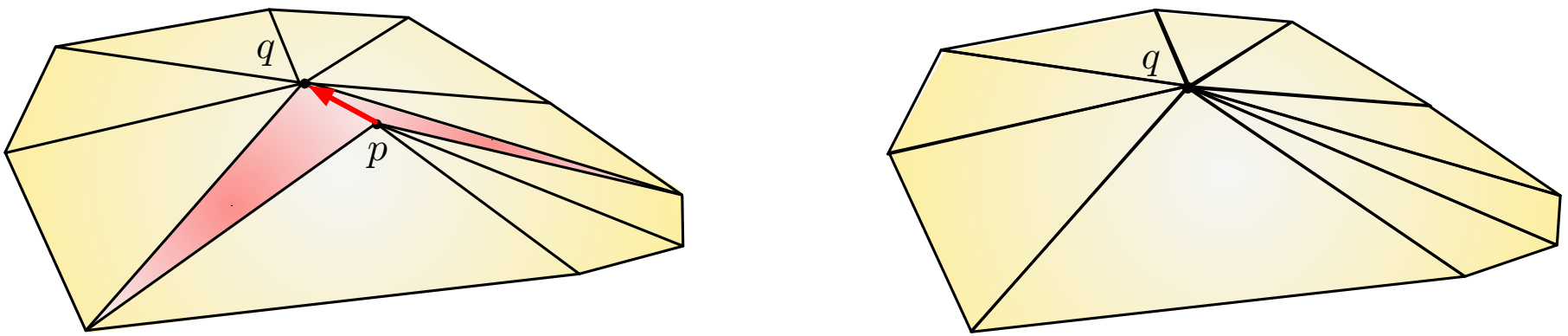


Meshing vs. remeshing (III)

Remeshing hinges on the intertwinement of four local operators, which are carefully driven to improve the mesh quality, and its geometric approximation capability.



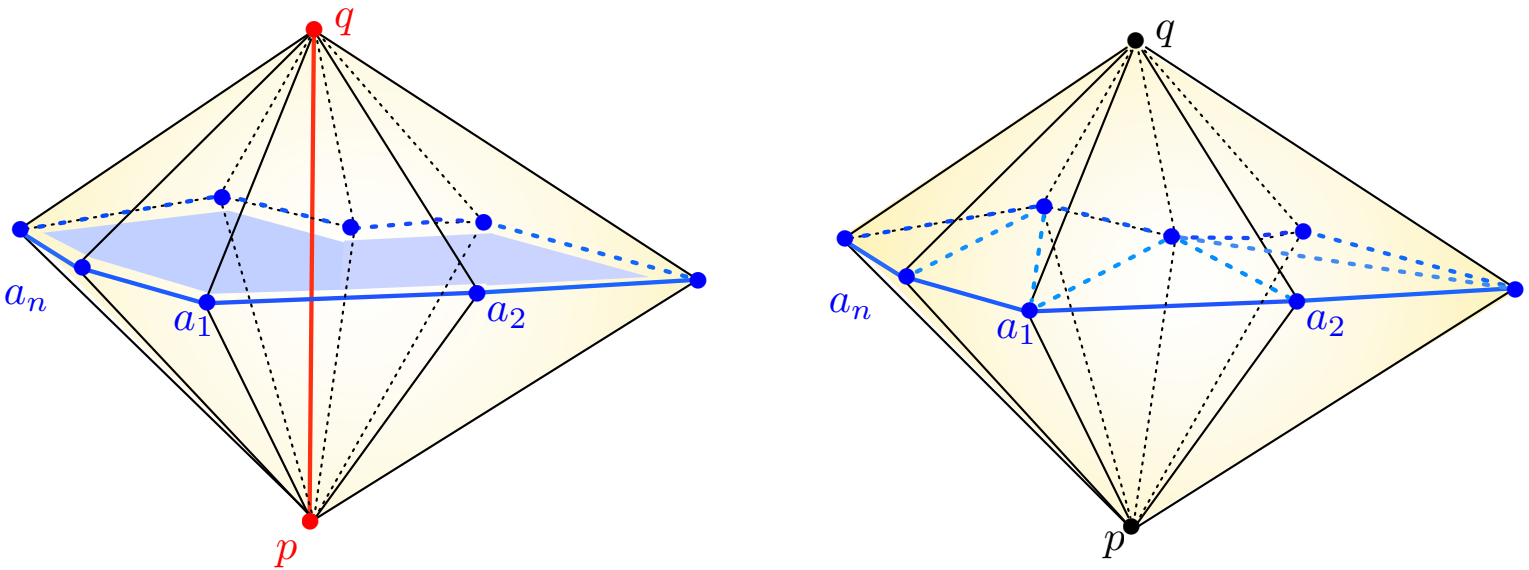
Splitting of the three edges of a surface triangle $T \in \mathcal{S}_{\mathcal{T}}$, placing the new points on the continuous surface $\partial\Omega$.



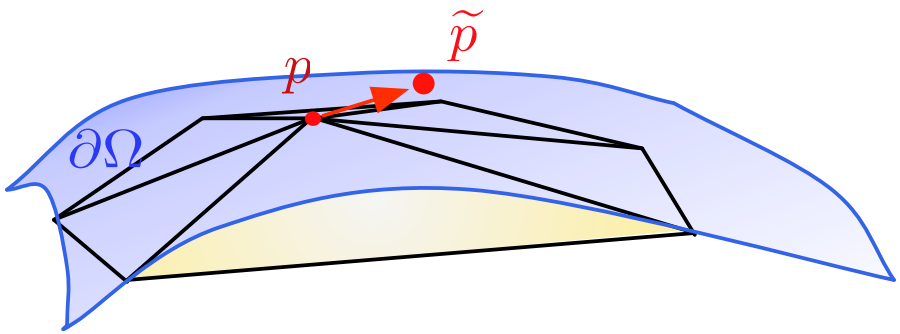
Collapse of point p onto q in a surface configuration.

Meshing vs. remeshing (III)

Remeshing hinges on the intertwinement of four local operators, which are carefully driven to improve the mesh quality, and its geometric approximation capability.



3d edge swap.



Relocation of node $p \in \mathcal{S}_{\mathcal{T}}$.

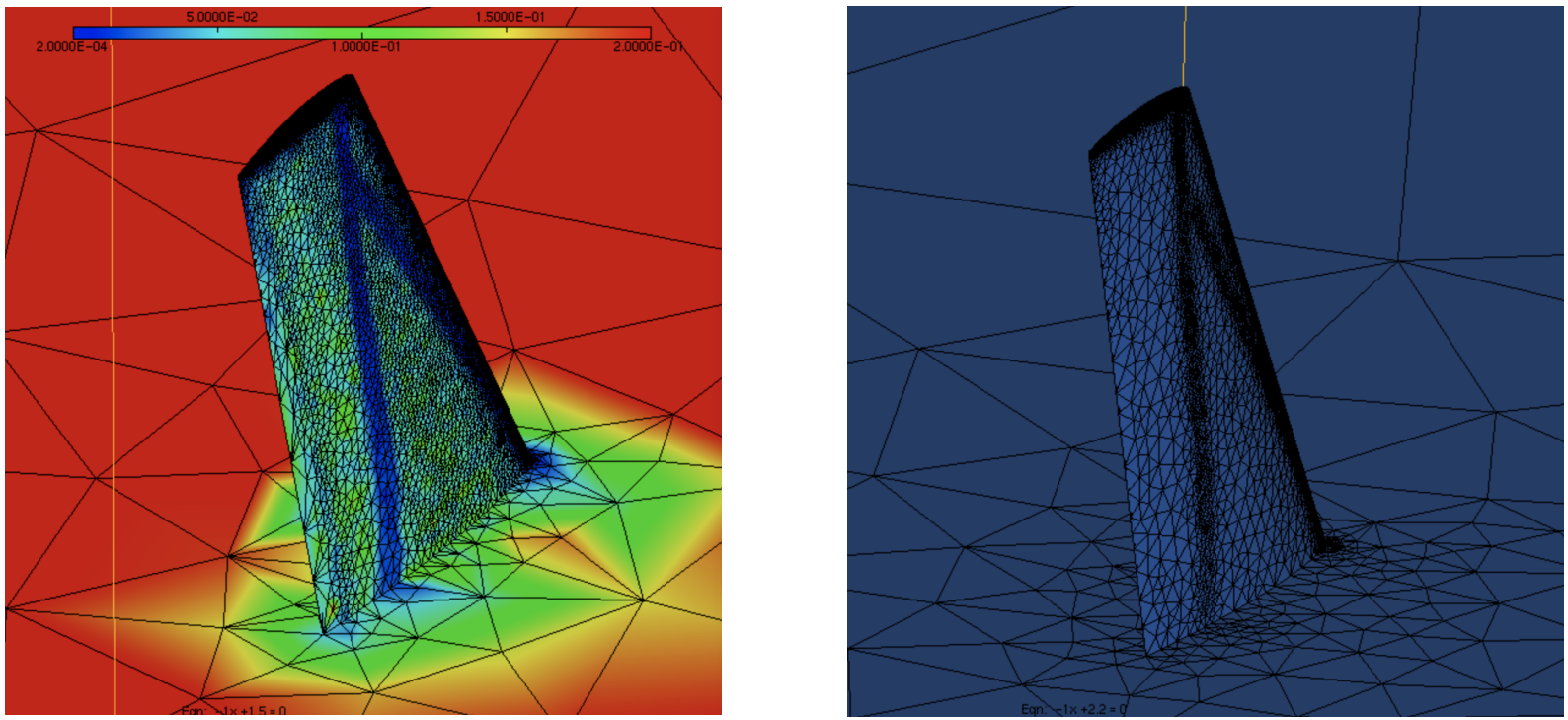
Remeshing with respect to a local size prescription (I)

Remeshing is often guided by a local size prescription, which may stem from

- A priori or a posteriori finite element estimators;
- Geometric approximation requirements of Ω.

When the size prescription is isotropic, it is encoded in a size map $h : \Omega \to \mathbb{R}$:

$$\text{For each } p \in \Omega, \quad h(p) = \text{desired size for the edges near } p.$$



(Left) Size map h defined at the vertices of a mesh; (right) Modified mesh.

Remeshing with respect to a local size prescription (II)

- An anisotropic size prescription can be encoded in a metric tensor $M : \Omega \to \mathbb{R}^{d \times d}$.
- **Rationale:** The length $\ell_M(e)$ of an edge $e = pq$ with respect to M is defined by:

$$\ell_M(e) = \int_0^1 \sqrt{M(p + t(q-p))(q-p) \cdot (q-p)}\, \mathrm{d}t.$$

- Introducing the eigenvalue decomposition of $M(p)$

$$M(p) = O(p) \begin{pmatrix} d_1(p) & 0 & 0 \\ 0 & d_2(p) & 0 \\ 0 & 0 & d_3(p) \end{pmatrix} O(p)^T,$$

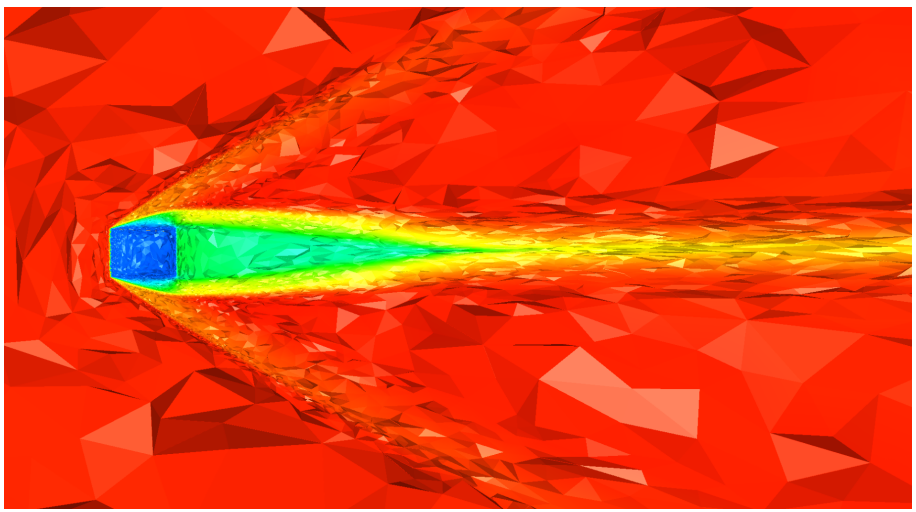
the quantities $d_i(p)$ and $O(p)$ are defined so that:

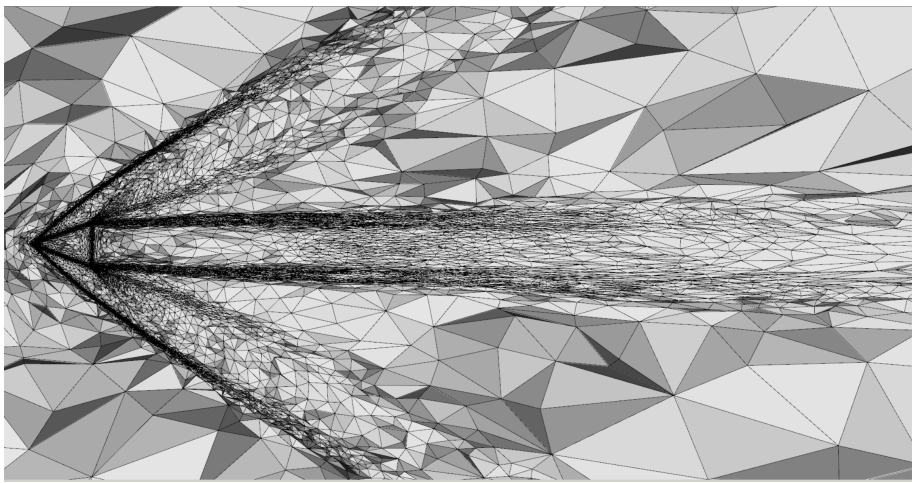
$d_i(p) = \dfrac{1}{h_i^2(p)}$, $h_i(p) =$ desired size in the direction of the ith column vector $O_i(p)$.

- The mesh $\mathcal{T}$ then fulfills the size prescription if

$$\forall \text{ edge } \ e \in \mathcal{T}, \quad \ell_M(e) \approx 1.$$

Remeshing with respect to a local size prescription (III)





(Left) Velocity field of a supersonic flow; (right) Adapted mesh.

