The level set method for structural topology optimization: Part I

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Foreword

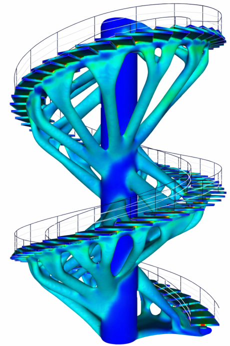
- Shape and topology optimization is a ubiquitous concern in the design of physical systems.

- Many "topology optimization" approaches represent the design variable by a density function, e.g. the homogenization method or the SIMP method.

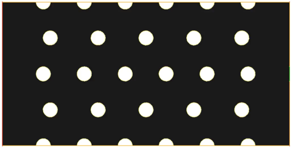
- Alternatively, the Level Set Method allows for an explicit representation of the optimized shape, which is a domain Ω in $\mathbb{R}^d$, $d = 2$ or 3.

- The Level Set Method for shape and topology optimization relies on two ingredients:

  - The Hadamard method for geometric shape variations,
  - The Osher-Sethian algorithm for the robust description of arbitrary shape evolution, including topological changes.



Optimization of a staircase (courtesy of Ansys).



Dramatic evolution of a shape under optimization.

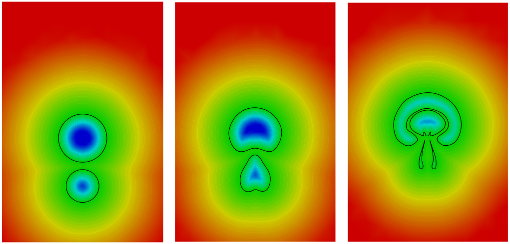
Contents of the webinar series

- Webinar 1 Introduction to the Level Set Method for shape and topology optimization.
  - The Level Set Method for interface capturing.
  - Deployment of the Level Set Method for shape and topology optimization.
  - First numerical experiments.

- Webinar 2 Mathematical and technical details of the Level Set Method.
  - Calculation of shape derivatives by the adjoint method.
  - Advanced practice of the Level Set Method: Hilbertian method, optimization algorithm, use of simplicial meshes.

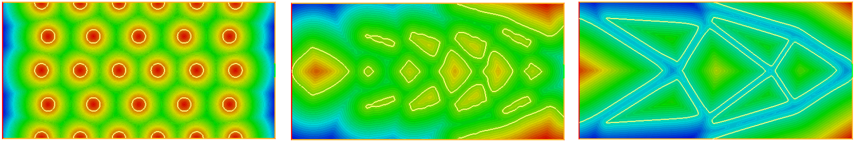
- Webinar 3 The body-fitted Level Set Method, based on remeshing.
  - A taste of meshing and remeshing techniques.
  - Multi-physics applications and beyond.

- The Level Set Method for interface capturing.



- Deployment of the Level Set Method for shape and topology optimization.

- First numerical experiments.



Part I

Basic principles

A general formulation

We consider a shape and topology optimization problem of the form:

$$\min_{\Omega} J(\Omega) \text{ s.t. } G(\Omega) = 0. \qquad (\mathcal{P})$$

In this formulation:

- The design variable is a shape, i.e. a domain Ω in $\mathbb{R}^d$, $d = 2$ or 3.
- $J(\Omega)$ is an objective function.
- $G(\Omega) = (G_1(\Omega), \ldots, G_p(\Omega))$ is a collection of p (equality) constraints (for simplicity, no inequality constraints).
- $J(\Omega)$ or some of the $G_i(\Omega)$ depend on Ω via a state u_Ω, solution to a physical boundary value problem posed on Ω.

In this first `Webinar`, we focus on the unconstrained problem:

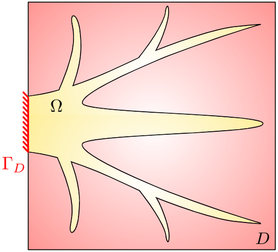
$$\min_{\Omega} J(\Omega).$$

(Possibly, volume constraints are incorporated to $J(\Omega)$ by fixed penalization)

An example in thermal conduction

Let $D \subset \mathbb{R}^d$ be made of two materials with conductivities $0 < \alpha < \beta$.

- The temperature is set to 0 on $\Gamma_D \subset \partial D$.
- The remaining boundary $\partial D \setminus \overline{\Gamma_D}$ is insulated.
- A source $f : D \to \mathbb{R}$ is acting in the medium.
- The design variable is the shape $\Omega \subset D$ of the phase β.



The temperature u_Ω inside D is the solution of the two-phase conductivity equation:

$$\left\{ \begin{array}{cl} -\mathrm{div}(\gamma_\Omega \nabla u_\Omega) = f & \text{in } D \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ \gamma_\Omega \frac{\partial u_\Omega}{\partial n} = 0 & \text{on } \partial D \setminus \overline{\Gamma_D}, \end{array} \right. \quad \text{where} \quad \gamma_\Omega(x) = \left\{ \begin{array}{cl} \beta & \text{if } x \in \Omega, \\ \alpha & \text{if } x \in D \setminus \overline{\Omega}. \end{array} \right.$$

The mean temperature within D is minimized, for a given amount of good conductor:

$$J(\Omega) = \frac{1}{|D|} \int_D u_\Omega \, \mathrm{d}x, \quad G(\Omega) = \mathrm{Vol}(\Omega) - V_T, \text{ where } \mathrm{Vol}(\Omega) := \int_\Omega \mathrm{d}x.$$

An example in solid mechanics

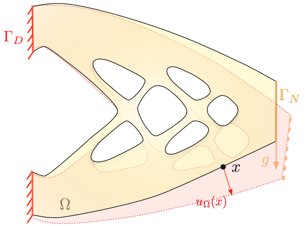
- The shape $\Omega \subset \mathbb{R}^d$ is a mechanical structure.
  - It is fixed on a subsett Γ_D of its boundary,
  - Surface loads $g : \Gamma_N \to \mathbb{R}^d$ are applied on a disjoint region $\Gamma_N \subset \partial\Omega$.
- The displacement $u_\Omega : \Omega \to \mathbb{R}^d$ is the solution to the linear elasticity system:

$$\left\{ \begin{array}{rcll} -\mathrm{div}(Ae(u_\Omega)) & = & 0 & \text{in } \Omega, \\ u_\Omega & = & 0 & \text{on } \Gamma_D, \\ Ae(u_\Omega)n & = & g & \text{on } \Gamma_N, \\ Ae(u_\Omega)n & = & 0 & \text{on } \Gamma, \end{array} \right.$$

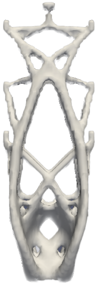
  where $e(u) = \frac{1}{2}(\nabla u^T + \nabla u)$ is the strain tensor, and A is the Hooke's law of the material.

- One common objective function is the compliance:

$$J(\Omega) = \int_\Omega Ae(u_\Omega) : e(u_\Omega)\,\mathrm{d}x = \int_{\Gamma_N} g \cdot u_\Omega \,\mathrm{d}s.$$



The linear elasticity model

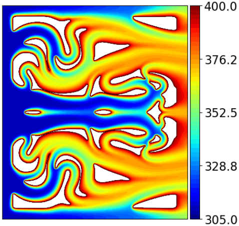


Optimized design of a pylon

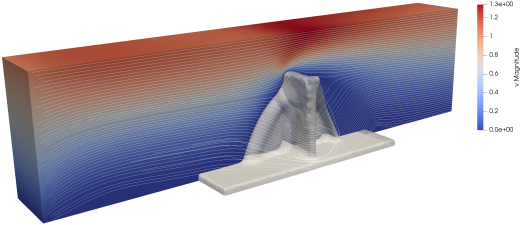
A wide variety of applications beyond

More recently, optimal design appears in such diverse fields as:

- Fluid mechanics: external aerodynamics, fluid transport, mixing devices, etc.

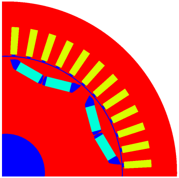


Optimized 2d section of a heat exchanger.

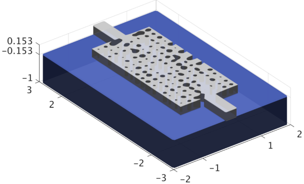


Optimized shape of a solid obstacle to a fluid flow.

- Electromagnetism: electric machines, current sensors, photonic crystals...



Optimized section of a rotor.



Optimized shape of a nanophotonic polarization converter.

- Quantum chemistry, with the theory of Maximum Probability Domains.

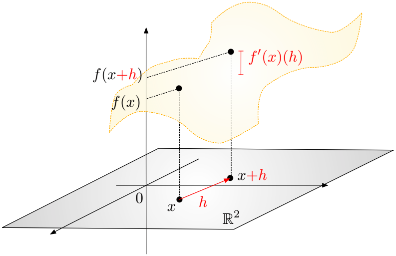
Part I

Basic principles

Differentiation with respect to the shape

- The solution of $(\mathcal{P})$ calls for a notion of derivative of a function $J(\Omega)$ with respect to the shape Ω.