A(nother) word of advertising

Mmg PLATFORM

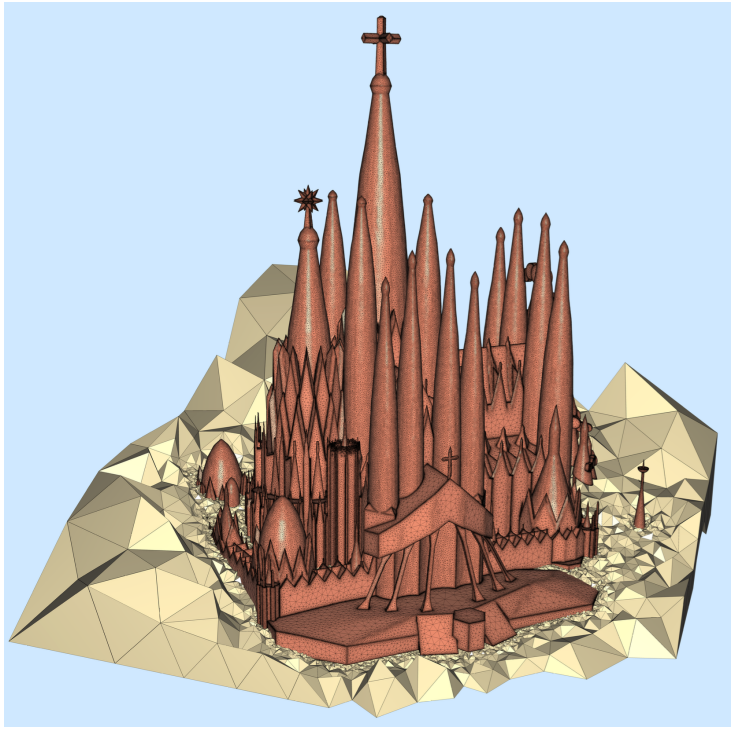
Robust, Open-source & Multidisciplinary Software for Remeshing

upgrade your meshes

Most of the above features are integrated in the free, open-source environment Mmg.

https://www.mmgtools.org

https://github.com/MmgTools/mmg



Part I

A look at meshing and remeshing

1. A look at meshing and remeshing
   - A few definitions and key concepts
   - Isosurface discretization
2. The body-fitted variant of the Level Set Method
3. A few applications and extensions

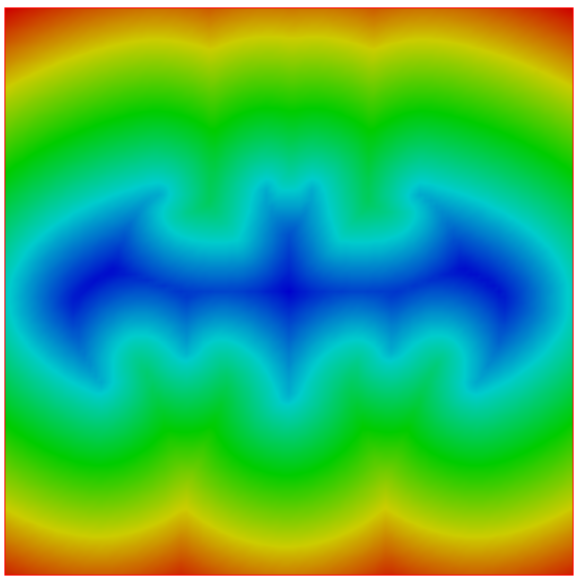
Isosurface discretization

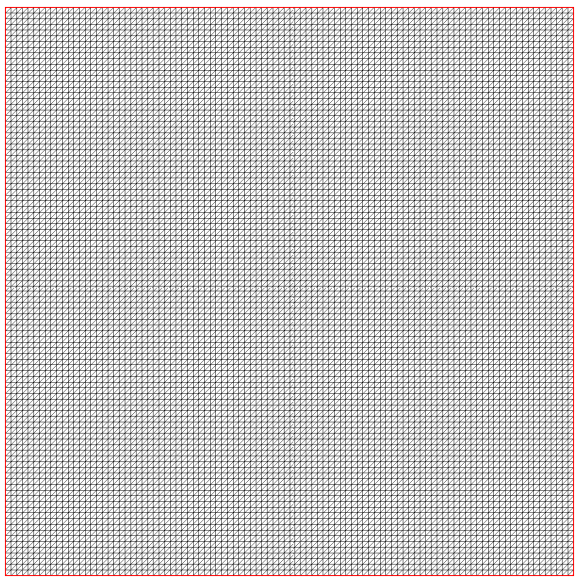
In many applications,

- The hold-all domain D is equipped with a computational, simplicial mesh $\mathcal{T}$.
- A Level Set function $\phi : D \to \mathbb{R}$ is defined at the vertices of $\mathcal{T}$.
- We wish to construct a new mesh of D in which the 0 level set Γ of ϕ and its negative subdomain Ω,

$$\Gamma := \{x \in D,\ \phi(x) = 0\}, \quad \Omega := \{x \in D,\ \phi(x) < 0\},$$

appear explicitly.





(Left) Isolines of a level set function ϕ defined at the vertices of a mesh $\mathcal{T}$ of a computational box D (right).

Isosurface discretization (II)

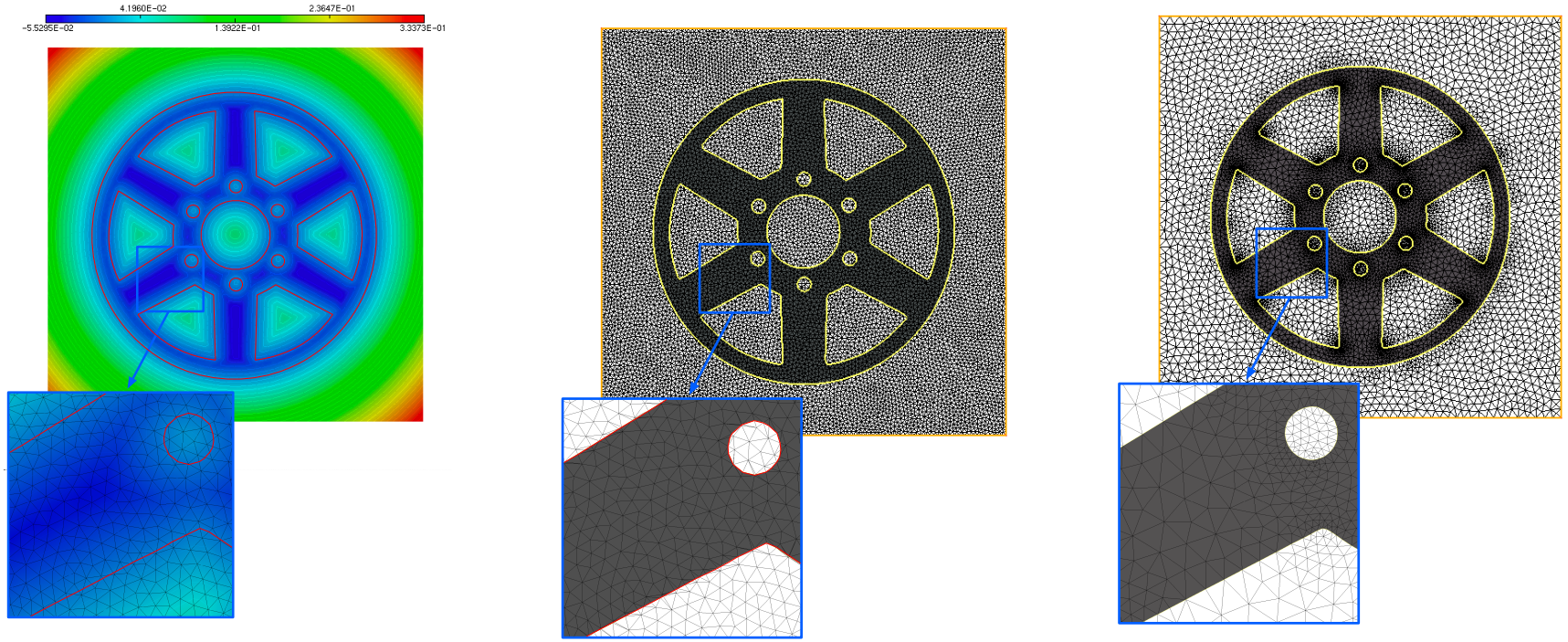
A two-step solution:

1. Discretize explicitly the 0 level set of ϕ into $\mathcal{T}$ by using patterns.

   $\Rightarrow$ A new valid, conforming mesh $\mathcal{T}_{\text{temp}}$ is obtained, which is of very low quality.

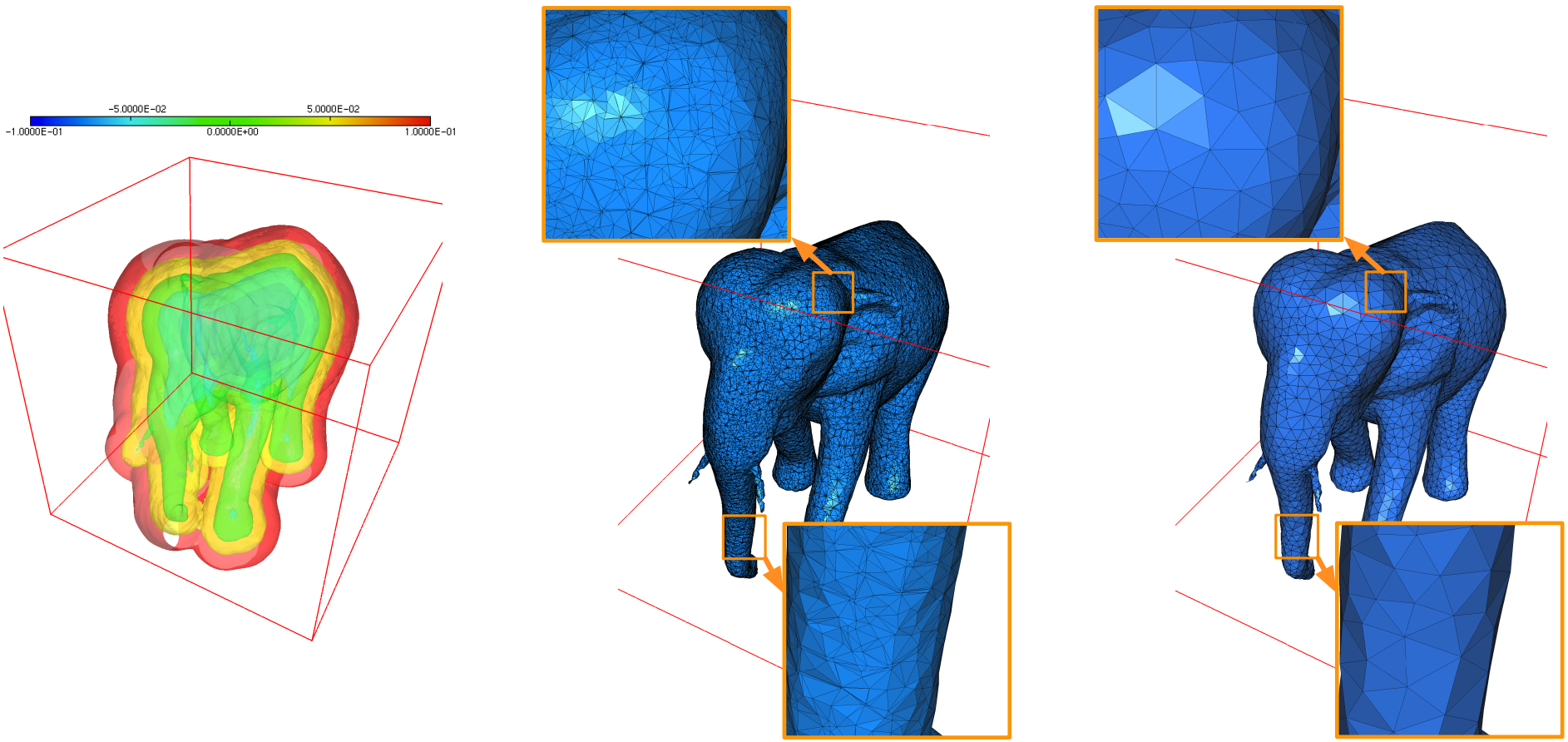
2. Improve the quality of $\mathcal{T}_{\text{temp}}$ by remeshing, to obtain $\widetilde{\mathcal{T}}$.

$\Rightarrow$ A high-quality mesh $\widetilde{\mathcal{T}}$ is obtained, where Ω and $D \setminus \overline{\Omega}$ are explicitly discretized.



(Left) One level set function ϕ defined at the vertices of $\mathcal{T}$; (middle) Low-quality mesh $\mathcal{T}_{\text{temp}}$ obtained from the discretization of the 0 level set of ϕ into $\mathcal{T}$; (right) High-quality mesh $\widetilde{\mathcal{T}}$ obtained after remeshing $\mathcal{T}_{\text{temp}}$.

Isosurface discretization (III)



(Left) Some isosurfaces of an implicit function defined in a cube; (middle) Result after rough discretization in the ambient mesh; (right) Result after local remeshing.

Part II

The body-fitted variant of the Level Set Method

1. A look at meshing and remeshing
2. The body-fitted variant of the Level Set Method
   - Presentation of the strategy
   - A few benchmark examples
3. A few applications and extensions

The body-fitted Level Set Method (I)

Let us consider a generic shape and topology optimization problem, of the form:

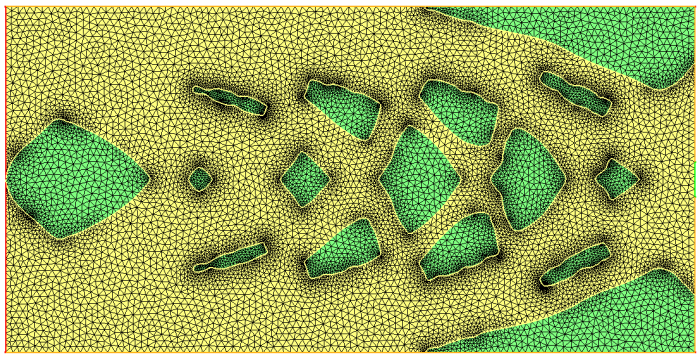
$$\min_{\Omega} J(\Omega) \ \text{ s.t. } \ G(\Omega) = 0. \qquad (\mathcal{P})$$

In this formulation:

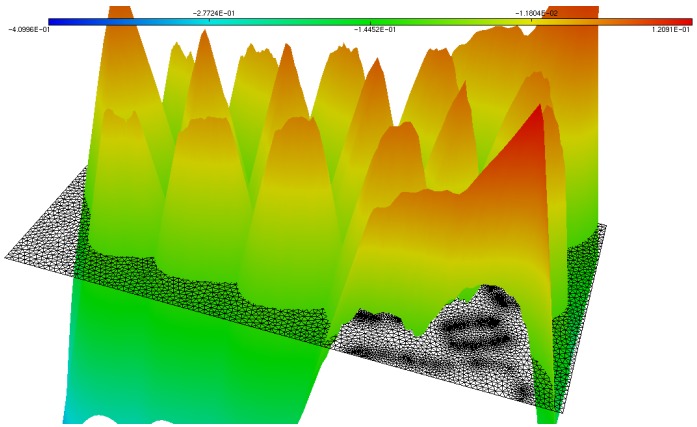
- The shape is included in a fixed "hold-all" domain D in $\mathbb{R}^d$, $d = 2$ or 3.
- $J(\Omega)$ is an objective function.
- $G(\Omega) = (G_1(\Omega), \ldots, G_p(\Omega))$ is a collection of p (equality) constraints.
- $J(\Omega)$ or some of the $G_i(\Omega)$ depend on Ω via the solution u_Ω to a physical boundary value problem posed on Ω.

The body-fitted Level Set Method (II)

- **Key idea:** At each iteration $n = 0, \ldots$ of the optimization, D is equipped with a simplicial mesh $\mathcal{T}^n$, in which Ω^n is explicitly discretized.
- The shape Ω^n is described at the same time by:



The mesh made of the interior elements of $\mathcal{T}^n$.