- In a vector space, derivatives are defined from "small variations" of the variables.



The derivative $f'(x)$ of $f : \mathbb{R}^2 \to \mathbb{R}$ is defined from the fact that, for "small" $h \in \mathbb{R}^2$, $f(x+h) \approx f(x) + f'(x)(h)$.

- Likewise, shape derivatives build on "small variations" of a given shape Ω.

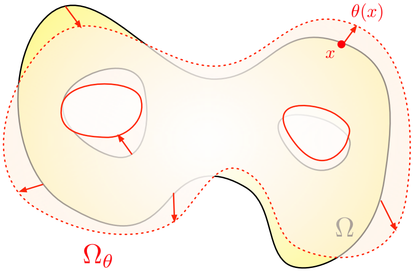
- What is a convenient small increment for a shape ?

- Hadamard's boundary variation method features variations of a shape Ω of the form:

$$\Omega_\theta := (\mathrm{Id} + \theta)(\Omega),$$

for "small" vector fields $\theta : \mathbb{R}^d \to \mathbb{R}^d$.

- In the mathematical theory $\theta \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}^d)$.



- **Idea:** Perturbations of a reference shape Ω are parametrized by the vector field θ.

- When θ is "small", Ω and Ω_θ have the same topology.

A. Henrot and M. Pierre, *Shape Variation and Optimization*, EMS Tracts in Mathematics Vol. 28, 2018.

Definition.

Let $\Omega \subset \mathbb{R}^d$ be a smooth domain. A function $\Omega \mapsto J(\Omega)$ is shape differentiable at Ω if the mapping

$$\theta \mapsto J(\Omega_\theta)$$

is differentiable at 0, i.e. the following expansion holds:

$$J(\Omega_\theta) = J(\Omega) + J'(\Omega)(\theta) + \mathrm{o}(\theta), \text{ where } \frac{o(\theta)}{||\theta||} \xrightarrow{\theta \to 0} 0.$$

The mapping $\theta \mapsto J'(\Omega)(\theta)$ is the shape derivative of $J(\Omega)$ at Ω.

A descent direction for $J(\Omega)$ is a vector field θ such that $J'(\Omega)(\theta) < 0$:

$$\text{For “small” } \tau > 0, \quad J(\Omega_{\tau\theta}) \approx J(\Omega) + \tau J'(\Omega)(\theta) < J(\Omega);$$

$\Rightarrow$ A “small” perturbation of Ω according to θ entails a decrease of $J(\Omega)$.

Lemma.

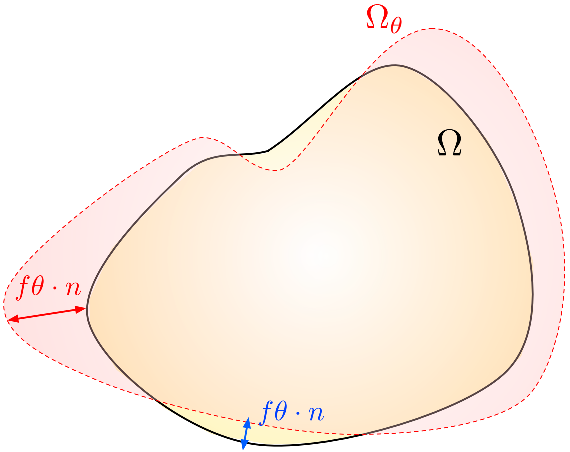
Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain, and let f be a fixed smooth function. Consider the functional:

$$J(\Omega) = \int_\Omega f(x)\, \mathrm{d}x$$

Then $J(\Omega)$ is shape differentiable at Ω and its shape derivative is:

$$J'(\Omega)(\theta) = \int_{\partial\Omega} f(x) \left(\theta(x) \cdot n(x)\right) \mathrm{d}s(x).$$

First examples of shape derivatives (II)



<u>Intuition:</u> *For "small" θ, the variation of $J(\Omega_\theta)$ with respect to $J(\Omega)$ is proportional to the distance $\theta \cdot n$ between both sets and to the magnitude of f.*

First examples of shape derivatives (III)

- This formula is a particular case of the Transport (or Reynolds) theorem, used to derive the equations of motion from conservation principles in fluid mechanics:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega(t)} f(t,x)\,\mathrm{d}x\right)}_{\substack{\textbf{Total amount of } f(t,x) \\ \textbf{within moving domain } \Omega(t)}} = \underbrace{\int_{\Omega(t)} \frac{\partial f}{\partial t}(t,x)\,\mathrm{d}x}_{\substack{\textbf{``Eulerian'' change of } f(t,x) \\ \textbf{over time}}} + \underbrace{\int_{\partial\Omega(t)} f(t,x)(\theta\cdot n_t)(x)\,\mathrm{d}s(x)}_{\substack{\textbf{Variation of } f \textbf{ due to} \\ \textbf{the motion of } \Omega(t)}}.$$

- This result gives the shape derivative of the volume functional

$$\mathrm{Vol}(\Omega) = \int_\Omega 1\,\mathrm{d}x.$$

Indeed,

$$\mathrm{Vol}'(\Omega)(\theta) = \int_{\partial\Omega} \theta\cdot n\,\mathrm{d}s = \int_\Omega \mathrm{div}\theta\,\mathrm{d}x.$$

$\Rightarrow$ If $\mathrm{div}\theta = 0$, the volume is unchanged (at first order) when Ω is perturbed by θ.

Lemma.

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain, and let $g : \mathbb{R}^d \to \mathbb{R}$ be a smooth function. Consider the functional:

$$J(\Omega) = \int_{\partial\Omega} g(x)\,\mathrm{d}s.$$

Then $J(\Omega)$ is shape differentiable at Ω and its shape derivative is:

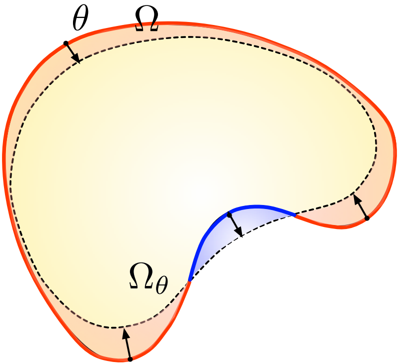
$$J'(\Omega)(\theta) = \int_{\partial\Omega} \left(\frac{\partial g}{\partial n} + \kappa g \right) (\theta \cdot n)\,\mathrm{d}s,$$

where κ is the mean curvature of $\partial\Omega$.

Example: The shape derivative of the perimeter $\mathrm{Per}(\Omega) = \int_{\partial\Omega} 1\,\mathrm{d}s$ is:

$$\mathrm{Per}'(\Omega)(\theta) = \int_{\partial\Omega} \kappa\,(\theta \cdot n)\,\mathrm{d}s.$$

First examples of shape derivatives (V)



Intuition: *$\theta = -\kappa n$ is a descent direction for $\mathrm{Per}(\Omega)$ which is reduced by smearing the bumps of $\partial\Omega$ (i.e. $\theta \cdot n < 0$ when $\kappa > 0$), and sealing its holes (i.e. $\theta \cdot n > 0$ when $\kappa < 0$).*

Structure of shape derivatives (I)

Theorem.

The shape derivative of a function $J(\Omega)$ has the following Hadamard structure:

$$\boxed{J'(\Omega)(\theta) = \int_{\partial\Omega} v_\Omega(x)\, \theta(x) \cdot n(x) \, \mathrm{d}s(x),}$$

where the scalar field $v_\Omega : \partial\Omega \to \mathbb{R}$ depends on:

- *The state u_Ω, solution to the physical boundary value problem at play (if applicable).*
- *An adjoint state p_Ω, solution to a problem depending on $J(\Omega)$.*
- *Geometric quantities related to $\partial\Omega$ (normal vector, curvature).*

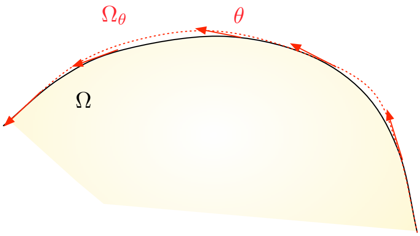
This may hold in the sense of distributions and depend on surface derivatives of $\theta \cdot n$.

The practical computation of shape derivatives by the adjoint method will be presented in `Webinar 2`.

Structure of shape derivatives (II)

- The structure of $J'(\Omega)(\theta)$ encodes that it only depends on the normal component of θ on the boundary:

$$\theta \cdot n = 0 \text{ on } \partial\Omega \;\Rightarrow\; J'(\Omega)(\theta) = 0.$$



A tangential vector field θ, (i.e. $\theta \cdot n = 0$) only accounts for a convection of the shape Ω and $J'(\Omega)(\theta) = 0$.