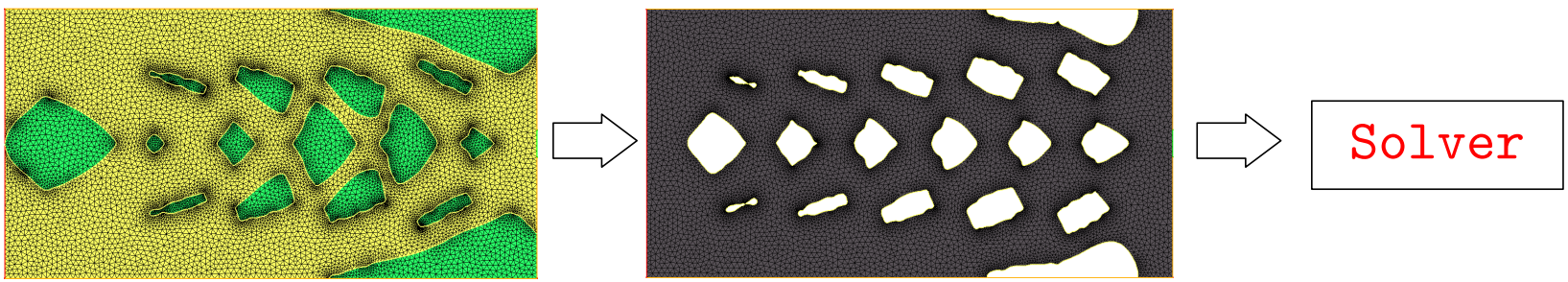
A Level Set function $\phi^n : D \to \mathbb{R}$, discretized on $\mathcal{T}^n$.

- Efficient algorithms allow to switch from one representation to the other:
  - The Fast Marching Method calculates the Level Set function $\phi^n = d_{\Omega^n}$ for Ω^n from any mesh of Ω^n, on any discretization of D.
  - Isosurface discretization yields a mesh of D body-fitted to Ω^n, from any Level Set function ϕ^n for Ω^n, discretized on any mesh of D.

The body-fitted Level Set Method (III)

Each operation of the nth iteration is carried out on the most suitable representation.

- The meshed representation is used to calculate u_{Ω^n}, p_{Ω^n} and a descent direction θ^n thanks to a "black box" use of an arbitrary Finite Element solver.



- The Level Set representation is used to update the shape $\Omega^n \to \Omega^{n+1}$, by solving the Hamilton-Jacobi equation on $\mathcal{T}^n$:

$$\phi^{n+1}(x) = \psi(\tau^n, x), \text{ where}$$

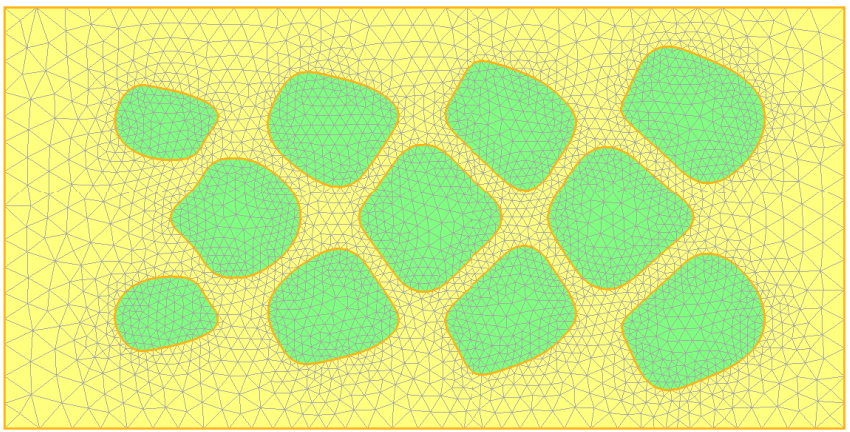
$$\begin{cases} \frac{\partial \psi}{\partial t}(t,x) + \theta^n(x)|\nabla \psi(t,x)| = 0 & \text{for } t \in (0,\tau^n),\ x \in D, \\ \psi(t=0,x) = \phi^n(x). \end{cases}$$

The algorithm in motion (I)

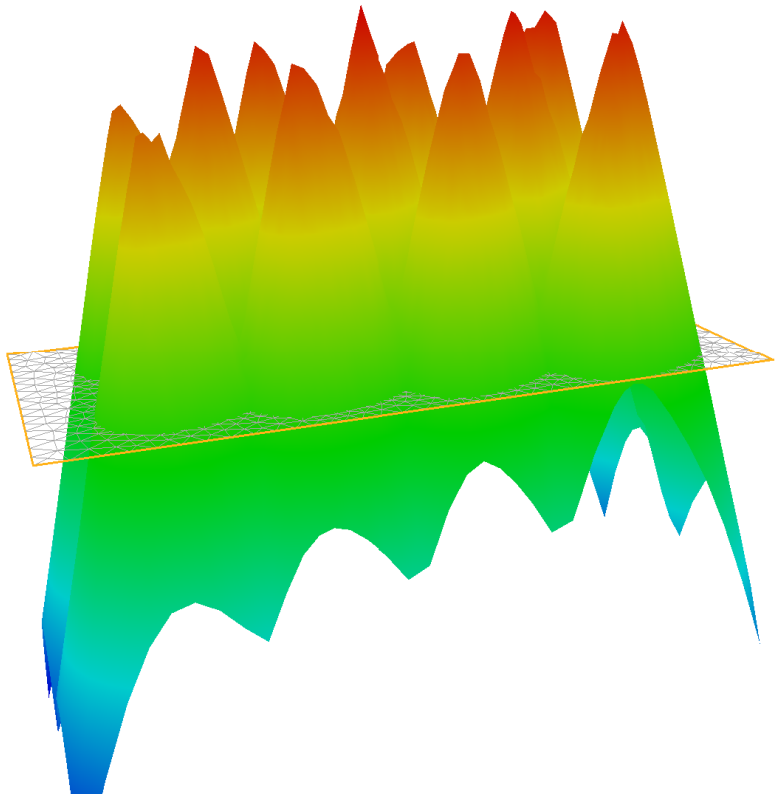
Let $n = 0, \ldots$ be a given iteration of the optimization process.

Step 1:

Compute the signed distance function d_{Ω^n} to the shape Ω^n on the mesh $\mathcal{T}^n$ of D.



(a) The initial shape.

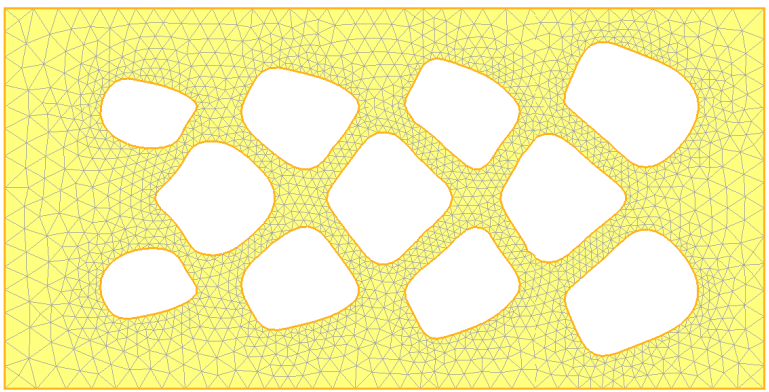


(b) Graph of d_{Ω^n}.

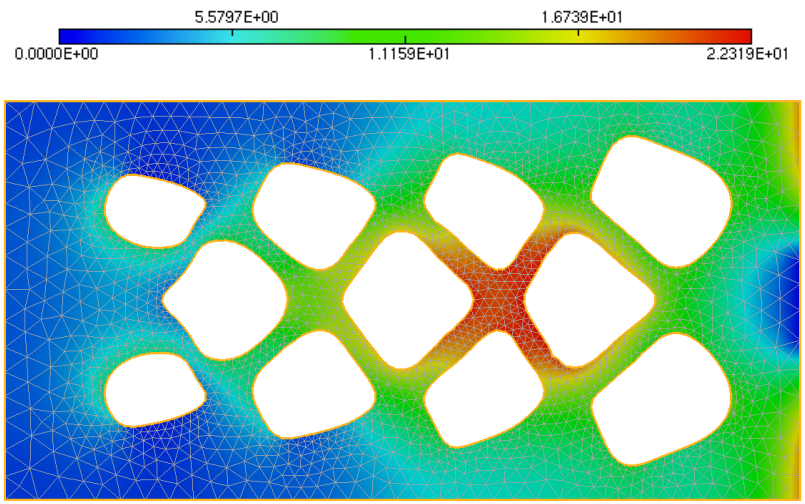
The algorithm in motion (II)

Step 2:

- Discard the exterior part $D \setminus \overline{\Omega^n}$;
- Calculate the descent direction θ^n on (the mesh $\mathcal{T}^n$ of) Ω^n.



(a) The "interior mesh".

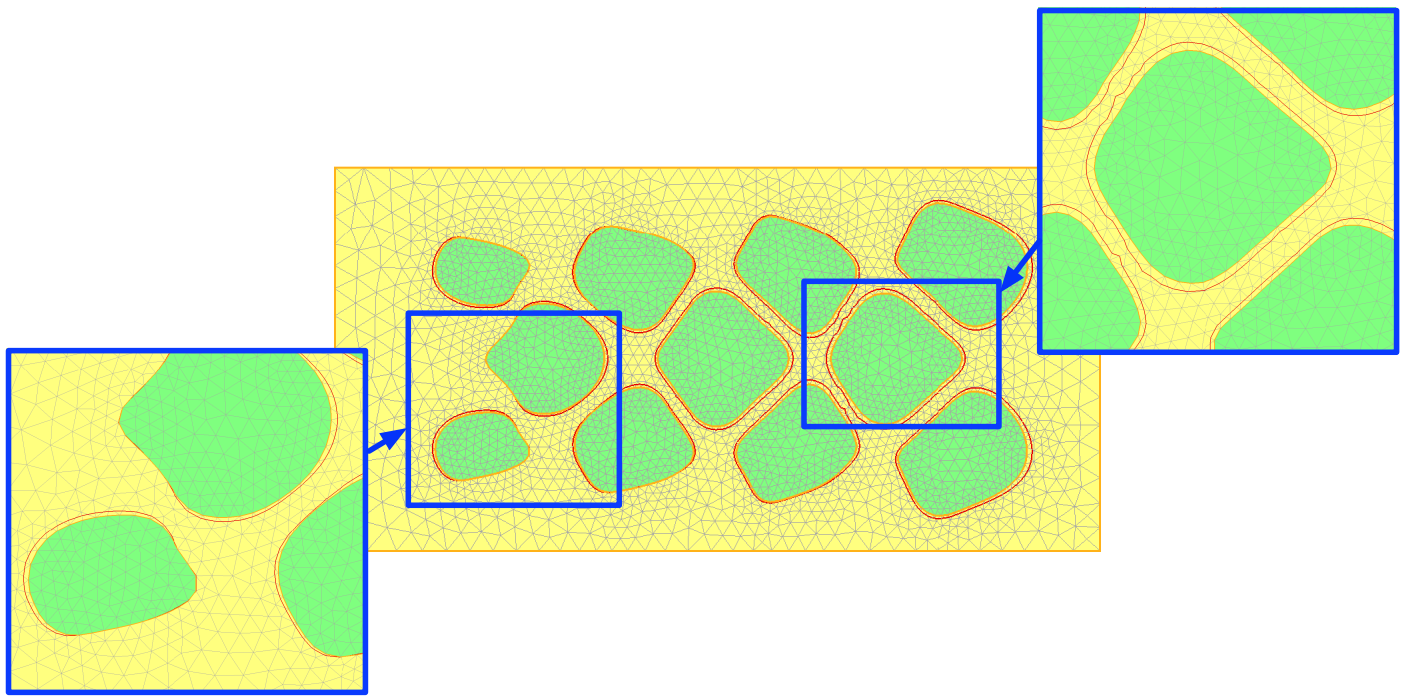


(b) Computation of θ^n.

The algorithm in motion (III)

Step 3:

- "Retrieve" the whole mesh $\mathcal{T}^n$ of D.
- Extend the velocity field θ^n to the whole mesh;
- Advect d_{Ω^n} along θ^n for a (small) time step τ^n.
- A new level set function ϕ^{n+1} is obtained on $\mathcal{T}^n$, for the new shape Ω^{n+1}.

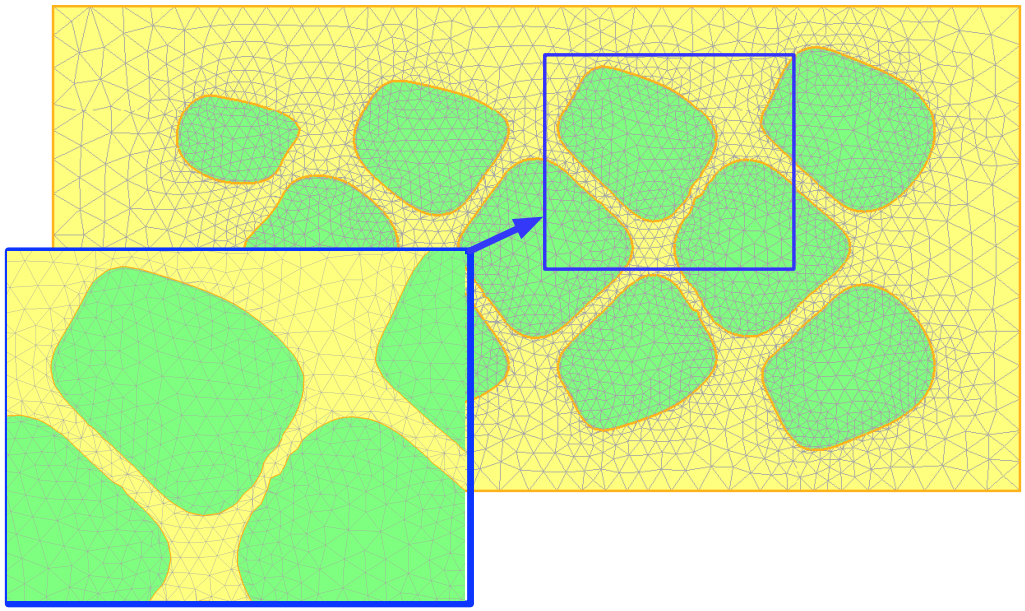


The shape Ω^n, discretized in the mesh (in yellow), and the "new", advected 0-level set (in red).

The algorithm in motion (IV)

Step 4:

- The 0 level set of ϕ^{n+1} is explicitly discretized in the mesh $\mathcal{T}^n$.
- As expected, roughly "inserting" this line in $\mathcal{T}^n$ yields a very ill-shaped mesh.