- A descent direction for $J(\Omega)$ is readily revealed from this expression:

$$\boxed{\theta = -v_\Omega n \;\Rightarrow\; J'(\Omega)(\theta) < 0.}$$

Part I

Basic principles

A generic numerical algorithm

Hadamard's boundary variation method paves the way to a gradient algorithm for the (unconstrained) problem

$$\min_{\Omega} J(\Omega). \qquad (\mathcal{P})$$

- **Input:** Initial guess Ω^0.
- **For** $n = 0, ...,$ **until convergence**,
  1. Compute the solution u_{Ω^n} (and the adjoint p_{Ω^n}) on the domain Ω^n.
  2. From the shape derivative $J'(\Omega^n)$, infer a descent direction θ^n for $J(\Omega)$.
  3. Update the shape Ω^n according to θ^n, so as to get

     $$\Omega^{n+1} := (\mathrm{Id} + \tau^n \theta^n)(\Omega^n),$$

     with a "small" descent step $\tau^n > 0$ such that $J(\Omega^{n+1}) \leq J(\Omega^n)$.
- **Return:** Optimized shape Ω^*.

This geometric shape optimization algorithm faces two main difficulties:

- The shape representation has to be updated during the iterations $\Omega^n \mapsto \Omega^{n+1}$;
- The topology of Ω^n never changes.

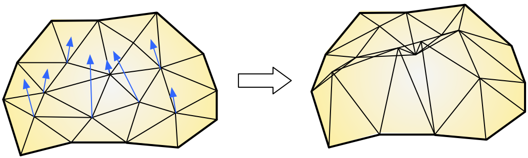
The classical framework: "Lagrangian" approach

- Each shape Ω^n is represented by a (triangular) mesh $\mathcal{T}^n$.
- The Finite Element method is applied on $\mathcal{T}^n$ for computing u_{Ω^n} (and p_{Ω^n}).
- The descent direction θ^n is obtained from the surface form of the shape derivative:

$$J'(\Omega^n)(\theta) = \int_{\partial\Omega^n} v_{\Omega^n} \, \theta \cdot n \, ds \quad \Rightarrow \quad \theta^n = -v_{\Omega^n} n \text{ on } \partial\Omega^n.$$

- The update $\Omega^n \xrightarrow{(\mathrm{Id}+\tau^n\theta^n)} \Omega^{n+1}$ is realized by pushing the nodes of $\mathcal{T}^n$ along $\tau^n\theta^n$, to obtain the new mesh $\mathcal{T}^{n+1}$:

$$\forall \text{ vertex } x \in \mathcal{T}^n, \;\; x \longmapsto x + \tau^n\theta^n(x).$$



Pushing nodes according to the velocity field may result in an invalid configuration.

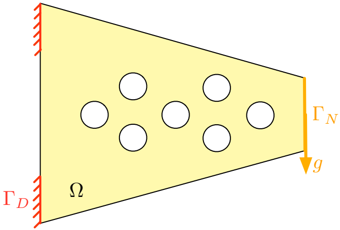
An example of the "Lagrangian" approach

- We minimize the compliance $C(\Omega)$ of a cantilever beam:

$$C(\Omega) = \int_\Omega Ae(u_\Omega) : e(u_\Omega)\,\mathrm{d}x,$$

where u_Ω is the elastic displacement of Ω.

- A volume constraint is imposed: $G(\Omega) = \mathrm{Vol}(\Omega) - V_T$.



The topology does not change and the algorithm stops because it wants to, but cannot break a thin bar.

Part II

The Level Set algorithm

Foreword about the Level Set Method

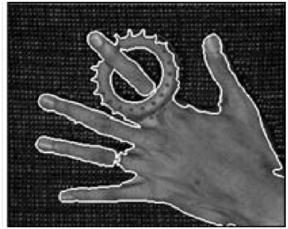
- The Level Set algorithm was pioneered by Osher and Sethian [OSeth] to represent moving interfaces.
- A moving shape $\Omega(t)$ is described implicitly, via a Level Set function

$$\phi(t,x), \;\; t > 0, \;\; x \in D,$$

defined on a fixed "hold-all" domain D.

- The shape $\Omega(t)$ is not meshed, and can undergo dramatic evolutions, including topological changes.
- This viewpoint has progressively been adopted in applicative fields so diverse as fluid mechanics, material science, imaging, solid mechanics,...
- ... and shape and topology optimization!

Collapse of a water column



Active contour algorithm for image segmentation [CreRouDe].

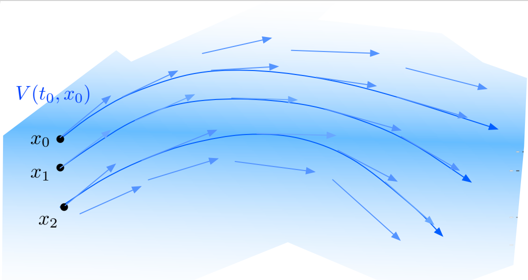
Origin of the Level Set algorithm (I)

- Let $\Omega(t)$ be a shape with boundary $\Gamma(t)$, evolving via a given velocity field $V(t,x)$.
- Early Lagrangian approaches to represent the motion of $\Omega(t)$ track the motion of its individual points.

Definition (Characteristic curve).

Let $V : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}^d$ be a smooth velocity field. The characteristic curve $t \mapsto \chi(x_0,t)$ emerging from a point $x_0 \in \mathbb{R}^d$ at time $t=0$ is defined by the ODE:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\chi(x_0,t)) = V(t,\chi(x_0,t)) & \text{for } t \in (0,T), \\ \chi(x_0,0) = x_0. \end{cases}$$



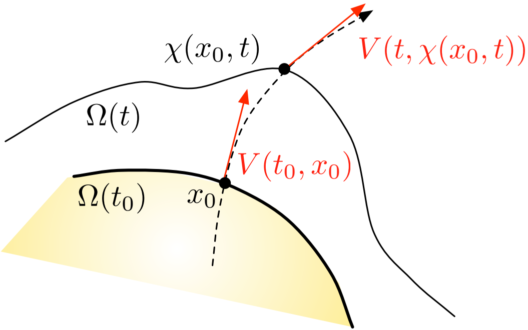
Three characteristic curves of the velocity field V starting at $t=0$ from different points x_0, x_1, x_2.

Origin of the Level Set algorithm (II)

"Intuitive" definition of an evolving domain

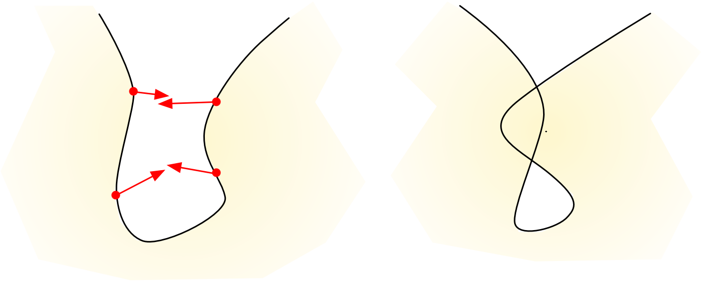
A domain $\Omega(t)$ evolves from an initial configuration $\Omega(0)$ according to a velocity field $V(t,x)$ if it is obtained by advection of its points along V:

$$\Omega(t) = \Big\{ \chi(x_0, t), \; x_0 \in \Omega(0) \Big\}.$$



Origin of the level set algorithm (III)

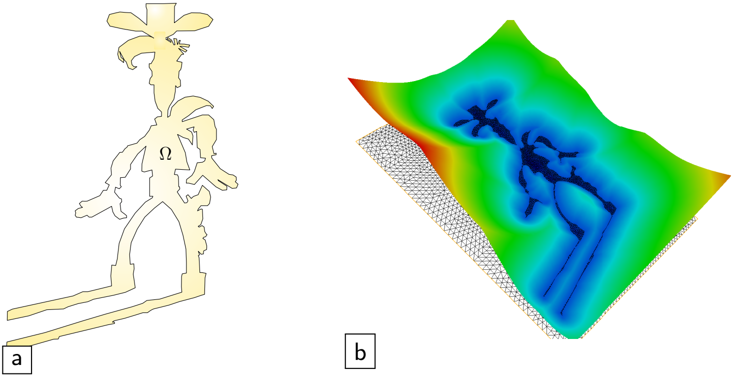
- This Lagrangian viewpoint is unable to accommodate the onset of singularities and self-intersections.



- A robust method for dealing with such phenomena should be:
  - Irreversible: the algorithm "selects" a behavior when $\Omega(t)$ becomes singular.
  - Capable of handling general velocity fields $V(t,x)$ depending on $\Gamma(t)$ through its normal $n(t,x)$, mean curvature $\kappa(t,x)$, etc.
- The Level Set Method is such a Eulerian, shape capturing strategy.

Paradigm: *A shape Ω is implicitly defined by a "Level Set" function $\phi : \mathbb{R}^d \to \mathbb{R}$:*

$$\forall x \in \mathbb{R}^d, \quad \begin{cases} \phi(x) = 0 & \text{if } x \in \Gamma := \partial\Omega, \\ \phi(x) < 0 & \text{if } x \in \Omega, \\ \phi(x) > 0 & \text{otherwise.} \end{cases}$$



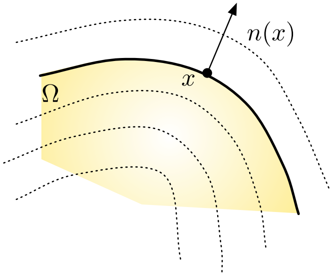
(a) A bounded domain $\Omega \subset \mathbb{R}^2$; (b) Graph of an associated Level Set function.

Implicit geometries

Most operations on shapes $\Omega \subset \mathbb{R}^d$ can be realized via Level Set functions $\phi(x)$.

1. Calculation of the unit normal vector n to Γ pointing outward Ω:

$$\forall x \in \Gamma, \;\; n(x) = \frac{\nabla \phi(x)}{|\nabla \phi(x)|}.$$



Normal vector to a domain Ω; some isolines of ϕ are dotted.

2. Calculation of the mean curvature κ of Γ:

$$\forall x \in \Gamma, \;\; \kappa(x) = \mathrm{div}\left(\frac{\nabla \phi(x)}{|\nabla \phi(x)|}\right).$$

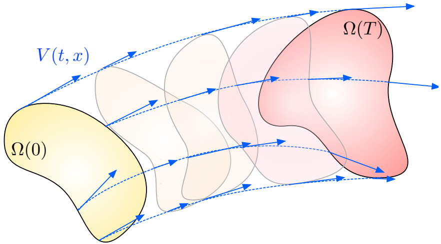
3. Calculation of integrals on Ω and Γ with an Heaviside function.

Part II

The Level Set algorithm

Evolving domains (I)

- Let $\Omega(t) \subset \mathbb{R}^d$ be a domain with boundary $\Gamma(t)$, evolving over a time period $(0, T)$ according to a velocity field $V(t, x)$.



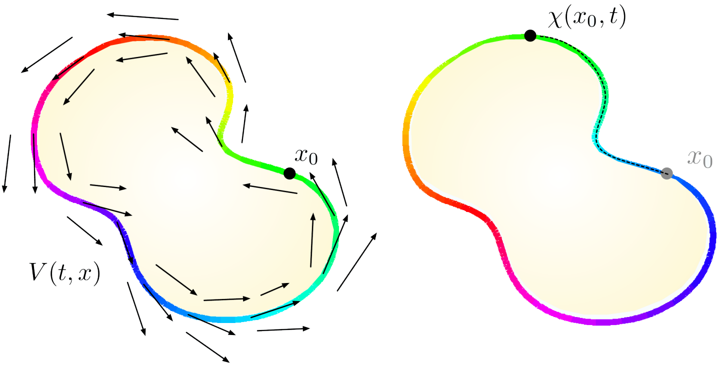
- Let $\phi(t, \cdot)$ be a Level Set function for $\Omega(t)$: $\begin{cases} \phi(t,x) < 0 & \text{if } x \in \Omega(t), \\ \phi(t,x) = 0 & \text{if } x \in \Gamma(t), \\ \phi(t,x) > 0 & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega(t)}. \end{cases}$

Questions

- How does the motion of $\Omega(t)$ translate in terms of $\phi(t, \cdot)$?
- To start with, what does it even mean for $\Omega(t)$ to evolve according to $V(t, x)$?

Evolving domains (II)

- The motion of $\Omega(t)$ depends only on the normal component of $V(t,x)$.



A tangential velocity $V(t,x)$, (i.e. $V \cdot n = 0$) only accounts for a sliding of $\Omega(t)$.

- The tangential component of $V(t,x)$ can thus be ignored.

Evolving domains (III)

The motion of $\Omega(t)$ may be classified into three categories depending on the nature of the velocity field $V(t,x)$.

1. $\Omega(t)$ is passively transported when $V(t,x)$ does not depend on $\Omega(t)$, i.e. it is a prescribed datum.

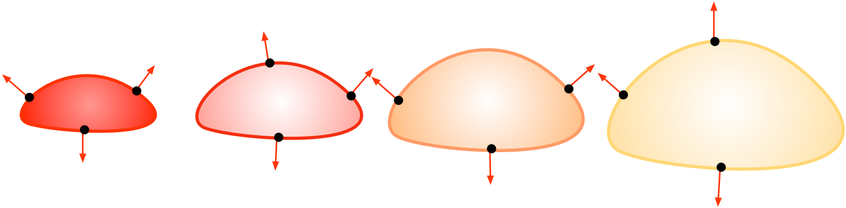
2. The velocity $V(t,x)$ depends on local features of $\Omega(t)$ or $\Gamma(t)$, such as:
   - The normal vector $n(t,x)$ at $x \in \Gamma(t)$;
   - The mean curvature $\kappa(t,x)$ of $\Gamma(t)$.

3. The field $V(t,x)$ depends on global features of the domain $\Omega(t)$, e.g. it depends on the solution to a partial differential equation (PDE) posed on $\Omega(t)$.

Example (I): The flame propagation model

The shape $\Omega(t)$ is a burnt region, which expands with constant, normal velocity c:

$$V(t,x) = c\, n(t,x), \text{ where } c > 0 \text{ is a constant.}$$



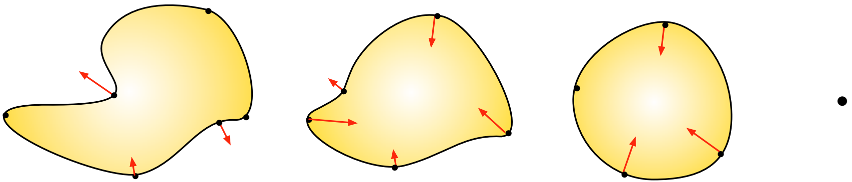
An example of the dynamics in the flame propagation model.

Example (II): The mean curvature flow

The velocity field $V(t,x)$ reads:

$$V(t,x) = -\kappa(t,x)\, n(t,x),$$

that is, $\Omega(t)$ evolves by "resorption of its bumps", and "filling of its creases".



An example of the dynamics of the mean curvature flow: Grayson's theorem [Grayson].

Deriving an equation for the Level Set evolution (I)

- Let $\Omega(t)$ be a shape moving with velocity field $V(t,x)$ (that may depend on $\Omega(t)$).
- Let $\phi(t,\cdot)$ be a smooth Level Set function for $\Omega(t)$, i.e:

$$\forall t \in (0,T),\ x \in \mathbb{R}^d, \begin{cases} \phi(t,x) < 0 & \text{if } x \in \Omega(t), \\ \phi(t,x) = 0 & \text{if } x \in \Gamma(t), \\ \phi(t,x) > 0 & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega(t)}. \end{cases}$$

- Let $x_0 \in \Gamma(0)$ be fixed; by the intuitive definition of an evolving domain, it comes:

$$\forall t \in (0,T),\ \ \phi(t, \underbrace{\chi(x_0,t)}_{\in \Gamma(t)}) = 0.$$

- Differentiating and using the chain rule, we obtain:

$$\frac{\partial \phi}{\partial t}(t, \chi(x_0,t)) + \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}(\chi(x_0,t))}_{=V(t,\chi(x_0,t))} \cdot \nabla \phi(t, \chi(x_0,t)) = 0.$$

Deriving an equation for the Level Set evolution (II)

- Since this holds for any point $x_0 \in \Gamma(0)$, we obtain the Level Set advection equation

$$\boxed{\forall t \in (0,T), \;\; x \in \mathbb{R}^d, \;\; \frac{\partial \phi}{\partial t}(t,x) + V(t,x) \cdot \nabla \phi(t,x) = 0. \qquad \text{(LS-ADV)}}$$

- Since the tangential component of the velocity does not play any role, one may assume that $V(t,x)$ is aligned with the normal vector $n(t,x)$ to $\Omega(t)$, that is:

$$V(t,x) = v(t,x) \underbrace{\frac{\nabla \phi(t,x)}{|\nabla \phi(t,x)|}}_{=n(t,x)}, \text{ for some scalar field } v(t,x).$$

The equation (LS-ADV) rewrites as the Level Set Hamilton-Jacobi equation

$$\boxed{\forall t \in (0,T), \;\; x \in \mathbb{R}^d, \;\; \frac{\partial \phi}{\partial t}(t,x) + v(t,x)|\nabla \phi(t,x)| = 0. \qquad \text{(LS-HJ)}}$$

Evolving domains: comments (I)

- Strictly speaking, (LS-ADV) and (LS-HJ) only hold for pairs (t, x) with $x \in \Gamma(t)$. However, the previous analysis applies *mutatis mutandis* when

$$x_0 \in \Gamma_c(0) := \left\{ x \in \mathbb{R}^d, \ \phi(0, x) = c \right\}, \text{ for arbitrary } c \in \mathbb{R}.$$

  Thus, the equations (LS-ADV) and (LS-HJ) actually encode that all the level sets of ϕ move according to $V(t, x)$.

- The velocity fields $V(t, x)$ and $v(t, x)$ often make sense only for $x \in \Gamma(t)$. In the above derivation, it is assumed that they are extended to the whole space $\mathbb{R}^d$.