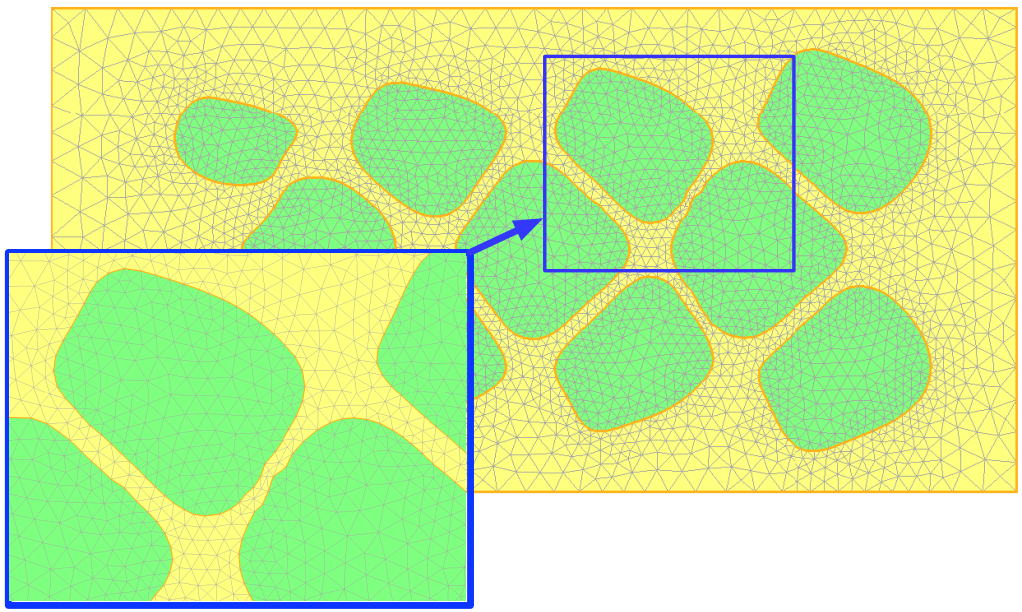


Rough discretization of the 0 *level set of* ϕ^{n+1} *into* $\mathcal{T}^n$*; the resulting mesh of* D *is ill-shaped.*

The algorithm in motion (V)

Step 5:

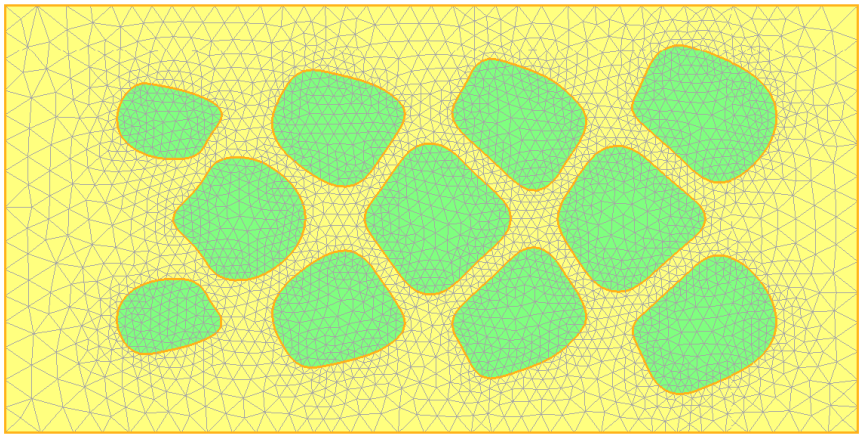
- Remeshing is then conducted, so as to enhance the overall quality of the mesh, while respecting the geometry of the shape.
- The new mesh $\mathcal{T}^{n+1}$ is eventually obtained.

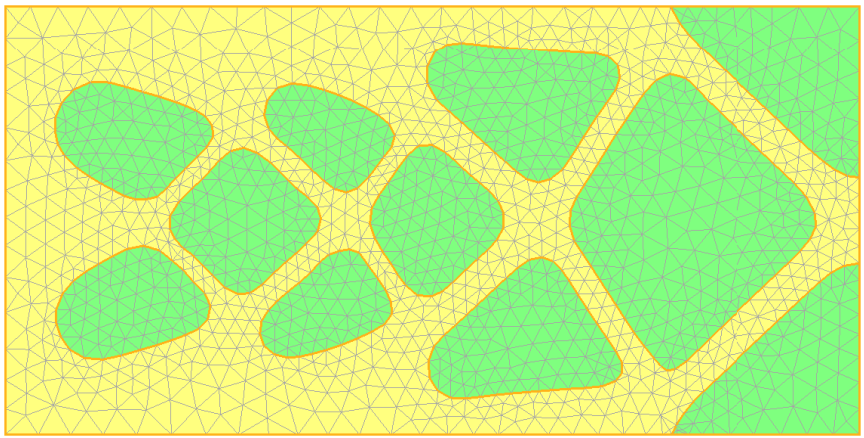


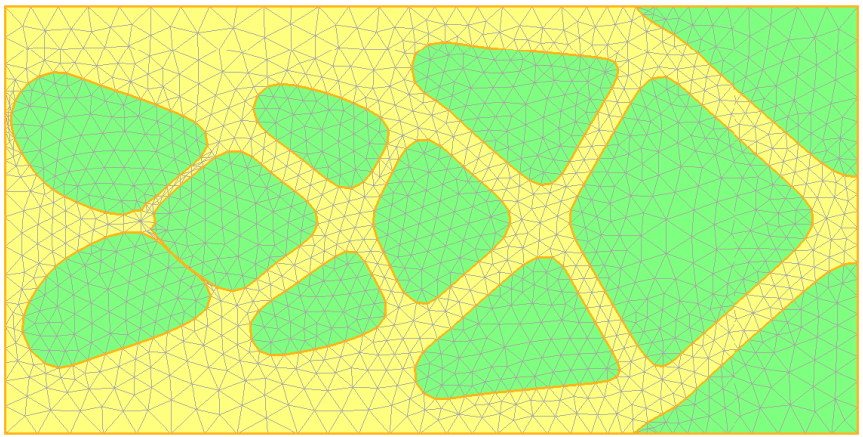
Quality-oriented remeshing ends with the well-shaped mesh $\mathcal{T}^{n+1}$ of D in which Ω^{n+1} is explicitly discretized.

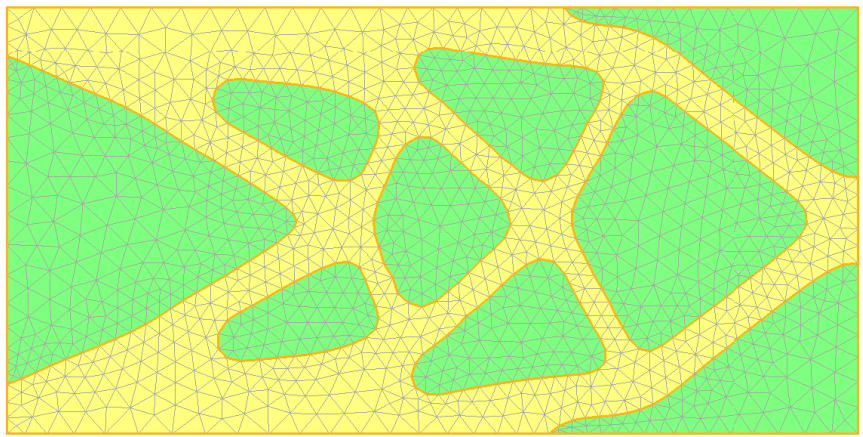
The algorithm in motion (VI)

The procedure is repeated until convergence.







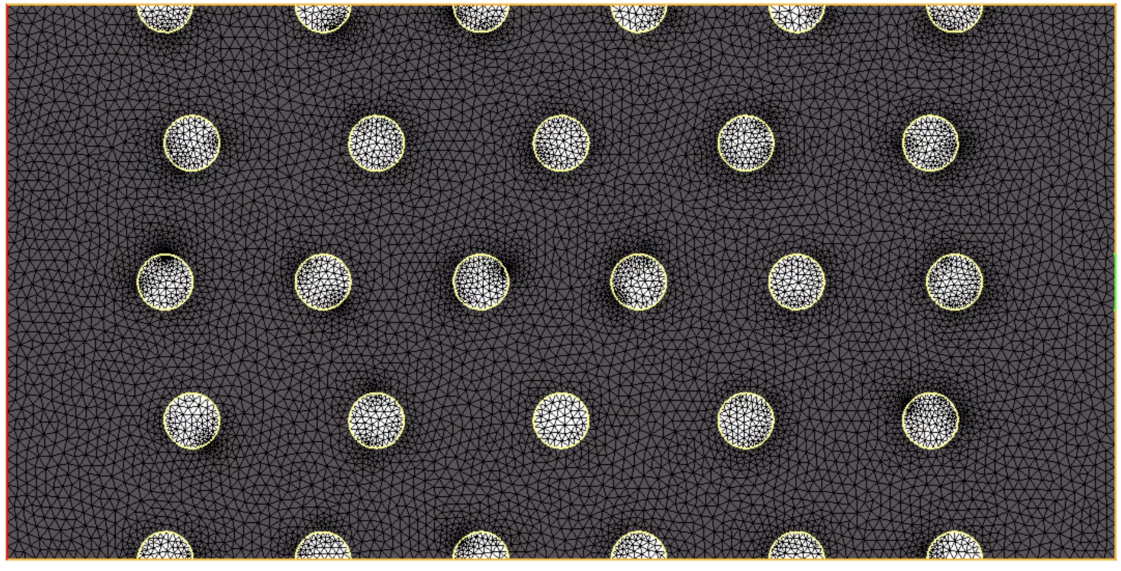


The body-fitted Level Set Method: summary (I)

At each iteration $n = 0, \ldots,$ the hold-all domain D is equipped with a mesh $\mathcal{T}^n$,

- In which Ω^n is explicitly discretized;
- Bearing a Level Set function ϕ^n for Ω^n.

- The Finite Element computations for u_{Ω^n} and p_{Ω^n} are performed on the sub-mesh of $\mathcal{T}^n$ corresponding to Ω^n.
- A descent direction θ^n is obtained from $J'(\Omega^n)$, $G'(\Omega^n)$.
- The Level Set function $\phi^n \to \phi^{n+1}$ for Ω is updated.
- A new mesh $\mathcal{T}^{n+1}$ of D, body-fitted to Ω^{n+1}, is created.



The body-fitted Level Set Method: summary (II)

- This method features an exact mesh of the shape Ω^n at each iteration:
  - Accurate and cheaper resolution of the physical equations.
    - In linear elasticity, no ersatz material is needed.
    - No approximation at interfaces between physical subdomains.
    - Fewer elements in the sub-mesh of Ω^n than in the box D.
  - Any mechanical solver can be used, in a "black-box" fashion: `FreeFem`, `Fenics`, `Firedrake`, industrial solvers...
  - Accurate modeling of geometric quantities: curvature, thickness constraints...

- The Level Set Method guarantees a robust and efficient tracking of arbitrarily large motions of the shape, including topological changes.

- The computational overhead incurred by remeshing is low when compared to mechanical computations.

Part II

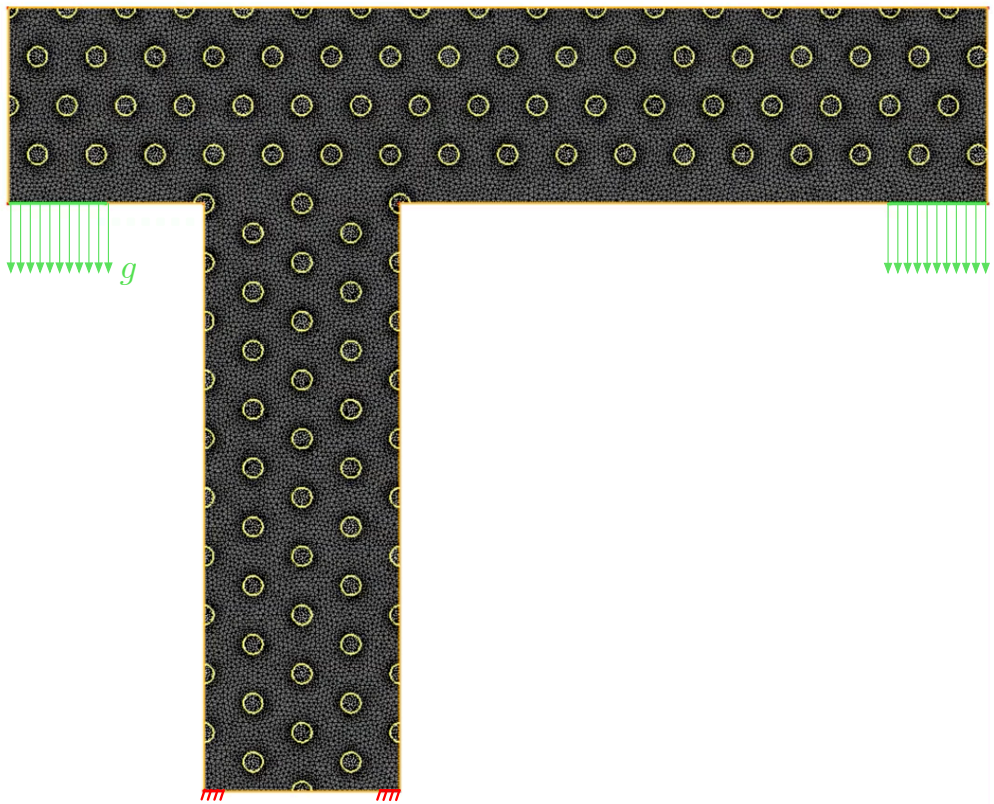
The body-fitted variant of the Level Set Method

1. A look at meshing and remeshing
2. The body-fitted variant of the Level Set Method
   - Presentation of the strategy
   - A few benchmark examples
3. A few applications and extensions

Optimization of a 2d crane

We minimize the compliance of a 2d crane under a volume constraint:

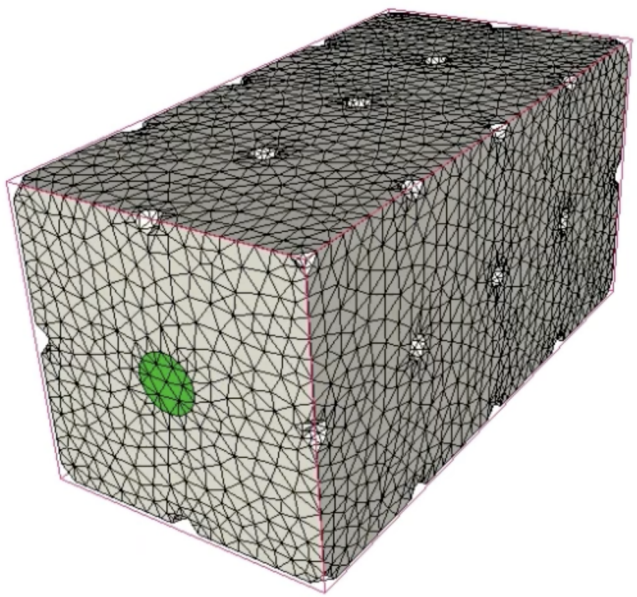
$$\min_{\Omega} C(\Omega) \text{ s.t. } \mathrm{Vol}(\Omega) = V_T, \text{ where } C(\Omega) = \int_{\Omega} Ae(u_\Omega) : e(u_\Omega)\,\mathrm{d}x.$$