Evolving domains: comments (II)

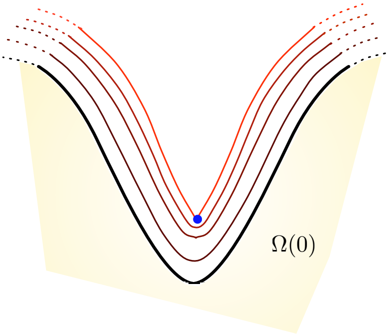
- Critically, this derivation rests upon the assumption that $\Omega(t)$, $V(t,x)$, $\phi(t,x)$ stay smooth over $(0, T)$.

Question

How is it possible to account for the evolution of $\Omega(t)$ when either the domain $\Omega(t)$, or the velocity field $V(t,x)$ has developed a singularity?

This problem is not a pure mathematicality: even in the simplest models, $\Omega(t)$ and $V(t,x)$ (thus $\phi(t,x)$) become singular in finite time.

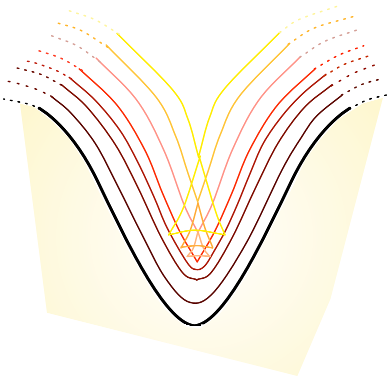
In the flame propagation model, a domain $\Omega(t)$ initially featuring a concave region evolves according to the velocity field $V(t,x) = n(t,x)$.



A few positions of $\Gamma(t)$; at a critical time, $\Gamma(t)$ develops a singularity where $n(t,x)$ (thus $V(t,x)$) is not defined (blue dot).

Emergence of singularities (II)

There are then several ways of giving a sense to the evolution of the front once a singularity has appeared.



(Left) Evolution of $\Omega(t)$ obtained by "pursuing the motion" of all the points of $\Gamma(t)$ where the normal vector is defined; (right) Evolution of $\Omega(t)$ obtained by imposing an "entropy criterion": "a burnt point stays burnt".

Emergence of singularities (III)

- Singularities are inevitable, even when the initial situation is smooth.
- What happens after the onset of singularities is actually a matter of defining the motion of a possibly non smooth domain, under a possibly non smooth velocity.
- The theory of viscosity solutions [Ba, CIL] offers a notion of solutions which selects the correct physical evolution.

Mathematical definition of an evolving domain

1. Start from any Level Set function $\phi_0(x)$ for the initial domain $\Omega(0)$.
2. Find the unique viscosity solution of the Level Set evolution equation

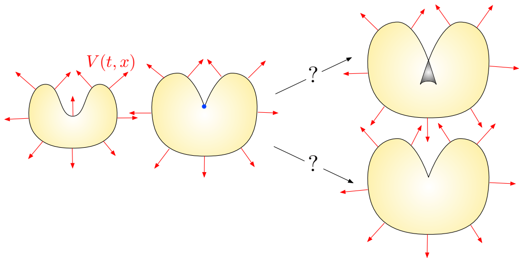
$$\begin{cases} \frac{\partial \phi}{\partial t}(t,x) + v(t,x)|\nabla\phi(t,x)| = 0 & \text{for } t \in (0,T),\ x \in \mathbb{R}^d, \\ \phi(0,x) = \phi_0(x) & \text{for } x \in \mathbb{R}^d. \end{cases}$$

3. Define the domain $\Omega(t)$ by $\Omega(t) = \left\{x \in \mathbb{R}^d,\ \phi(t,x) < 0\right\}$.

Emergence of singularities: final comments

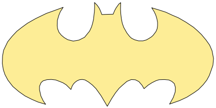
Why solve the non-linear Hamilton-Jacobi equation (LS-HJ) instead of using the simple Lagrangian evolution scheme?

- The Lagrangian method is unable to define the motion of $\Omega(t)$ after the onset of singularities.

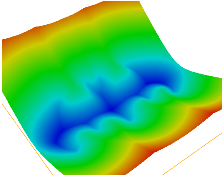


- The theory of viscosity solutions for Hamilton-Jacobi equations goes beyond the singularities and defines fronts for all times t.
- Viscosity solutions naturally accommodate topology changes.
- This is at the price of loosing time-reversibility of the evolution.

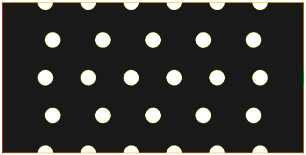
- Domain $\Omega \subset \mathbb{R}^d$.



- Level Set function $\phi(t,x)$.

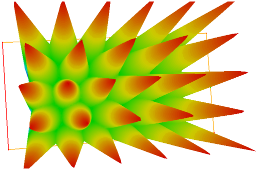


- Evolution w.r.t. a normal vector field $V(t,x) = v(t,x)n(t,x)$.



- Resolution of the Level Set equation.

$$\frac{\partial \phi}{\partial t}(t,x) + v(t,x)|\nabla \phi(t,x)| = 0.$$



Part II

The Level Set algorithm

Solution of the Level Set equation (I)

Let $\Omega(t)$ be a shape, described by a Level Set function $\phi(t,x)$

$$\Omega(t) = \left\{ x \in \mathbb{R}^d, \ \phi(t,x) < 0 \right\},$$

evolving through a normal velocity $v(t,x)$ (a descent direction for its optimization).

- The Level Set Hamilton-Jacobi equation is

$$\begin{cases} \frac{\partial \phi}{\partial t}(t,x) + v(t,x)|\nabla\phi(t,x)| = 0 & \text{for } (t,x) \in (0,T) \times \mathbb{R}^d, \\ \phi(t=0,x) = \phi_0(x) & \text{for } x \in \mathbb{R}^d. \end{cases}$$

- We discretize $(0,T)$ into a series of sub-intervals $0 = t^0 < t^1 < \ldots < t^N = T$, where the velocity is assumed to be constant in time:

$$v(t,x) \approx v^n(x) := v(t^n,x), \ \text{for } t \in (t^n, t^{n+1}).$$

- Doing so, it is enough to consider a steady-state velocity $v(x)$:

$$\begin{cases} \frac{\partial \phi}{\partial t}(t,x) + v(x)|\nabla\phi(t,x)| = 0 & \text{for } (t,x) \in (0,T) \times \mathbb{R}^d, \\ \phi(t=0,x) = \phi_0(x) & \text{for } x \in \mathbb{R}^d. \end{cases}$$

Solution of the Level Set equation (II)

- We present numerical schemes for first-order Hamilton-Jacobi equations of the form:
$$\left\{ \begin{array}{ll} \frac{\partial \phi}{\partial t}(t,x) + v(x)|\nabla \phi(t,x)| = 0 & \text{for } (t,x) \in (0,T) \times \mathbb{R}^d, \\ \phi(t=0,x) = \phi_0(x) & \text{for } x \in \mathbb{R}^d. \end{array} \right.$$

- The theory of viscosity solutions for Hamilton-Jacobi equations gives existence and uniqueness of solutions for this equation.

- It also proposes adequate numerical schemes to compute viscosity solutions.

- The key idea of these numerical schemes is to rely on upwinding, as in fluid mechanics.

- For simplicity, we focus on the 2d situation, where the space is equipped with a Cartesian grid.

The need for upwinding (I)

Example: When $v(x) \equiv c \in \mathbb{R}$, (LS-HJ) is the Eikonal equation:

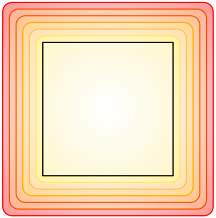
$$\frac{\partial \phi}{\partial t}(t,x) + c|\nabla \phi(t,x)| = 0.$$

- The viscosity solution to this equation equals:

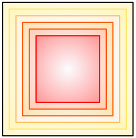
$$\phi(t,x) = d_{\Omega(0)}(x) - c\,t, \text{ where } d_{\Omega(0)}(x) = \begin{cases} -d(x,\Gamma(0)) & \text{if } x \in \Omega(0), \\ 0 & \text{if } x \in \Gamma(0), \\ d(x,\Gamma(0)) & \text{otherwise,} \end{cases}$$

  is the signed distance function to the initial domain $\Omega(0)$.

- This solution is irreversible: some corners remain sharp, others get rounded.



$c = 1$



$c = -1$

- An upwind scheme is needed to distinguish both situations.

The need for upwinding (II)

- Let us consider the one-dimensional Eikonal equation:

$$\frac{\partial \phi}{\partial t}(t,x) + c\left|\frac{\partial \phi}{\partial x}(t,x)\right| = 0, \quad t > 0,\ x \in \mathbb{R},$$

  with constant, positive velocity $c > 0$.

- (Formally) Setting $u = \frac{\partial \phi}{\partial x}$ and taking derivatives, this becomes:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \quad \text{with } f(u) = c\,|u| \text{ convex } ,$$

  which is a one-dimensional hyperbolic equation.