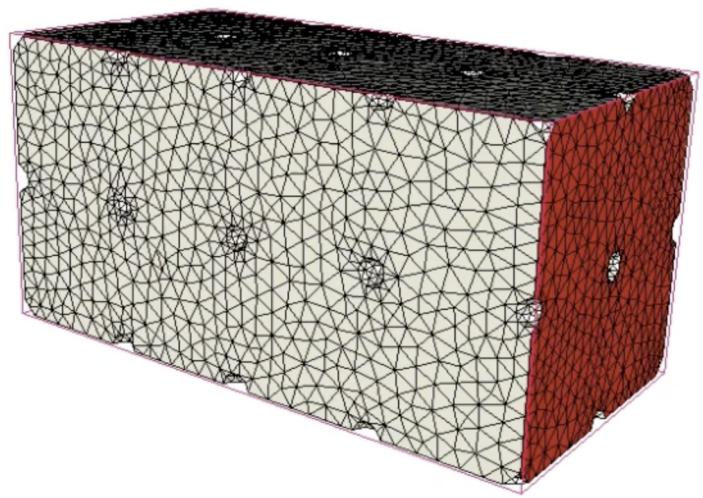


The benchmark 3d cantilever

We minimize the compliance of a 3d beam under a volume constraint:

$$\min_{\Omega} C(\Omega) \text{ s.t. } \mathrm{Vol}(\Omega) = V_T.$$

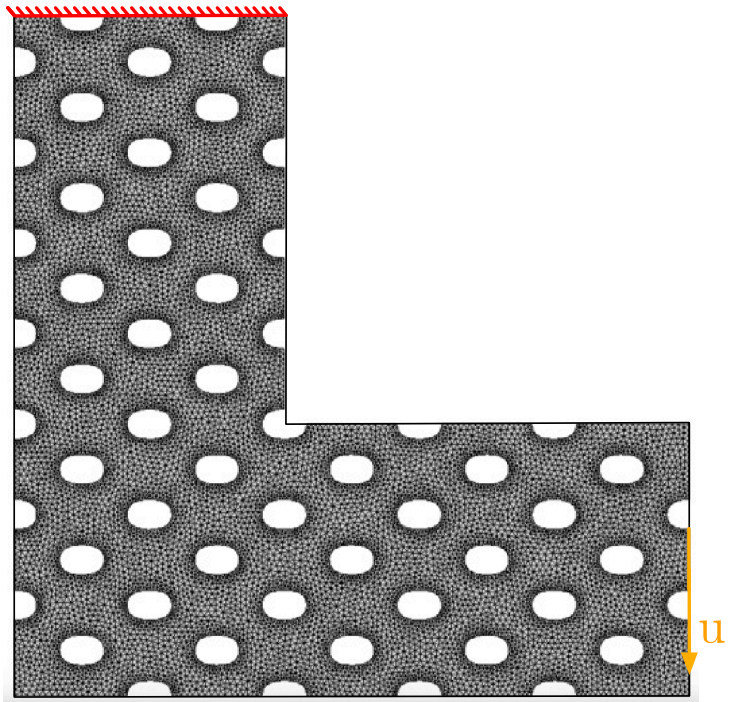




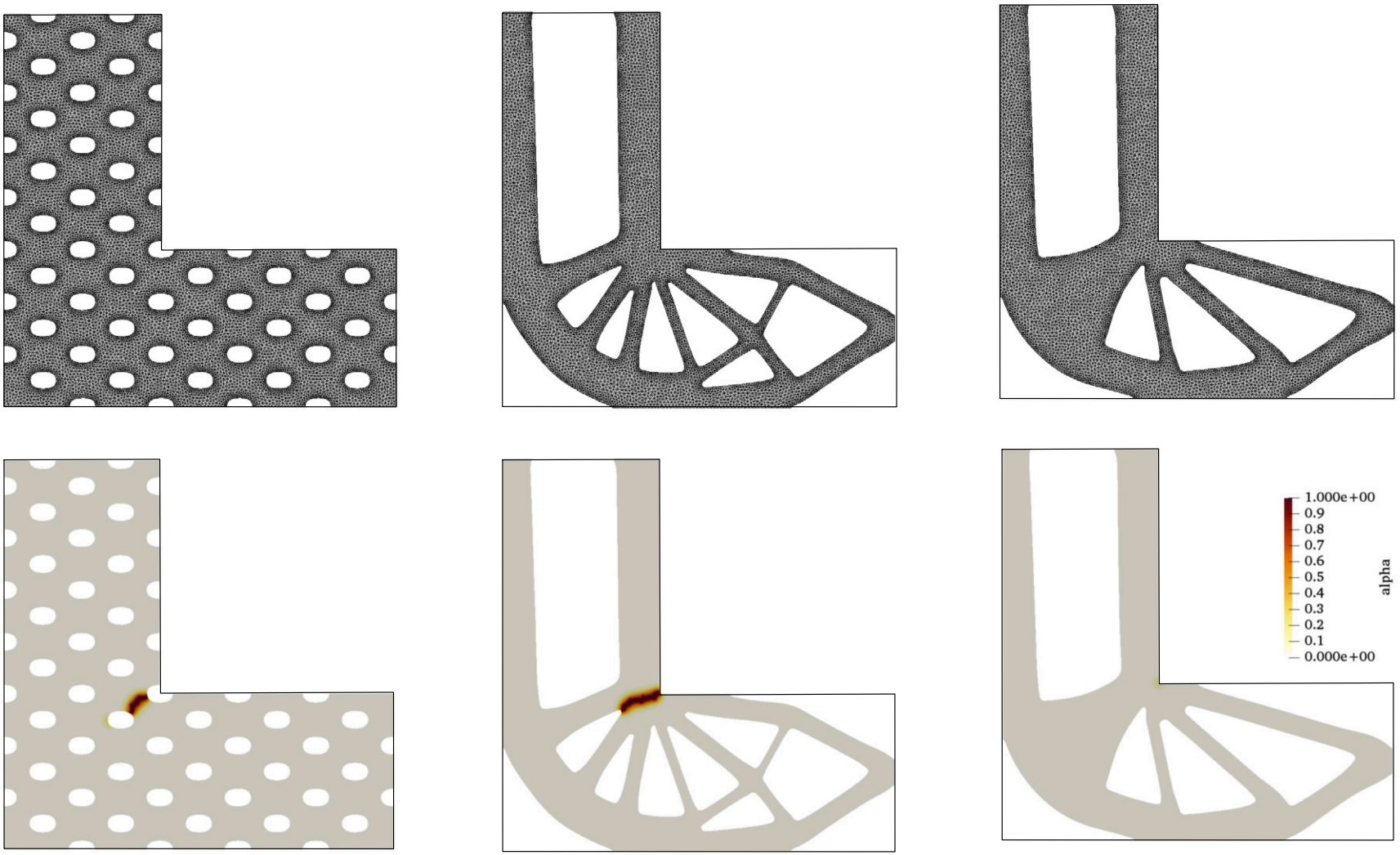
We minimize the total energy of an L-shaped beam

$$\min_{\Omega} J(\Omega) \text{ s.t. } \mathrm{Vol}(\Omega) = V_T, \text{ where } J(\Omega) = -\int_0^T \int_\Omega A(\alpha_\Omega) e(u_\Omega) : e(u_\Omega)\, \mathrm{d}x\mathrm{d}t,$$

and $(\alpha_\Omega, u_\Omega)$ is the time-dependent solution to a phase-field crack propagation model.



Optimization of an L-beam against crack propagation (II)



(Upper row, left) Initial design, (center) Optimized shape for the "classical" compliance minimization problem; (right) Optimized shape for the crack propagation model; (Bottom row) Distribution of the damage variable α within the structure.

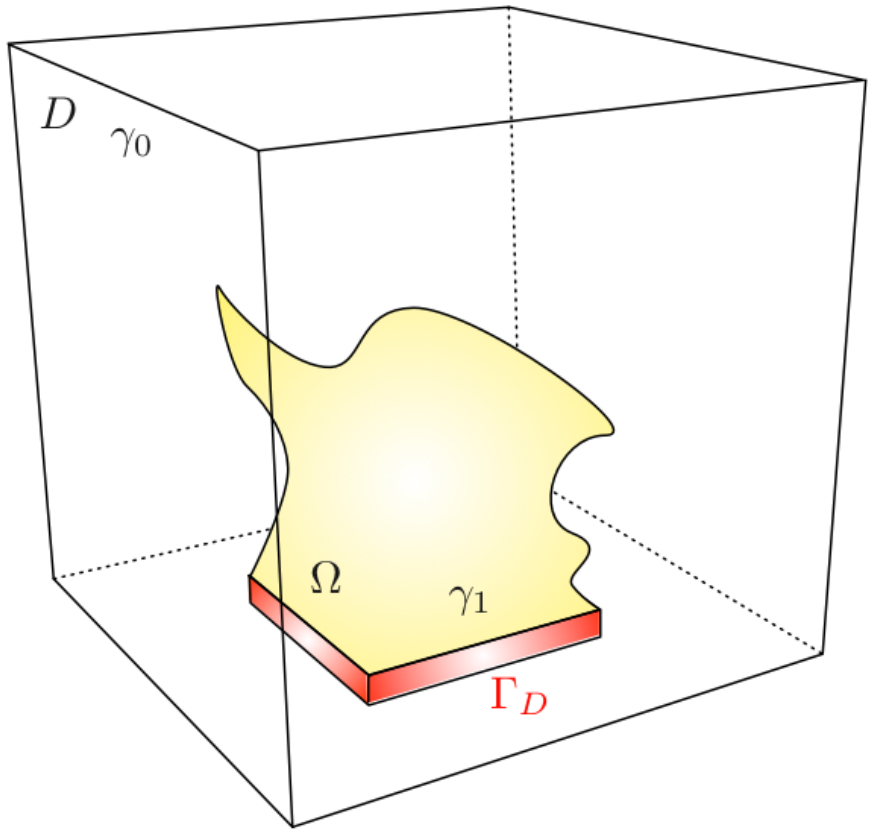
Optimization of the shape of a heat diffuser (I)

G. Allaire, C. Dapogny, F. Feppon & P. Jolivet

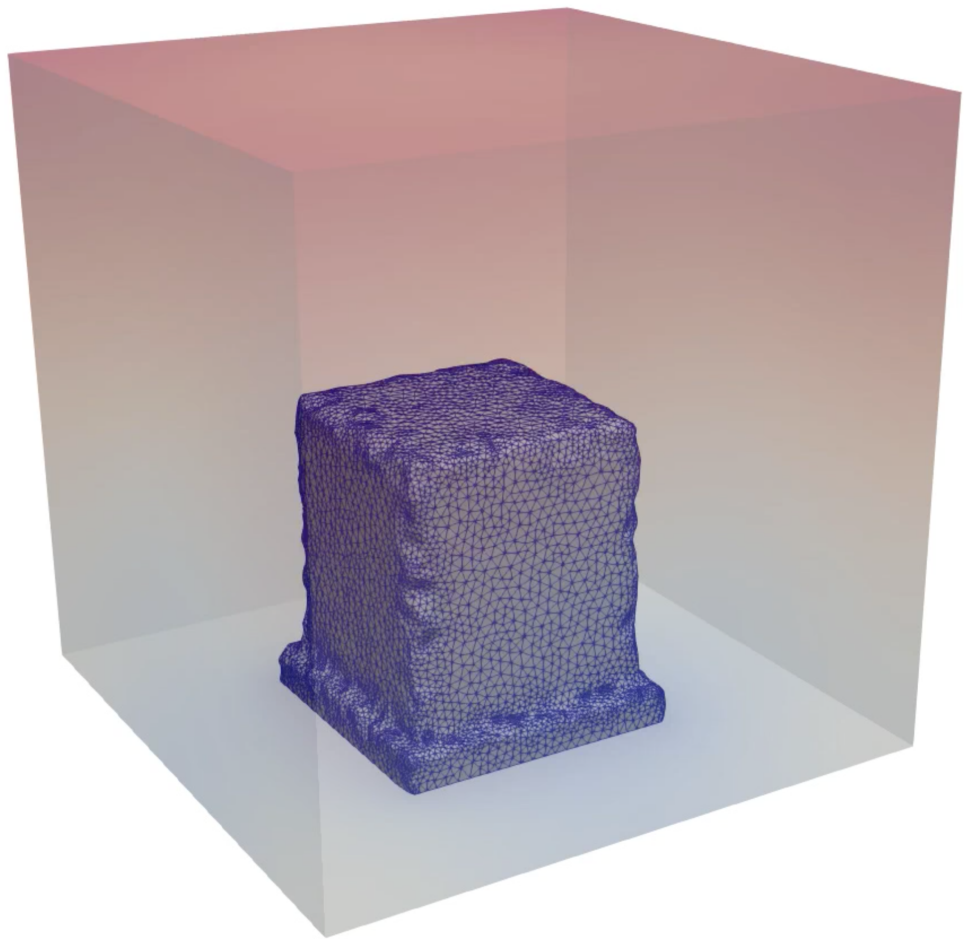
- A thermal chamber D is made of
  - A phase Ω with high conductivity γ_1
  - A phase $D \setminus \overline{\Omega}$ with low conductivity γ_0.
- A temperature $T_0 = 0$ is imposed on Γ_D.
- The remaining boundary $\partial D \setminus \overline{\Gamma_D}$ is insulated from the outside.
- A heat source is acting inside D.
- The temperature u_Ω inside D is solution to the two-phase Laplace equation.
- The average temperature inside D,

$$J(\Omega) = \frac{1}{|D|} \int_D u_\Omega \, \mathrm{d}x$$

is minimized under a volume constraint.



Optimization of the shape of a heat diffuser (II)

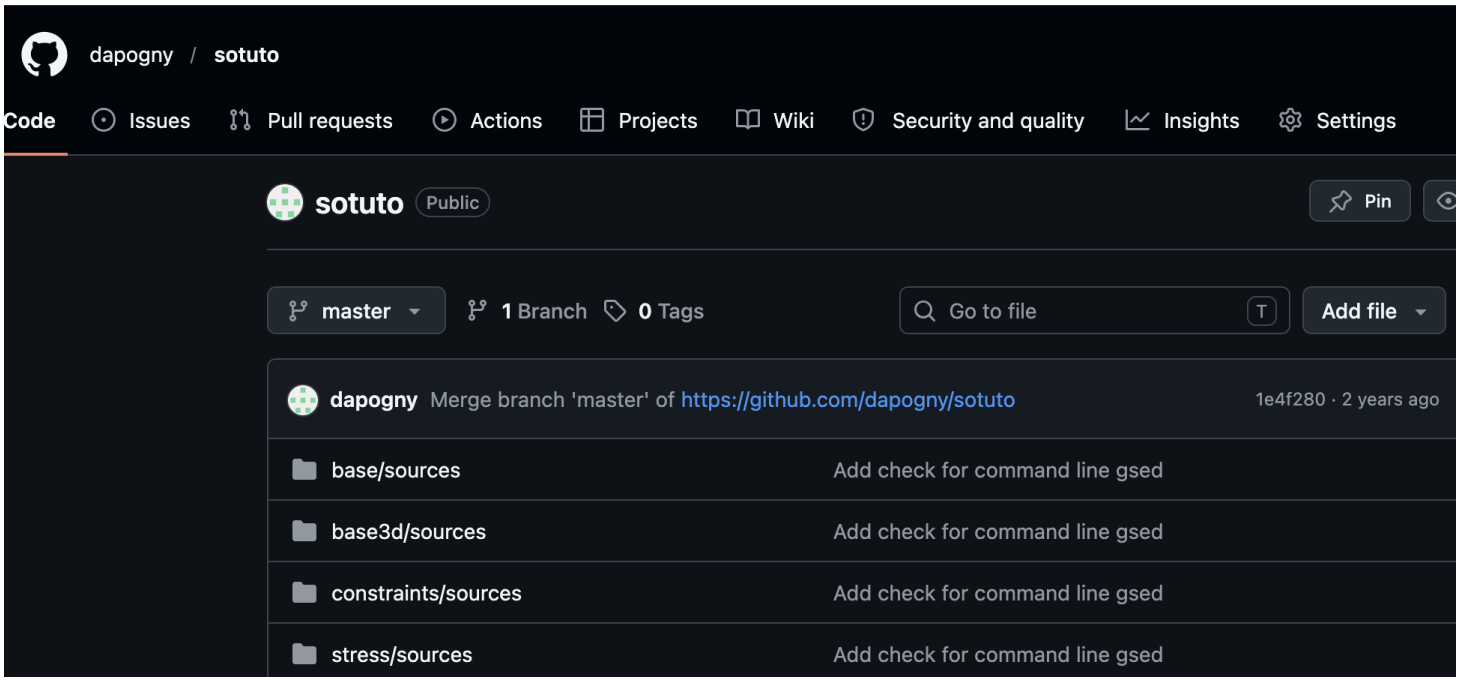


Optimization of the shape of a heat diffuser.

Yet another word of advertising

- A fully open-source implementation of this strategy:

`https://github.com/dapogny/sotuto`



- See the companion article for a documentation:

C. Dapogny and F. Feppon, *Shape optimization using a level set based mesh evolution method: an overview and tutorial* , C. R. Acad. Sci. Paris, 361, (2023), pp. 1267–1332.

Part III

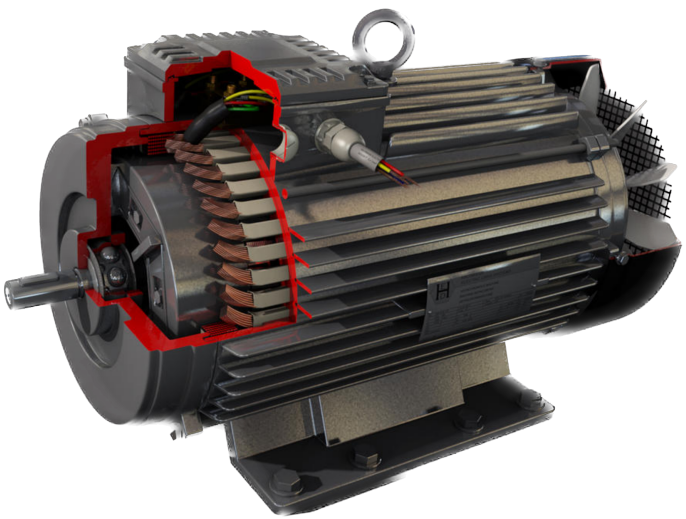
A few applications and extensions

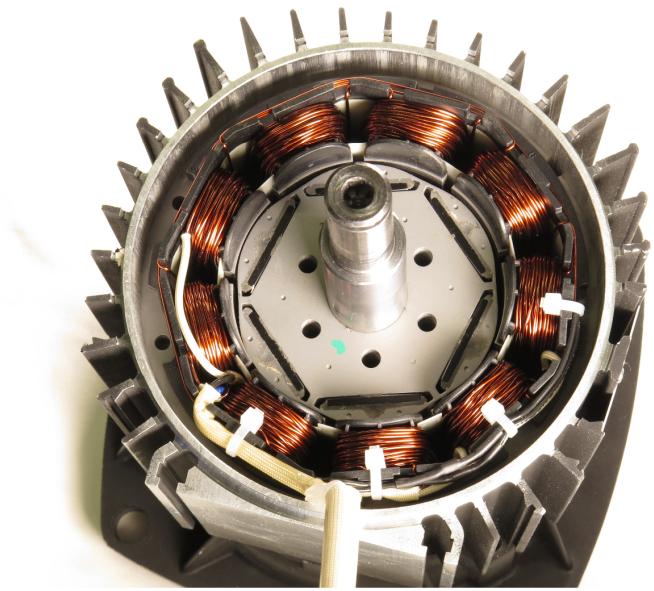
1. A look at meshing and remeshing
2. The body-fitted variant of the Level Set Method
3. A few applications and extensions
   - Optimization of the internal structure of an electric motor
   - Optimization of multi-physics systems
   - Shape optimization of boundary regions

Optimization of the shape of an electric motor (I)

G. Allaire, F. Bordeu, T. Gauthey, M. Hage-Hassan & X. Mininger

- Electric machines convert electrical energy into mechanical work.
- The physics at play is magneto-statics electromagnetism:
  - An electric current is powered in the coils of the outermost stator part.
  - It induces a magnetic field in the inner, rotor part, provoking its rotation.
  - The resulting mechanical energy is collected by a central shaft.
- Electric motors are seen as promising answers to the environmental crisis and the needs for energy savings.





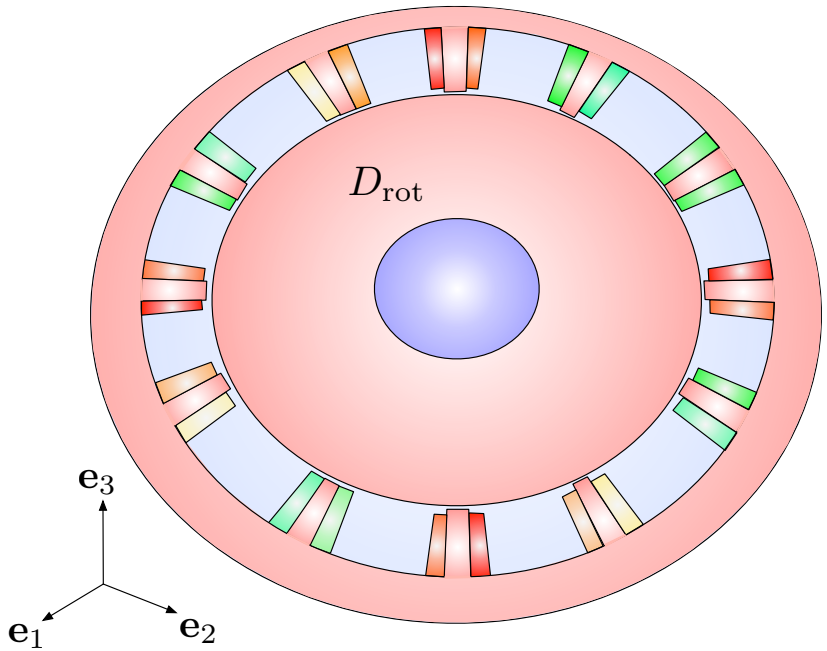
(Left) Overview of a motor [Hau]; (right) Cross-section of a motor, by Hanning Elektro-Werke GmbH & Co KG.

Optimization of the shape of an electric motor (II)

- The 2d cross-section D of a motor is made of the stator, the rotor, separated by an air gap:

$$\overline{D} = \overline{D_{\mathrm{rot}}} \cup \overline{D_{\mathrm{gap}}} \cup \overline{D_{\mathrm{stat}}}.$$

- The stator D_{stat} consists of:
  - A region filled with ferromagnetic material;
  - A region made of air;
  - The coils, featuring copper wires.
- At rest, the rotor D_{rot} is composed of:
  - A ferromagnetic core, occupying the optimized region Ω;
  - Air, occupying the region Ω_{a}.



- The internal geometry of the rotor undergoes a rotating motion φ_t:

$$\Omega(t) = \varphi_t(\Omega),\ \Omega_a(t) = \varphi_t(\Omega_a),\ \text{etc.}$$

Optimization of the shape of an electric motor (III)

- The values $u_{\Omega(t)}$ of the transverse component of the vector potential at different times t satisfy independent 2d static problems:

$$\begin{cases} -\mathrm{div}\left(\nu_{\Omega(t)}(x, |\nabla u_{\Omega(t)}|)\nabla u_{\Omega(t)}\right) = f & \text{in } D, \\ u_{\Omega(t)}(x) = 0 & \text{for } x \in \partial D, \end{cases}$$

  where ν is the phase-dependent reluctivity coefficient.

- These problems are non linear due to the behavior of the ferromagnetic material.

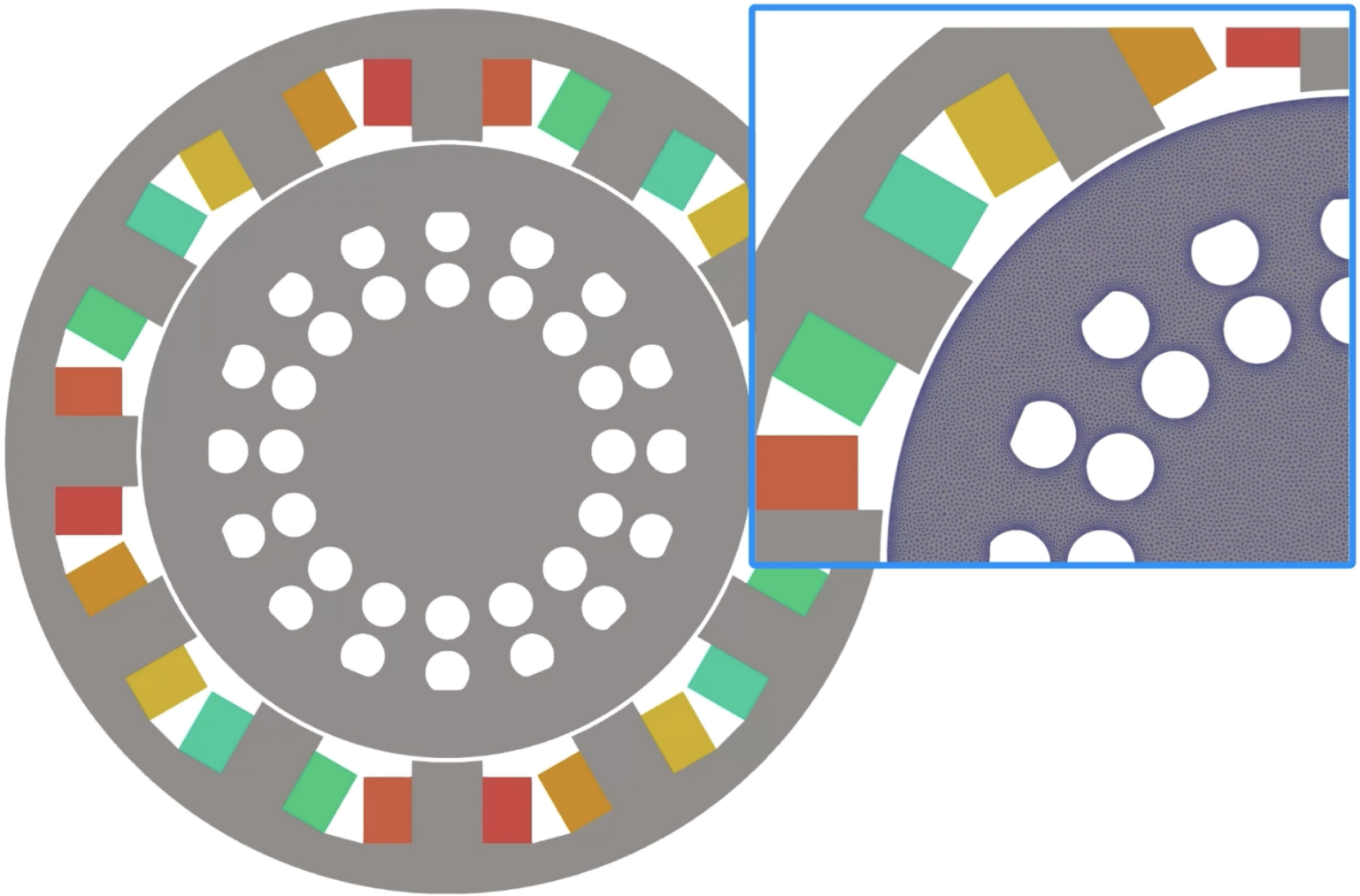
- The mean mechanical torque of the motor is maximized:

$$T_{\text{mean}}(\Omega) := \sum_{k=1}^{N} \int_D j(x, \nabla u_{\Omega(t^k)})\, \mathrm{d}x,$$

  involving the rotations $\Omega(t^k)$ of the configuration at rest Ω at several times t^k.

- Constraints are imposed on:
  - The volume $\mathrm{Vol}(\Omega)$,
  - The mechanical compliance of the structure,
  - The distance between different members of Ω.

Optimization of the shape of an electric motor (IV)



Optimization of the shape of a motor.

Part III

A few applications and extensions

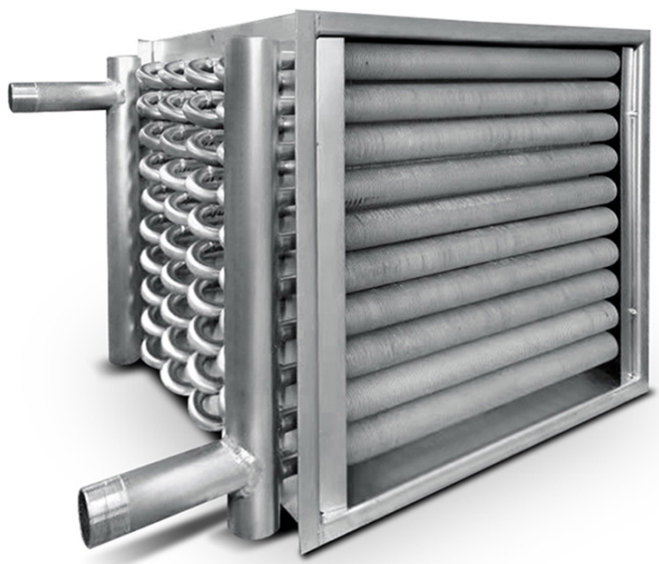
1. A look at meshing and remeshing
2. The body-fitted variant of the Level Set Method
3. A few applications and extensions
   - Optimization of the internal structure of an electric motor
   - Optimization of multi-physics systems
   - Shape optimization of boundary regions

Complex engineering devices D are often described by coupled physical effects:

- A solid phase $\Omega_s = \Omega$ is characterized by its elastic displacement $u_\Omega : \Omega \to \mathbb{R}^d$;
- The velocity v_Ω and pressure p_Ω of the fluid inside $\Omega_f := D \setminus \overline{\Omega}$ obey the Navier-Stokes equations;
- The temperature $T_\Omega : D \to \mathbb{R}$ depends on convection effects within the fluid, and diffusion through the total device.



a



b

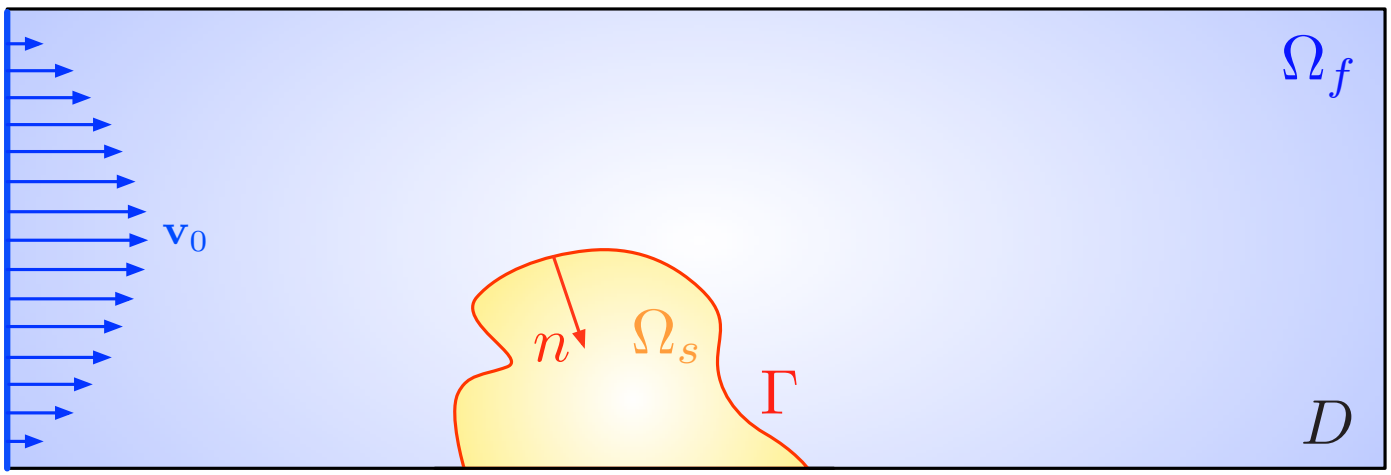
(a) A water turbine rotor; (b) A heat exchanger from Bangwin.