- According to the "classical" theory of shock waves, the solution of the Riemann problem with initial data $u(0,x) = \begin{cases} u_L & x < 0, \\ u_R & x > 0. \end{cases}$ is

$$u(t,x) = \begin{cases} \text{shock} & \text{if } u_L > u_R, \\ \text{rarefaction} & \text{if } u_L < u_R. \end{cases} \quad \Rightarrow \quad \begin{cases} \text{sharp corner} & \text{for } \psi, \\ \text{rounded corner} & \text{for } \psi. \end{cases}$$

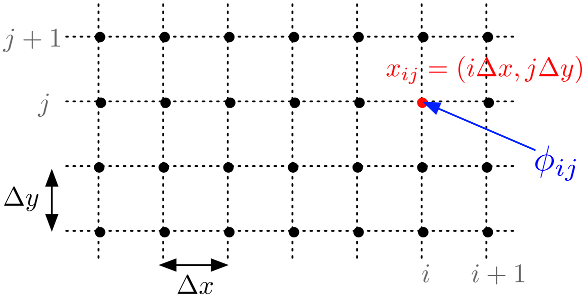
- This behavior is preserved numerically if one uses upwind schemes.

Osher-Sethian's first-order upwind scheme (I)

- The time interval $(0, T)$ is split into $N = T/\Delta t$ subintervals:

$$(t^n, t^{n+1}), \text{ where } t^n = n\Delta t, \;\; n = 0, ..., N, \text{ and } \Delta t \text{ is a time step.}$$

- The space is discretized by a Cartesian grid with steps $\Delta x, \Delta y$.



- For $i, j \in \mathbb{Z}$, we introduce the finite difference quantities:

$$D_{ij}^{+x}\phi = \frac{\phi_{i+1j} - \phi_{ij}}{\Delta x} \quad ; \quad D_{ij}^{-x}\phi = \frac{\phi_{ij} - \phi_{i-1j}}{\Delta x},$$

and:

$$D_{ij}^{+y}\phi = \frac{\phi_{ij+1} - \phi_{ij}}{\Delta y} \quad ; \quad D_{ij}^{-y}\phi = \frac{\phi_{ij} - \phi_{ij-1}}{\Delta y}.$$

Osher-Sethian's first-order upwind scheme (II)

The Level Set Hamilton-Jacobi equation is solved by the following numerical scheme:

$$\boxed{\begin{cases} \forall n \in \mathbb{N}, i,j \in \mathbb{Z}, & \phi_{ij}^{n+1} = \phi_{ij}^{n} - \Delta t \left(\max(v_{ij},0)\nabla_{ij}^{+}\phi^{n} + \min(v_{ij},0)\nabla_{ij}^{-}\phi^{n} \right), \\ \forall i,j \in \mathbb{Z}, & \phi_{ij}^{0} = \phi_0(i\Delta x, j\Delta y), \end{cases}}$$

with the discretizations $\nabla_{ij}^{+}\phi$ and $\nabla_{ij}^{-}\phi$ of the gradient norm $|\nabla\phi|$ defined by:

$$\nabla_{ij}^{+}\phi = \begin{pmatrix} \max(\max(D_{ij}^{-x}\phi, 0), -\min(D_{ij}^{+x}\phi, 0))^2 \\ + \max(\max(D_{ij}^{-y}\phi, 0), -\min(D_{ij}^{+y}\phi, 0))^2 \end{pmatrix}^{\frac{1}{2}},$$

and

$$\nabla_{ij}^{-}\phi = \begin{pmatrix} \max(\max(D_{ij}^{+x}\phi, 0), -\min(D_{ij}^{-x}\phi, 0))^2 \\ + \max(\max(D_{ij}^{+y}\phi, 0), -\min(D_{ij}^{-y}\phi, 0))^2 \end{pmatrix}^{\frac{1}{2}}.$$

Osher-Sethian's first-order upwind scheme (III)

- The quantity $\nabla^+_{ij}\phi$ (resp. $\nabla^-_{ij}\phi$) is upwind (resp. downwind): it involves only those values among $\{\phi_{i-1j}, \phi_{i+1j}, \phi_{ij-1}, \phi_{ij+1}\}$ which are smaller (resp. larger) than ϕ_{ij}.
- The discretization of the Hamiltonian $H(x,p) = v(x)|p|$ by the quantity

$$\max(v_{ij}, 0)\nabla^+_{ij}\phi^n + \min(v_{ij}, 0)\nabla^-_{ij}\phi^n$$

  is upwind: for given i, j, n, the update $\phi^n_{ij} \to \phi^{n+1}_{ij}$ is only carried out using information coming from

  - smaller values than ϕ^n_{ij} if $v_{ij} \geq 0$,
  - larger values than $\phi^n_{ij} \leq 0$.

- This scheme is convergent, provided the following CFL-like condition is fulfilled:

$$\left(\sup_{i,j} v_{ij}\right) \frac{\Delta t}{\min(\Delta x, \Delta y)} \leq 1, \text{ i.e.}$$

  "The information cannot travel more than one cell during one time step".

- High-order variants of this strategy are available.

Part II

The Level Set algorithm

The need for calculating / re-initializing distance functions (I)

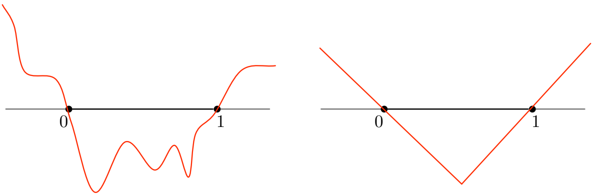
The signed distance function is one preferred Level Set function for a domain $\Omega \subset \mathbb{R}^d$.

Definition (Signed distance function).

The signed distance function d_Ω to a domain $\Omega \subset \mathbb{R}^d$ is:

$$d_\Omega(x) = \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \partial\Omega, \\ d(x, \partial\Omega) & \text{if } x \in {}^c\overline{\Omega}, \end{cases}$$

where $d(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$ is the usual Euclidean distance to $\partial\Omega$.



Two Level Set functions for the domain $\Omega = (0,1) \subset \mathbb{R}$. The signed distance function (right) satisfies $|\nabla d_\Omega(x)| = 1$ for a.e. $x \in \mathbb{R}^d$.

The need for calculating / re-initializing distance functions (II)

- For numerical stability, it is crucial that the Level Set function $\phi(t,\cdot)$ of $\Omega(t)$ stay "close" to a signed distance function for $t \geq 0$:

$$|\nabla\phi(t,x)| = 1 \text{ for a.e. } x \in \mathbb{R}^d \;\Rightarrow\; \phi(t,\cdot) \text{ is not "too steep", nor "too flat".}$$

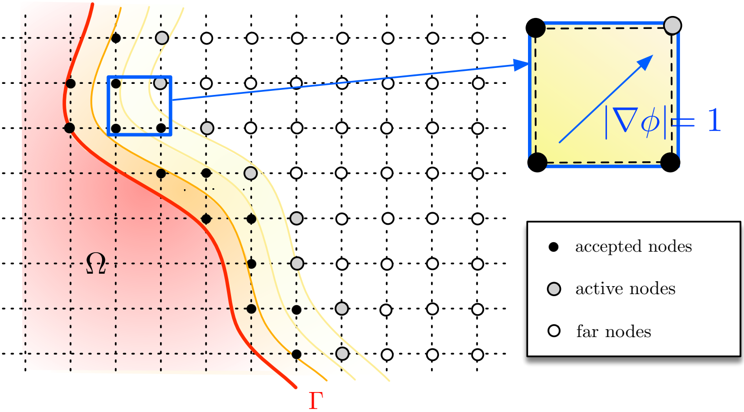
- Unfortunately, even if the initial Level Set function ϕ_0 is a signed distance function, $\phi(t,\cdot)$ is bound not to stay so during motion.

- In the practice of the Level Set Method, it is essential to
  - Convert the initial shape $\Omega(0)$ into a Level Set function by calculating its signed distance function, e.g. via the Fast Marching Method.
  - Periodically re-initialize a current, very steep or flat Level Set function into a signed distance function for the same domain.

- **Remark:** The use of the signed distance function as a Level Set function is not required by the theory: it is a (crucial) matter of numerical accuracy [Cho].

Initializing distance functions with the Fast Marching algorithm

- The Fast Marching Method was pioneered in [SethianFMM].
- Inspired by Dijkstra's algorithm, it mimicks the expansion of a front from Γ.
- An upwind discretization of the eikonal equation $|\nabla\phi| = 1$ is used to predict trial values at nodes "on the front", from accepted values, where it has already passed.
- The front advances from closer to further nodes.



Level Set redistancing

- Let $\Omega \subset \mathbb{R}^d$ be a domain, described with a possibly "ill-shaped" Level Set function ϕ_0 (with very steep or flat variations).