An example in fluid-structure interaction (I)

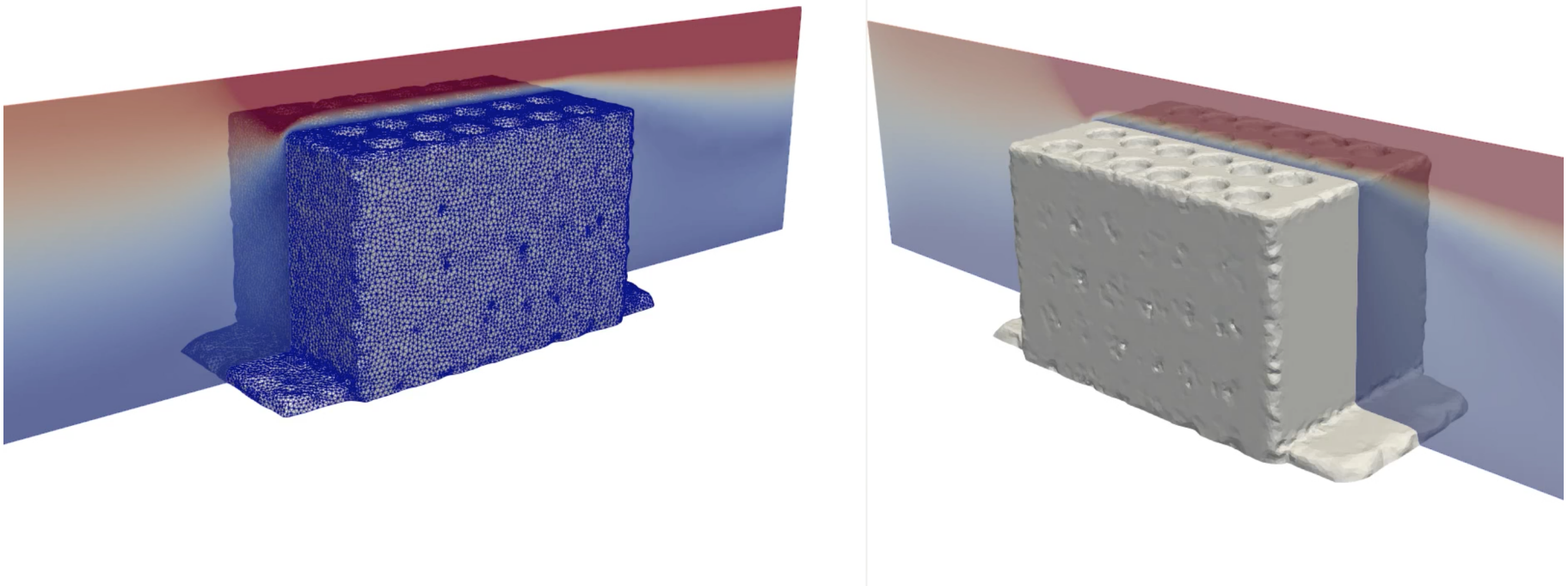
- A solid obstacle $\Omega_s := \Omega$ is placed inside a fixed cavity D where a fluid is flowing, occupying the phase $\Omega_f := D \setminus \overline{\Omega_s}$.
- The fluid obeys the Navier-Stokes equations ($Re = 60$), and the solid is governed by the linearized elasticity system.
- Weak coupling between Ω_f and Ω_s: the fluid exerts a traction on the interface Γ.
- We optimize the shape of Ω_s with respect to the solid compliance

$$J(\Omega) = \int_{\Omega_s} Ae(u_{\Omega_s}) : e(u_{\Omega_s}) \, \mathrm{d}x,$$

under a volume constraint.



An example in fluid-structure interaction (II)



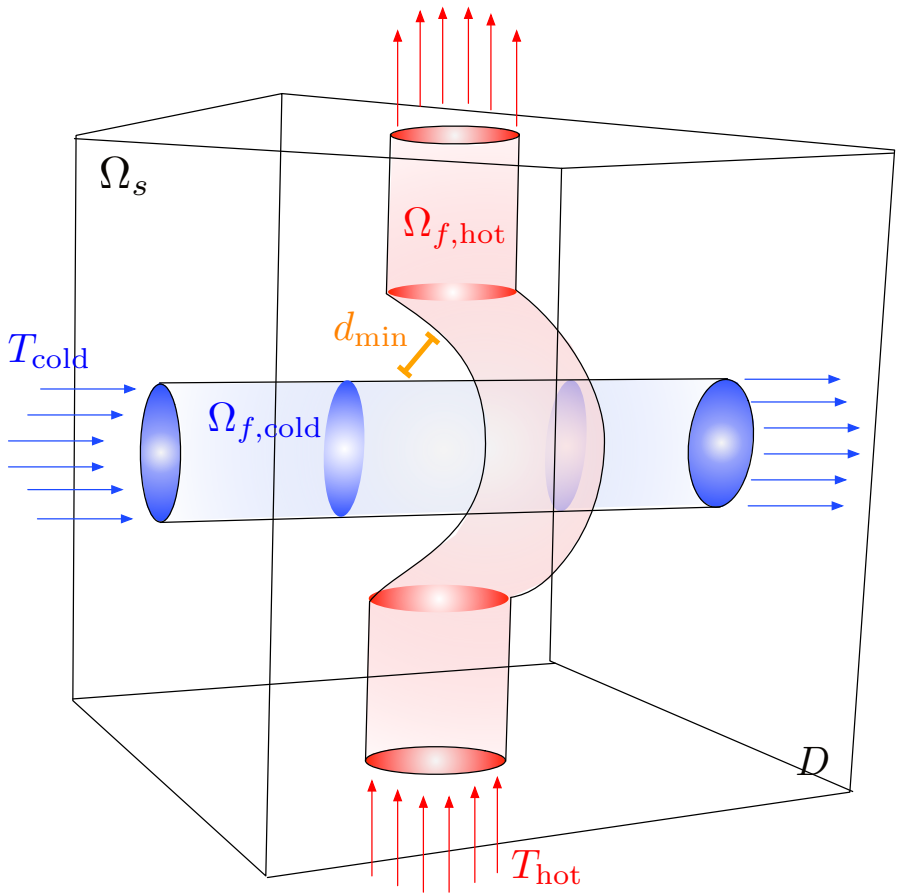
Optimization of the shape of a mast withstanding an incoming flow in 3d.

Optimization of the shape of a heat exchanger (I)

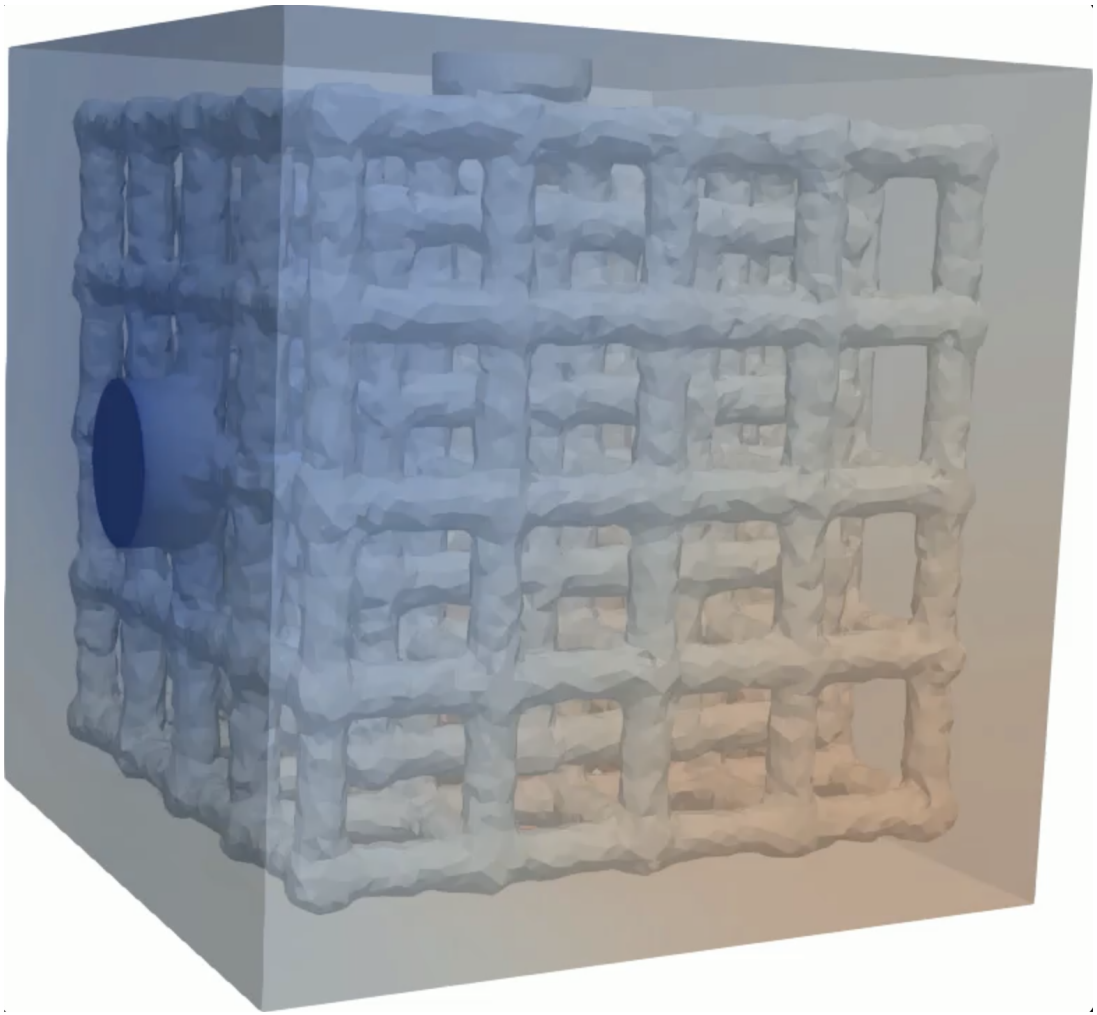
- A thermal chamber D is divided into
  - A phase $\Omega_{f,\text{hot}}$ conveying a hot fluid;
  - A phase $\Omega_{f,\text{cold}}$ conveying a cold fluid;
  - A solid phase Ω_s.
- The Navier-Stokes equations are satisfied in $\Omega_{f,\text{hot}}$, $\Omega_{f,\text{cold}}$.
- The stationary heat equation accounts for the temperature diffusion within D.
- The heat transferred from $\Omega_{f,\text{hot}}$ to $\Omega_{f,\text{cold}}$ is maximized.
- A constraint is imposed on the minimal distance between $\Omega_{f,\text{hot}}$ and $\Omega_{f,\text{cold}}$:

$$d(\Omega_{f,\text{hot}}, \Omega_{f,\text{cold}}) \geq d_{\text{min}}.$$

- Volume and pressure drop constraints are added on $\Omega_{f,\text{hot}}$, $\Omega_{f,\text{cold}}$.



Optimization of the shape of a heat exchanger (II)



Optimization of the shape of a heat exchanger.

Part III

A few applications and extensions

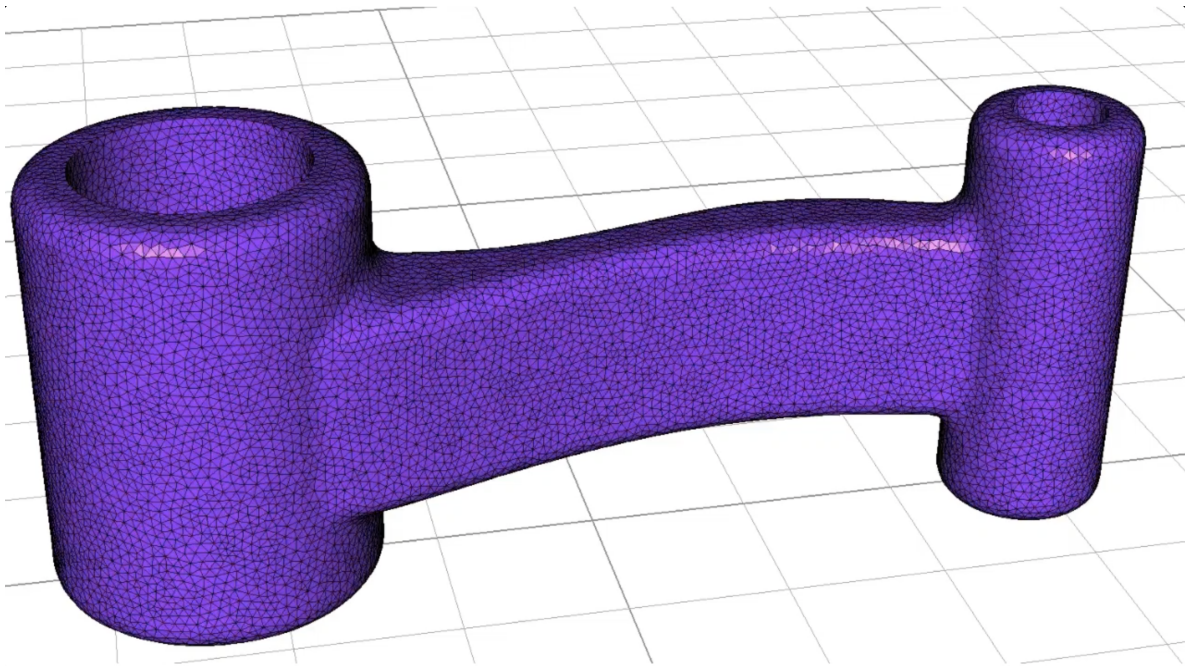
1. A look at meshing and remeshing
2. The body-fitted variant of the Level Set Method
3. A few applications and extensions
   - Optimization of the internal structure of an electric motor
   - Optimization of multi-physics systems
   - Shape optimization of boundary regions

This framework can be adapted to optimal design problems of the form:

$$\min_{G \subset \partial\Omega} J(G),$$

where:

- The optimized shape G is a region of the boundary of the fixed domain Ω;
- The objective $J(G)$ involves the solution u_G to a physical problem, where e.g. G is the support of boundary conditions.