- The function ϕ_0 is used as the initial state of the redistancing equation [Cho]:

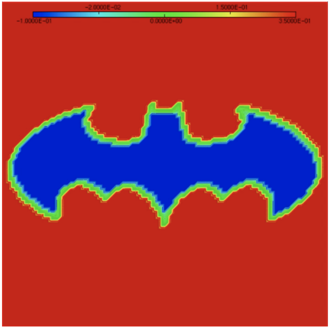
$$\begin{cases} \frac{\partial \psi}{\partial t}(t,x) + \text{sgn}(\phi_0(x)) \, (|\nabla \psi| - 1) = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^d \\ \psi(0,x) = \phi_0(x) & \text{for } x \in \mathbb{R}^d. \end{cases}$$

- Formally, its steady state $\widetilde{\psi}$ satisfies:

$$|\nabla \widetilde{\psi}| - 1 = 0, \text{ and}$$
$$\widetilde{\psi}(x) = 0 \text{ on } \Gamma = \{x \in \mathbb{R}^d \text{ s.t. } \phi_0(x) = 0\},$$

i.e. $\widetilde{\psi}$ is the signed distance function to Ω.

- Its study reveals that ϕ_0 is steadily "regularized" into d_Ω, starting from Γ, to the "far" region [AuAu].



The Level Set Method: computational efficiency

- The numerical algorithms for the practice of the Level Set Method are very cheap: no linear systems need to be inverted.

- They are explicit algorithms on Cartesian meshes: low CPU, low memory requirement.

- Several time steps can be performed at each optimization iteration: the explicit time step is not the descent step of optimization!

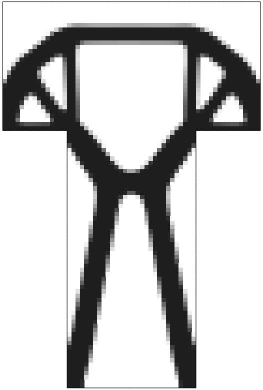
- In the next `Webinar 2`, we shall see how to handle the Level Set Method on simplicial (triangular) meshes.

Part III

The Level Set Method for shape and topology optimization

The level set Method for shape and topology optimization

- Preliminary forays of the Level Set Method in shape optimization appear in [SeWie, OSan].
- The method was later made more systematic in [AlJouToa, AlJouToa2, Wan].
- The shapes Ω^n are embedded in a working domain D equipped with a fixed (e.g. Cartesian) mesh $\mathcal{T}$.
- The successive shapes Ω^n are accounted for in the level set framework, i.e. via a function $\phi^n : D \to \mathbb{R}$ which implicitly defines them.
- Once a descent direction ($\equiv$ a velocity) θ^n is found, ϕ^n is updated to ϕ^{n+1} by solving the Level Set Hamilton-Jacobi equation on $\mathcal{T}$.
- This approach is versatile and does not require a mesh of the shapes at each iteration.

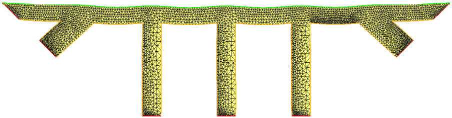


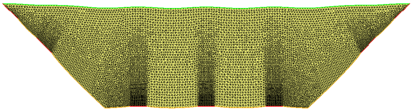
Shape accounted for by a Level Set description

Problem: At each iteration n, no mesh of Ω^n is available to solve the state and adjoint equations on Ω^n.

Remedy: Approximate these by problems posed on the total domain D.

Example: In the linear elasticity context, use the ersatz material method:





$$\left\{ \begin{array}{rl} -\mathrm{div}(Ae(u_\Omega)) = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ Ae(u_\Omega)n = g & \text{on } \Gamma_N, \\ Ae(u_\Omega)n = 0 & \text{on } \Gamma. \end{array} \right. \approx \left\{ \begin{array}{rl} -\mathrm{div}(A_\varepsilon e(u_{\Omega,\varepsilon})) = 0 & \text{in } D, \\ u_{\Omega,\varepsilon} = 0 & \text{on } \Gamma_D, \\ A_\varepsilon e(u_{\Omega,\varepsilon})n = g & \text{on } \Gamma_N, \\ Ae(u_{\Omega,\varepsilon})n = 0 & \text{on } \partial D \setminus (\Gamma_D \cup \Gamma_N), \end{array} \right.$$

(Problem posed on Ω) *(Problem posed on D)*

with the approximate Hooke's tensor $A_\varepsilon := \chi_\Omega A + (1-\chi_\Omega)\varepsilon A, \ \varepsilon \ll 1.$

Summary of the algorithm

- **Input:**
  - Mesh (e.g. Cartesian grid) $\mathcal{T}$ of the computational domain D;
  - Level set function ϕ^0 for the initial shape Ω^0, defined at the vertices of $\mathcal{T}$.
- **For** $n = 0, ...,$ **until convergence**
  1. Solve the state equation for u_{Ω^n} on $\mathcal{T}$.
  2. Solve the adjoint equation for p_{Ω^n} on $\mathcal{T}$.
  3. Identify a descent direction $\theta^n = v^n(x)n(x)$ for $J(\Omega)$ from Ω^n.
  4. Solve the Level Set Hamilton-Jacobi equation

$$\begin{cases} \frac{\partial \psi}{\partial t} + v^n |\nabla \psi| = 0 & \text{on } (0, \tau^n) \times D, \\ \psi(t = 0, \cdot) = \phi^n. \end{cases}$$

     for a "small" pseudo-time step τ^n, and set $\phi^{n+1} = \psi(t = \tau^n, \cdot)$.
  5. Stopping criterion (typically when $v^n \approx 0$).
- **Return:** Level set function ϕ^* for the optimized shape Ω^*.

Comments on the algorithm

- One may use different meshes $\mathcal{T}_{\rm FE}$ and $\mathcal{T}_{\rm LS}$ for the Finite Element analysis and the practice of the Level Set method, e.g.
  - A Cartesian grid $\mathcal{T}_{\rm LS}$ where the previous Finite Difference schemes are used for the Level Set Method,
  - A simplicial mesh $\mathcal{T}_{\rm FE}$ of the (complex) computational domain D for mechanical computations by a third-party solver.

  This leverages interpolation operators between $\mathcal{T}_{\rm FE}$ and $\mathcal{T}_{\rm LS}$.

- The derivative of $J(\Omega)$ is a shape derivative, which is completely different from a parametric derivative as in density-based methods.

- The Level Set function ϕ is used solely for the numerical capture of the shape evolution: it is not the optimization variable.

- In the next `Webinar 2`, we shall discuss the treatment of equality and inequality constraints in the optimization problem.

Part IV

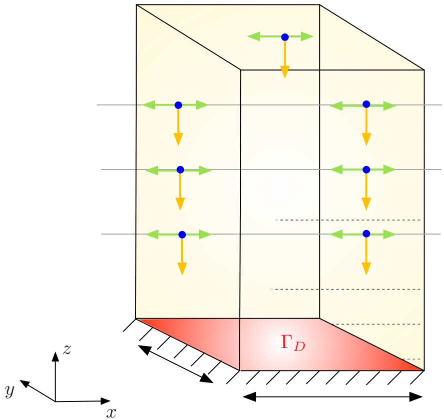
A few illustrations

We consider the optimization of an electric pylon $\Omega \subset \mathbb{R}^3$.

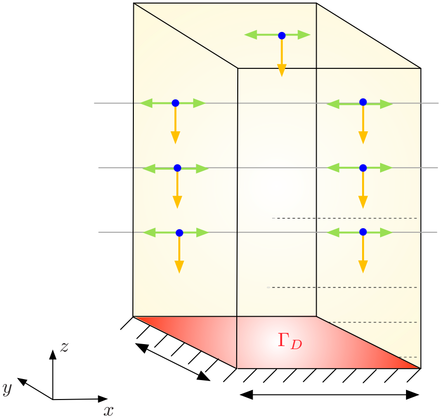
- At its basis Γ_D, the pylon is fixed to the ground.
- It is submitted to the weight of the cables attached to its arms, and to wind loads.
- The displacement $u_\Omega : \Omega \to \mathbb{R}^3$ is the solution to the linear elasticity system.
- The compliance of the structure,

$$J(\Omega) = \int_\Omega Ae(u_\Omega) : e(u_\Omega)\, \mathrm{d}x$$

is minimized under a volume constraint.



Optimization of a mast under multiple loads (II)

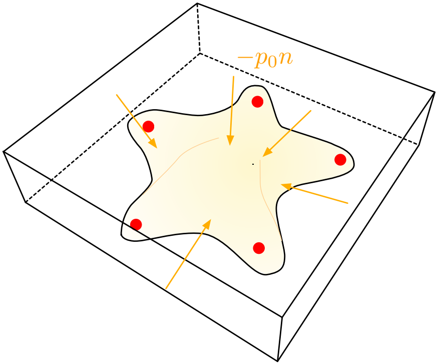


We consider the optimization of the shape of a submarine dome $\Omega \subset \mathbb{R}^3$.