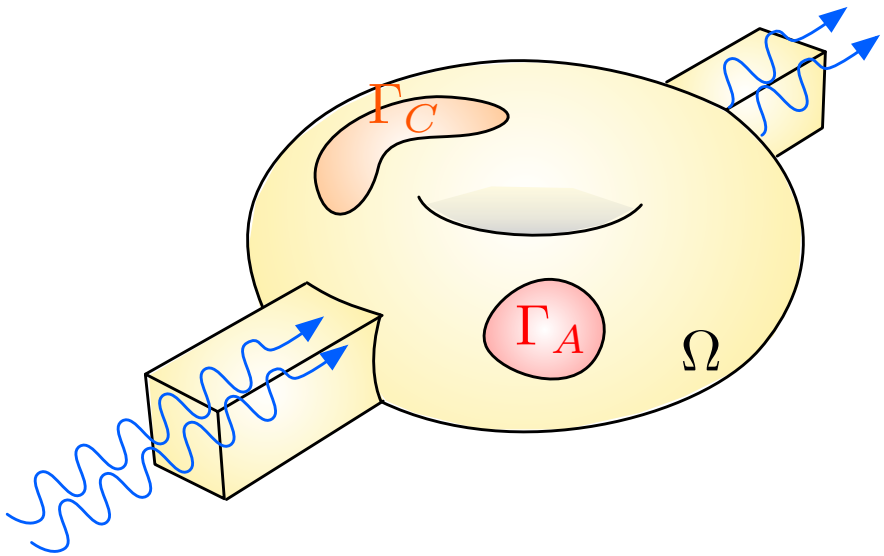


Optimization of a micro-osmotic mixer (I)

- Electro-osmotic mixers achieve the mixture of two fluids inside a device Ω by maximizing the electric field induced by electrodes on $\partial\Omega$.
- The boundary of Ω is decomposed as:

$$\partial\Omega = \overline{\Gamma_C} \cup \overline{\Gamma_A} \cup \overline{\Gamma},$$

where
- Γ_C is the cathode,
- Γ_A is the anode,
- Ω is insulated on Γ.



- The potential inside Ω is the solution to:

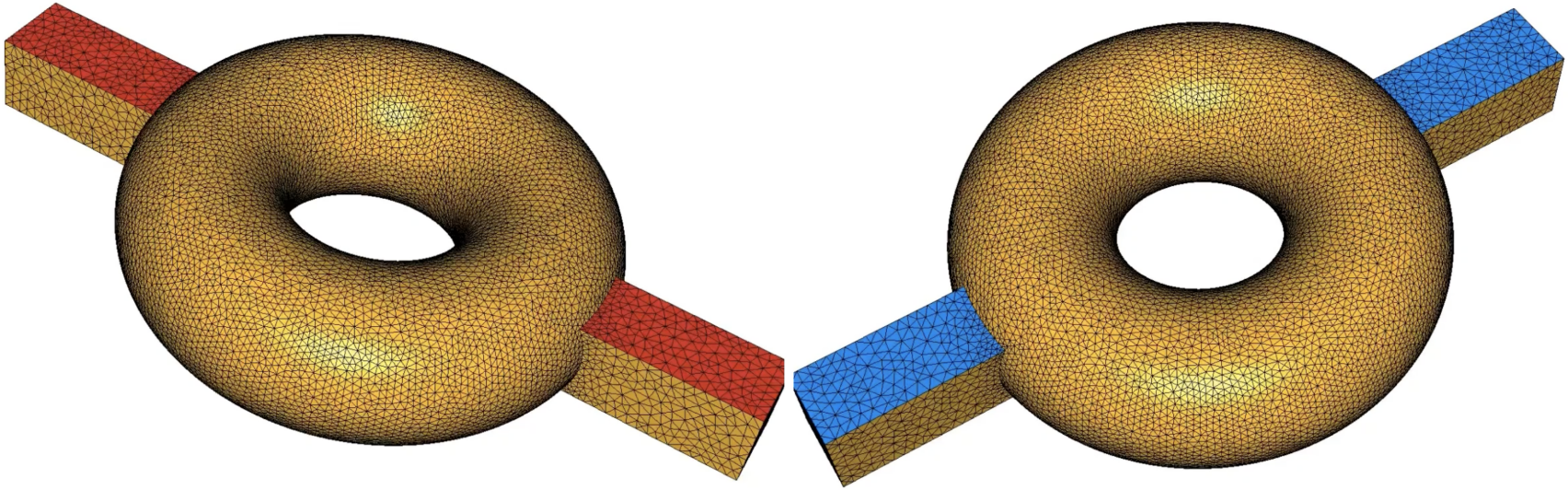
$$\begin{cases} -\mathrm{div}(\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_C, \\ u = u_{\mathrm{in}} & \text{on } \Gamma_A, \\ \gamma \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma. \end{cases}$$

- We aim to maximize the electric power inside Ω with respect to Γ_A and Γ_C:

$$J(\Gamma_A, \Gamma_C) = -\int_\Omega |\gamma \nabla u_{\Gamma_A, \Gamma_C}|^2 \, \mathrm{d}x,$$

under constraints on the surface measures of Γ_A and Γ_C.

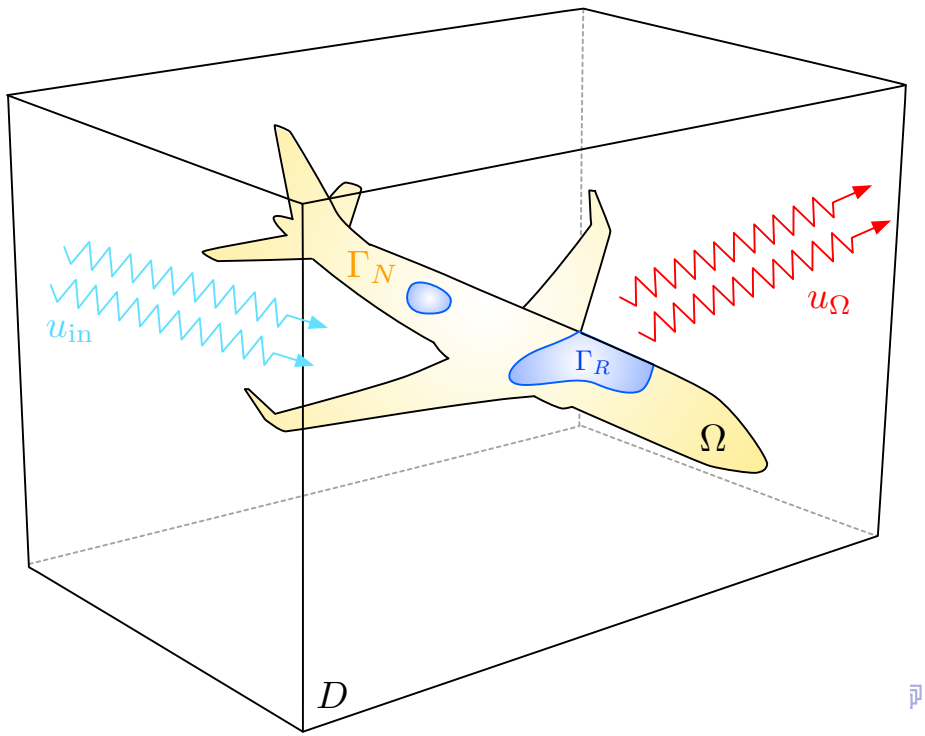
Optimization of a micro-osmotic mixer (II)



Optimization of (left) the anode, (right) the cathode of a micro-osmotic mixer.

Optimal repartition of sound-soft and sound-hard materials (I)

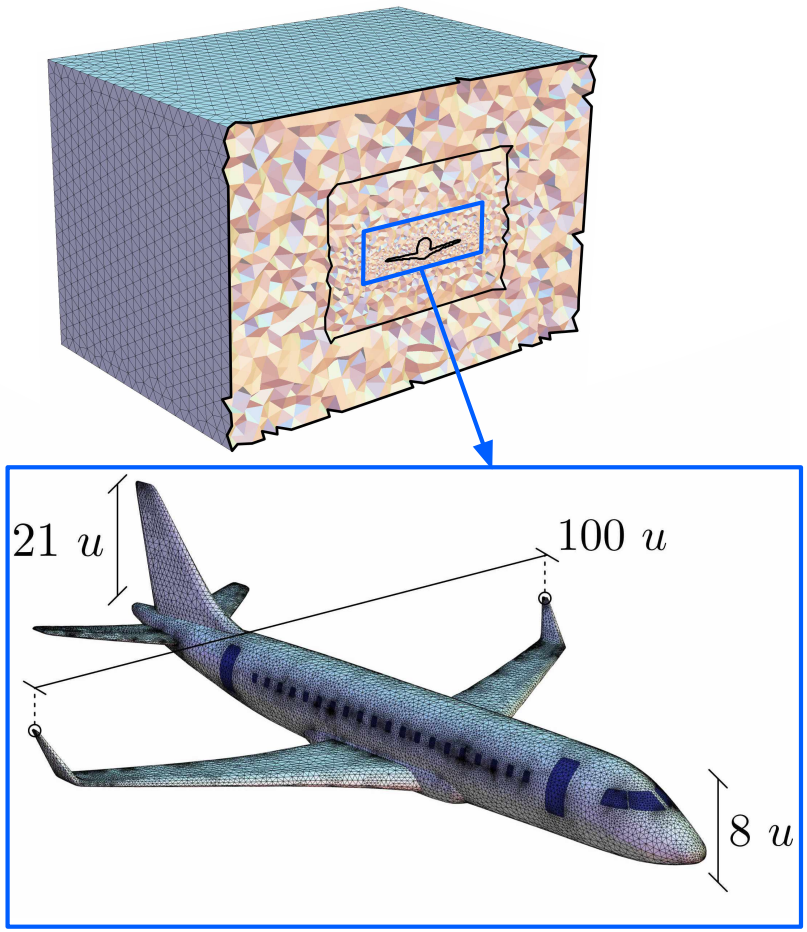
- The interaction of an incident wave u_{in} (e.g. sound) with an obstacle Ω inside the medium D induces a scattered wave u.
- This scattering effect can be used to detect or reconstruct Ω, which is either
  - Desirable, as in medical imaging (tomography, etc.),
  - Undesirable, e.g. when Ω is a stealth military submarine or aircraft.
- We aim to optimize the repartition of a sound-soft and a sound-hard materials on $\partial\Omega$ to cloak Ω, i.e. make it invisible to measurements of the scattered wave.



Optimal repartition of sound-soft and sound-hard materials (II)

- The obstacle Ω is an aircraft.
- The incoming wave reads $u_{\text{in}}(x) = e^{-i\omega\xi\cdot x}$.
- The boundary $\partial\Omega$ is decomposed as $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma_R}$, where:
  - Γ_N supports Neumann conditions;
  - Γ_R bears Robin conditions.
- The scattered wave $u(x)$ is solution to the Helmholtz equation:

$$\begin{cases} -\operatorname{div}(\gamma\nabla u) - \omega^2 u = 0 & \text{in } \Omega \\ \gamma\frac{\partial u}{\partial n} = -\gamma\frac{\partial u_{\text{in}}}{\partial n} & \text{on } \Gamma_N, \\ \gamma\frac{\partial u}{\partial n} + \frac{i\omega}{z}u = -\gamma\frac{\partial u_{\text{in}}}{\partial n} - \frac{i\omega}{z}u_{\text{in}} & \text{on } \Gamma_R, \\ + \text{ radiation boundary conditions} & \text{on } \partial D. \end{cases}$$



- We optimize Γ_R to minimize the amplitude of the scattered wave:

$$J(\Gamma_R) = \int_{D\setminus\overline{\Omega}} |u_{\Gamma_R}|^2 \, \mathrm{d}x,$$

under a constraint on its surface area.

Optimal repartition of sound-soft and sound-hard materials (III)



Optimization of the repartition of a sound-soft and a sound-hard material on the body of an aircraft.

The body-fitted Level Set Method: assets and drawbacks

Assets of the method:

- An exact mesh of the shape Ω^n and the exterior $D \setminus \overline{\Omega^n}$ are available at each stage of the optimization.
- The mechanical PDE involved in the optimal design problem can be solved:
  - Without approximation: no "ersatz material trick" is needed;
  - With an arbitrary numerical solver, used in a non intrusive fashion.

Drawbacks:

- The implementation rather delicate, although the most critical operations are implemented in open-source libraries.
- The method incurs a moderate additional cost due to the remeshing operations.

A few perspectives

- Treatment of complex multi-material patterns, as in e.g. electric motors.
- Ongoing implementation of a parallel remeshing library: `parmmg`.
- Development of a body-fitted Level Set Method for the optimization of open surfaces, e.g.
  - Shells in structure mechanics,
  - Refractive or reflective surfaces in optics,
  - Electromagnetic screens.

Thank you!



Appendix

A few time indicators for the body-fitted method

Operation	**Iteration 1**	**Iteration 20**
Removal of isolated mesh fluid parts	41 sec	21 sec
Signed distance function to Ω_f	31 sec	21 sec
Signed distance function to $\Omega_{f,\text{hot}}$	14 sec	10 sec
Signed distance function to $\Omega_{f,\text{cold}}$	22 sec	17 sec
State equations (30 processes)	344 sec	253 sec
Shape sensitivities (30 processes)	340 sec	259 sec
Advection of the solid–fluid interface	11 sec	8 sec
Remeshing (still sequential)	696 sec	521 sec
Total	26 mn	19 mn

CPU times of the main operations in the 3d heat exchanger test case.

Total CPU time (360 iterations): 8 days on 30 processing units of an Intel(R) Xeon(R) CPU E5-2407, 2.4GHz

Bibliography

References I



[AdDaFre1] G. Allaire, C. Dapogny, and P. Frey, *Topology and geometry optimization of elastic structures by exact deformation of simplicial mesh*, C. R. Acad. Sci. Paris, Série I, 349, (2011), pp. 999–1003.



[AdDaFre2] G. Allaire, C. Dapogny, and P. Frey, *A mesh evolution algorithm based on the level set method for geometry and topology optimization*, Struct. Multidisc. Optim., 48, 4, (2013), pp. 711–715.



[AdDaFre3] G. Allaire, C. Dapogny, and P. Frey, *Shape optimization with a level set based mesh evolution method*, Comput. Methods Appl. Mech. Eng., 282 (2014), pp. 22–53.



[BriDa] C. Brito-Pacheco and C. Dapogny, *Body-fitted tracking within a surface via a level set based mesh evolution method*, J Sci Comput 102, 73 (2025).



[BonBriDaEs] E. Bonnetier, C. Brito-Pacheco, C. Dapogny and E. Estevez, *Numerical shape and topology optimization of regions supporting the boundary conditions of a physical problem*, ESAIM: Control, Optimization and Calculus of Variations, 31, (2025), art. 75.

References II



[DaDobFre] C. Dapogny, C. Dobrzinsky and P. Frey, *Three-dimensional adaptive domain remeshing, implicit domain meshing, and applications to free and moving boundary problems*, J. Comput. Phys., 262, (2014), pp. 358–378.



[DaFe] C. Dapogny and F. Feppon, *Shape optimization using a level set based mesh evolution method: an overview and tutorial* , C. R. Acad. Sci. Paris, 361(G8), (2023), pp. 1267–1332.



[DeAlJou] J. Desai, G. Allaire and F. Jouve, *Topology optimization of structures undergoing brittle fracture*, Journal of Computational Physics, 458, (2022), 111048.



[FepAlBorCorDa] F. Feppon, G. Allaire, F. Bordeu, J. Cortial and C. Dapogny, *Shape optimization of a coupled thermal fluid-structure problem in a level set mesh evolution framework*, SeMA J., 76, 3, (2019), pp. 413–458.



[FepAlDa] F. Feppon, G. Allaire and C. Dapogny, *Topology optimization of thermal fluid–structure systems using body-fitted meshes and parallel computing*, J. Comput. Phys., 417, 109574, (2020).

References III



[FreGeo] P.J. Frey and P.L. George, *Mesh Generation : Application to Finite Elements*, Wiley, 2nd Edition, (2008).



[Hau] Haussmann, `https://www.deviantart.com/hausmann/art/Electric-Motor-Final-Cross-section-665887814`