- The dome is anchored at five points on the bottom of the design domain D.
- It is submitted to hydrostatic pressure forces:

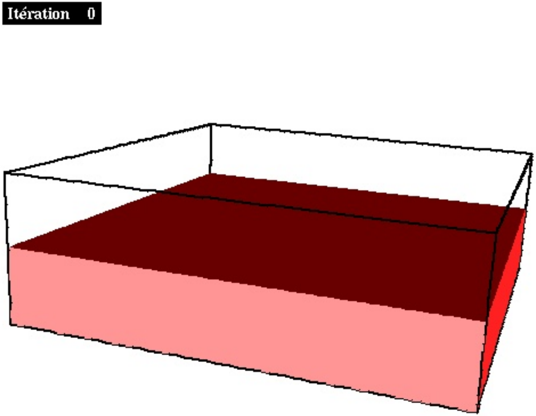
$$-p_0 n \text{ on } \partial\Omega, \text{ where } n : \partial\Omega \to \mathbb{R}^3 \text{ is the unit normal vector to } \partial\Omega.$$

- The compliance of Ω is minimized, under a volume constraint.



Optimization of a dome under hydrostatic pressure loads (II)

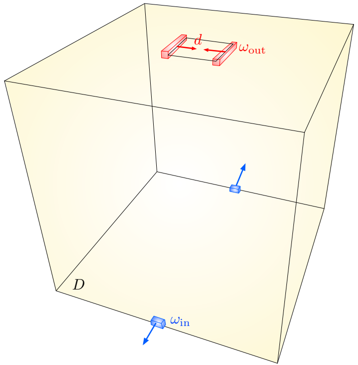
G. Allaire, F. Jouve & A.-M. Toader



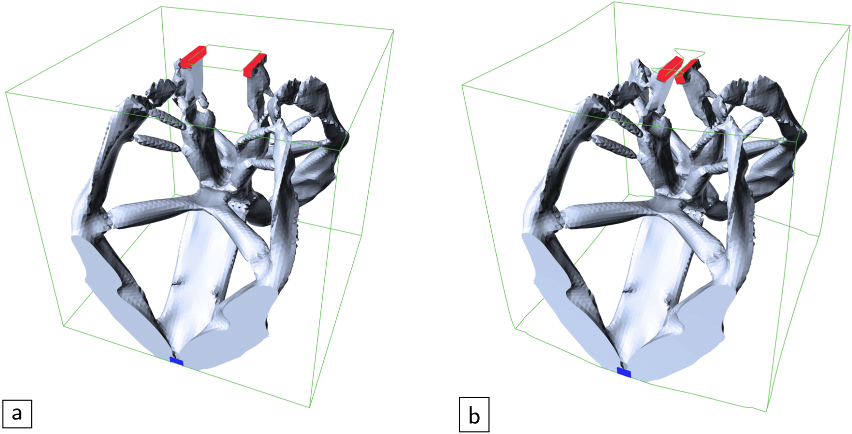
G. Allaire, C. Dapogny & F. Jouve

- We optimize a jointless compliant mechanism, converting an input displacement on an input region ω_{in} into a maximum output displacement on ω_{out}.
- We minimize the non self-adjoint "geometric advantage" criterion:

$$J(\Omega) = -\frac{\int_{\omega_{\text{out}}} |u_\Omega \cdot d| \, \mathrm{d}s}{\int_{\omega_{\text{in}}} |u_\Omega| \, \mathrm{d}s}.$$



Optimization of a gripping mechanism (II)

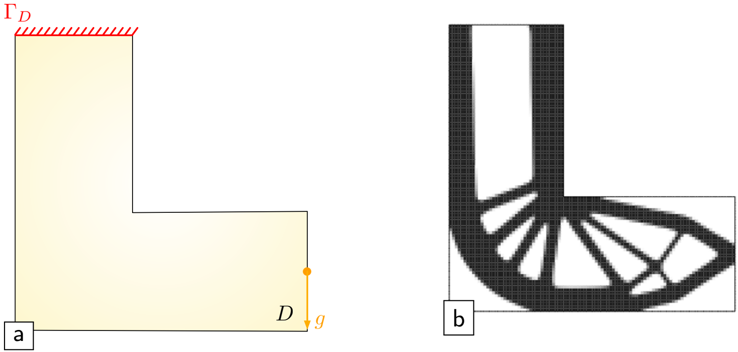


(a) Optimized gripping mechanism; (b) Deformed configuration.

- We minimize the L^p norm of the stress tensor within an L-shaped beam:

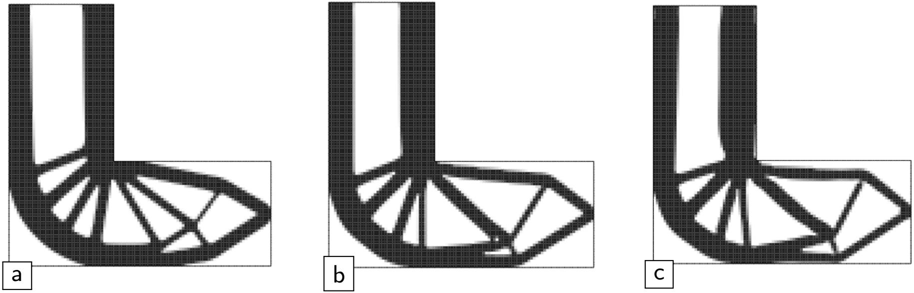
$$J(\Omega) = \int_\Omega k(x)||\sigma(u_\Omega)||^p \, \mathrm{d}x,$$

under a constraint on the volume $\mathrm{Vol}(\Omega)$ of the structure.



(a) Setting of the L-shaped beam problem; (b) Optimized shape with respect to the compliance.

Optimization of a stress-based functional (II)



Optimized shapes of the L-beam associated to the exponent values (a) $p = 2$; *(b)* $p = 5$, *and (c)* $p = 10$.

The Level Set Method: assets and drawbacks

Assets of the method:

- Optimal design problems are formulated in the exact language of shapes: the expressions of pressure loads, geometric constraints.... are natural.
- The Level Set Method allows to account for dramatic evolutions of the shape.

Drawbacks:

- The shape Ω is not meshed exactly: need to approximate the physical equations.

  The ersatz material method works pretty well for compliance or stress optimization, but much less so for the design of compliant mechanisms.

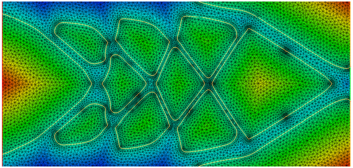
- The Level Set Method is quite sensitive to the initial design, especially in 2d.
- At first glance, its implementation is more complex than that of density methods:
  - Admittedly, the theory of shape derivatives is pretty involved;
  - Efficient open-source libraries exist for the practice of the Level Set Method.

Overview of the next webinars

- Webinar 2 Advanced methods for the practice of the Level Set Method.

  - Practical computation of shape derivatives;
  - Details on the optimization algorithms;
  - Making the Level Set Method efficient in practice;
  - A few applications.

- Webinar 3 A body-fitted variant of the Level Set Method.



  - A taste of remeshing;
  - Presentation of the method;
  - Multi-physics applications.

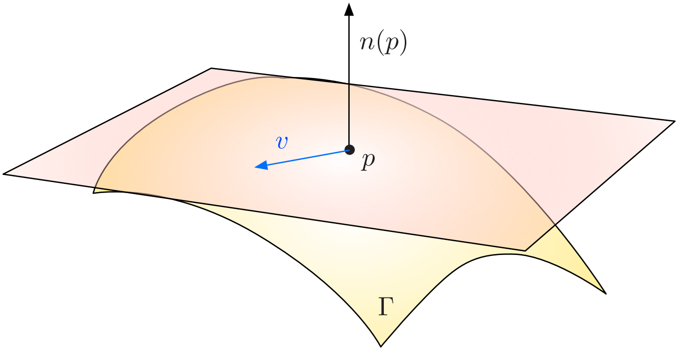
Thank you for your attention!

Technical appendix

Surfaces and curvature (I)

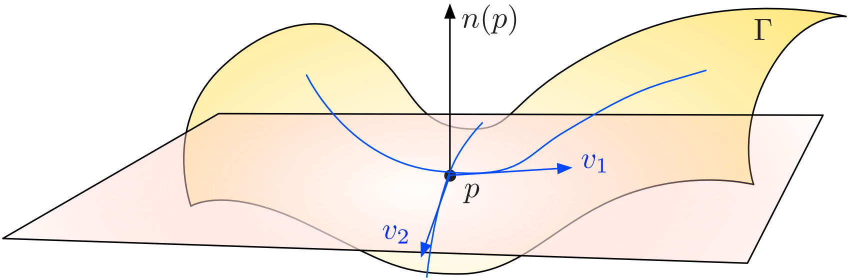
At first order, in the neighborhood of a point $p \in \Gamma$, a surface Γ behaves like a plane, the tangent plane,

- With normal vector $n(p)$,
- Which contains the tangential directions to Γ.



Surfaces and curvature (II)

- At second order in the neighborhood of $p \in \Gamma$, the surface Γ has one curvature in each tangential direction.
- The principal directions at p are those tangential directions $v_1(p)$ et $v_2(p)$ associated to the lower and larger curvatures $\kappa_1(p)$ et $\kappa_2(p)$.
- The mean curvature $\kappa(p)$ is the sum $\kappa(p) = \kappa_1(p) + \kappa_2(p)$.



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